Coset Construction of Parafermionic Hall States*

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Abstract

Fractional quantum Hall fluids with fillings $2 < \nu < 3$ have been recently proposed which generalize the Pfaffian state ($\nu = 2 + 1/2$) into a hierarchy of states with parafermionic excitations. We describe here the corresponding $\mathbb{Z}_k$-parafermion conformal field theory by means of the coset construction $\widehat{su(k)}_1 \oplus \widehat{su(k)}_1 / \widehat{su(k)}_2$. This extends our earlier derivation of the Pfaffian state from a "parent" state with abelian affine symmetry plus a projection of degrees of freedom. The numerator of the coset is actually a rational conformal field theory made out of $(2k-2)$ scalar fields and a specific extended symmetry. The coset construction projects out some neutral Hall edge excitations while preserving the filling fraction; it also respects the $\mathbb{Z}_k$ parity rule coupling neutral and charged excitations in the parent abelian theory.

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Abstract

Fractional quantum Hall fluids with fillings $2 < \nu < 3$ have been recently proposed which generalize the Pfaffian state ($\nu = 2 + 1/2$) into a hierarchy of states with parafermionic excitations. We describe here the corresponding $\mathbb{Z}_k$-parafermion conformal field theory by means of the coset construction $\widehat{su(k)}_1 \oplus \widehat{su(k)}_1 / \widehat{su(k)}_2$. This extends our earlier derivation of the Pfaffian state from a "parent" state with abelian affine symmetry plus a projection of degrees of freedom. The numerator of the coset is actually a rational conformal field theory made out of $(2k-2)$ scalar fields and a specific extended symmetry. The coset construction projects out some neutral Hall edge excitations while preserving the filling fraction; it also respects the $\mathbb{Z}_k$ parity rule coupling neutral and charged excitations in the parent abelian theory.

1 Introduction

Conformal field theories have been successfully applied to describe the universal properties of quantum Hall states [1], such as symmetries, quantum numbers and low-energy dynamics of edge excitations [2]. The simplest Laughlin states with filling fraction $\nu = 1, 1/3, 1/5, \ldots$ are well understood in terms of abelian conformal theories with central charge $c = 1$; most of our understanding of edge excitations, including their fractional statistics and dynamics,

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has been drawn from these examples. Furthermore, recent experiments have confirmed the predictions of the abelian conformal theories [3].

The theoretical studies have now addressed more involved Hall states, such as that occurring at $\nu = 5/2$. This *plateau* in the second Landau level has no analogue in the first level (i.e. at $\nu = 1/2$); thus, it should be caused by some new dynamical mechanism. In Ref.[4], it was proposed that the electrons form *pairs*, in a way similar to BCS pairs in superconductors; these bosonic pairs can then form a Laughlin fluid with even denominator filling fraction. The (p-wave) pairing of (spin-polarized) electrons is represented in the ground-state wave function by the Pfaffian term $\mathrm{Pf}\,(1/(z_i - z_j))$, which involves all possible pairs of electron coordinates. (Other pairings were also proposed with different relative angular momentum and spin polarization of electrons.)

There is a well-established relation between the Hall states and conformal field theory, which says that the analytic part of electron wave functions should correspond to correlators of conformal fields. Following Ref.[4], the Pfaffian term can be reproduced by the correlator of Majorana fermions in the $c = 1/2$ conformal field theory, i.e. the critical Ising model, plus the usual $c = 1$ boson theory accounting for the charge of excitations.

An interesting feature of the Pfaffian Hall state is that it possesses excitations with *non-abelian* fractional statistics: namely, the adiabatic transport of one excitation around another is described by a multi-dimensional unitary transformation acting on a multiplet of degenerate wave-functions, rather than by the multiplication by a sign (fermi statistics) or a phase (abelian fractional statistics). The non-abelian statistics is easily understood in the Ising conformal theory: the spin field σ possesses the operator-product expansion $\sigma \cdot \sigma \sim \mathrm{Id} + \psi$, with two terms in the right hand side; therefore, multi-spin correlators expand into several terms (the conformal blocks), which transform among themselves under monodromy.

Numerical analyses have shown that the Pfaffian state has a rather good overlap with the exact ground state at $\nu = 5/2$ [5][‖]; therefore, there is an exciting possibility that new phenomena such as pairing and non-abelian statistics could be experimentally observed at sufficiently low temperatures and $2 < \nu < 3$.

Read and Rezayi have recently proposed [6] a generalization of the Pfaffian to a hierarchy of states in the second Landau level, which are described by the

[‖] In contrast with the previous expectations favoring the Haldane-Rezayi paired state.

$\mathbb{Z}_k$-parafermion conformal theories [7]. They occur at filling fractions,

$$\nu \equiv 2 + \nu_k(M) = 2 + \frac{k}{kM+2}, \quad k = 2, 3, \ldots \quad M = 1, 3, 5, \ldots \ . \tag{1.1}$$

(The Pfaffian state corresponds to the value $k = 2$ (and $M = 1$) in this series.) The same authors found that the parafermionic states have good overlap with the numerical exact ground states at $\nu = 13/5, 8/3$, i.e. for $M = 1$ and $k = 3, 4$ [6]. Furthermore, Hall plateaux at these filling values have been experimentally observed [8] by cooling the sample at extremely low temperatures. As shown in Ref.[6], the $\mathbb{Z}_k$-parafermion Hall states describe an interesting dynamics: the Hall fluid is made by clusters of k electrons (e.g. pairs for $k = 2$) and there are excitations with non-abelian statistics; other properties were discussed in Ref.[9].

The problem we are addressing is to relate this (universal) Hall dynamics to the conformal field theory data of $\mathbb{Z}_k$ parafermions. For example, we would like to find conformal theory motivations for the quantum numbers of these Hall states and for the mechanism of clustering which actually yields the parafermions**.

Our general idea is to describe the non-abelian Hall states and the associated conformal theories by introducing a *parent* abelian theory (with the same filling fraction) and a projection leading to the non-abelian theory. The reason is that the Hall physics is well understood in the abelian theories, such as the Laughlin fluids; furthermore, the projection may have a physical interpretation, which could be useful to explain pairing and non-abelian statistics.

We recall that an abelian theory is a rational conformal field theory based on the $\widehat{u(1)}^n$-current algebra, spanned by n abelian currents $J^i(z)$, extended by vertex operators $Y(\underline{\lambda}, z) = :\exp(i\phi(\underline{\lambda}, z)):$ whose charges $\underline{\lambda}$ form an n-dimensional lattice Γ. This lattice, satisfying some physical conditions summarized in Section 2, has been called *chiral quantum Hall lattice* in Ref.[10]. The non-equivalent quasi-holes in the Hall fluid correspond to the irreducible representations of the extended algebra and are labeled by the charges $\underline{\lambda} \in \Gamma^*/\Gamma$ where Γ^* is the dual lattice. For example, the $\widehat{su(k)}_1$ theory is the extension of $\widehat{u(1)}^{k-1}$ theory with the A_{k-1} root lattice. A complete classification of (low-dimension) chiral Hall lattices has been obtained in Ref.[10].

In Ref. [11], we have already shown that the Pfaffian state can be described in terms of an abelian lattice theory by projecting out some neutral degrees

**On the other hand, the non-abelian statistics can be easily understood from the operator-product expansion of parafermion fields.

of freedom. A crucial step in this process is the decoupling of charged and neutral sectors of the theory ((iso)spin-charge separation), which cannot be done globally but give raise to a selection rule (the *parity rule*). In addition, the projection should preserve the locality of physical excitations, such as the electrons [11].

Here, we would like to briefly describe the generalization of this approach to the Read-Rezayi states[††]. We show that the $\mathbb{Z}_k$ parafermions can be obtained by coset construction [12] from a specific abelian Hall state, whose lattice is *maximally symmetric* in the sense of Ref.[10], where it has been denoted by $(M+2 \mid {}^{\Delta_1}A_{k-1} \; {}^{\Delta_1}A_{k-1})$. The associated extended symmetry includes the $\widehat{u(1)} \oplus \widehat{su(k)_1} \oplus \widehat{su(k)_1}$ affine algebra. The $\mathbb{Z}_k$-parafermion theory PF_k is obtained as follows:

$$
\begin{aligned}
\mathrm{PF}_k &= \frac{\widehat{su(k)_1} \oplus \widehat{su(k)_1}}{\widehat{su(k)_2}}, \\
c_{\mathrm{PF}_k} &= c_{\widehat{su(k)_1}} + c_{\widehat{su(k)_1}} - c_{\widehat{su(k)_2}} = \frac{2(k-1)}{(k+2)}.
\end{aligned} \qquad (1.2)
$$

The two $su(k)$ symmetries of the parent state can be associated to *layer* and *iso-spin* quantum numbers [13]. After the projection, there remain a Z_k charge, which is the parafermion "number"; the $\mathbb{Z}_k$ parity rule relates this number (modulo k) to the fractional charge of the quasi-holes of the resulting parafermion Hall fluid. The coset construction allows us to construct all superselection sectors corresponding to the topologically non-equivalent quasi-holes (the irreducible representations), compute their characters and write the partition functions for the Read–Rezayi states.

Other projective constructions of non-abelian Hall states can be found in the Refs.[14], including the different coset construction $PF_k = \widehat{su(2)_k}/\widehat{u(1)}$.

2 Parent abelian theory for the parafermion state

2.1 Maximally symmetric $(2k-1)$-dimensional lattice

In this Section, we describe the abelian theory $\widehat{u(1)} \oplus su(k)_1 \oplus su(k)_1$ with central charge $c = 2k-1$, which reproduces the same filling fraction (1.1)

[††]See Ref.[13] for a detailed account of this work.

of the parafermion theory; the corresponding chiral Hall lattice Γ is an odd integral lattice containing the electron charge vector $\underline{\mathrm{q}}$ of square length (twice the electron dimension):

$$|\underline{\mathrm{q}}|^2 \equiv (\underline{\mathrm{q}}|\underline{\mathrm{q}}) = M + 2 \ (= 2\Delta_{\mathrm{el}}). \tag{2.1}$$

In any chiral Hall lattice, one defines the *charge* vector $\underline{Q}$ of the dual lattice Γ^*, which sets the *electric charge* of any lattice point and satisfies the following defining conditions: (i) it is *primitive* (i.e., not a multiple of any other vector $\underline{\mathrm{q}}^* \in \Gamma^*_{2k-1}$); (ii) it is related to ν_k in (1.1) and $\underline{\mathrm{q}}$ by:

$$|\underline{Q}|^2 = \nu_k, \quad (\underline{Q}|\underline{\mathrm{q}}) = 1; \tag{2.2}$$

(iii) it obeys the *charge–statistics relation* for fermion excitations:

$$(-1)^{(\underline{Q}|\underline{\mathrm{q}})} = (-1)^{|\underline{\mathrm{q}}|^2} \qquad \text{for any } \underline{\mathrm{q}} \in \Gamma. \tag{2.3}$$

A convenient basis $\{\underline{\mathrm{q}}, \underline{\alpha}_i, \underline{\beta}_j\}$ for Γ is given by the electron charge vector $\underline{\mathrm{q}}$ (satisfying (2.1) (2.2)) and the root vectors $\underline{\alpha}_1, \ldots, \underline{\alpha}_{k-1}, \underline{\beta}_1, \ldots, \underline{\beta}_{k-1}$ of $su(k) \oplus su(k)$ with standard inner products: $(\underline{\alpha}_i|\underline{\alpha}_i) = (\underline{\beta}_i|\underline{\beta}_i) = 2$, $(\underline{\alpha}_i|\underline{\alpha}_j) = (\underline{\beta}_i|\underline{\beta}_j) = -1$, for $i \neq j = 1, \ldots, k-1$. The *symmetry algebra* $su(k) \oplus su(k)$ is assumed to be *electrically neutral*, i.e. $(\underline{Q}|\underline{\alpha}_i) = (\underline{Q}|\underline{\beta}_i) = 0$, $i = 1, \ldots, k-1$; therefore, $\underline{Q} = (1, 0 \ldots 0)$ in the dual basis. It then follows that the electron charge vector $\underline{\mathrm{q}}$ decomposes as follows:

$$\underline{\mathrm{q}} = \frac{1}{\nu_k}\underline{Q} + \underline{\omega}, \quad (\underline{Q}|\underline{\omega}) = 0, \quad (\underline{\alpha}_i|\underline{\omega}) = \delta_{i1} = -(\underline{\beta}_i|\underline{\omega}), \quad |\underline{\omega}|^2 = 2\frac{k-1}{k}; \tag{2.4}$$

thus, $\underline{\omega}$ is an $su(k) \oplus su(k)$ weight. (The sign convention in the relation $(\underline{\beta}_1|\underline{\mathrm{q}}) = (\underline{\beta}_1|\underline{\omega}) = -1$ differs from the one in [11] and is chosen to fit the standard coset projection – see Section 3.)

The Gram matrix in this basis takes the form:

$$G_\Gamma = \left[\begin{array}{c|c|c} M+2 & 1\,0\cdots\,0 & -1\,0\cdots\,0 \\ \hline \begin{matrix}1\\0\\ \vdots\\0\end{matrix} & C_{k-1} & 0 \\ \hline \begin{matrix}-1\\0\\ \vdots\\0\end{matrix} & 0 & C_{k-1} \end{array}\right], \tag{2.5}$$

with C_{k-1} the Cartan matrix of the A_{k-1} algebra.

In our previous analysis of the Pfaffian state ($k = 2$) [11], we actually considered *two* parent abelian states; the corresponding lattices were the maximally symmetric one described above and the the two-dimensional lattice $\tilde{\Gamma}$ of the (331) paired Hall state, with Gram matrix $G_{\tilde{\Gamma}} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. The former lattice contains the latter and one can devise a two-step projection connecting all the three theories: this is indeed the coset construction (1.2) for $k = 2$ [13]. In the general k case, $\tilde{\Gamma}$ extends to a k-dimensional lattice, which leads to another parent theory for the parafermions; this is, however, completely independent of Γ, and its projection to the parafermions is not known.

The unitary representations of the chiral algebra $\mathcal{A}(\Gamma)$ describe the edge excitations of the Hall fluid; they are labeled by the points of the dual lattice Γ^*. This is manifestly not decomposable into orthogonal sublattices of charge and neutral excitation. Nevertheless, this decomposition (physically meaning the isospin-charge separation) can be achieved at the expenses of enlarging the lattice and introducing a selection rule (the $\mathbb{Z}_k$ *parity rule*). We introduce the decomposable sublattice $L \subset \Gamma$ of index k spanned by the vectors:

$$\{\underline{\mathrm{e}} := k(\underline{\mathrm{q}} - \underline{\omega}) = (kM + 2)\underline{Q},\ \underline{\alpha}_i,\ \underline{\beta}_j\}. \tag{2.6}$$

It splits into 3 mutually orthogonal sublattices:

$$L = (kM + 2)\mathbb{Z}\underline{Q} \oplus A_{k-1} \oplus A_{k-1}\ . \tag{2.7}$$

We have

$$\begin{aligned} L \subset \Gamma \subset \Gamma^* \subset L^*, \qquad & \Gamma = \{\underline{\gamma} = \underline{\lambda} + l\underline{\mathrm{q}}\ ;\ \underline{\lambda} \in L,\ 0 \le l \le k-1\}, \\ & L^*/\Gamma^* \simeq \Gamma/L \simeq \mathbb{Z}_k; \end{aligned} \tag{2.8}$$

indeed, the determinants of the Gram matrices of L and Γ (which give the number of sectors of the corresponding RCFT) are:

$$|L| = (kM + 2)^2 |\underline{Q}|^2 |C_{k-1}|^2 = (kM + 2)k^3 = k^2 |\Gamma|. \tag{2.9}$$

The isospin-charge separation of excitations is achieved in the decomposable dual lattice L^*, whose physical points (corresponding to those of Γ^*) obey a selection rule *mod* k. This factorized description of the excitations in the abelian theory is crucial for allowing the projection of neutral degrees of freedom leading to the parafermion theory (see Section 3).

2.2 Unitary representations of the chiral algebra $\mathcal{A}(\Gamma)$

The irreducible unitary representations of the chiral algebra $\mathcal{A}(L)$ can be expressed as $\mathbb{Z}_k$-invariant products of fundamental representations of $\widehat{su(k)}_1 \oplus \widehat{su(k)}_1$ times chiral vertex operators carrying charge $(n/k)\underline{Q}$ ($n \in \mathbb{Z}/(k(kM+2)\mathbb{Z})$); these representations are labeled by the elements of the abelian group L^*/L. We shall choose a vector $\underline{\lambda} \in L^*$ in each coset in L^*/L of the form:

$$\underline{\lambda} = \frac{l}{k}\underline{Q} + \underline{\Lambda}_\mu^{(\alpha)} - \underline{\Lambda}_\nu^{(\beta)}, \quad \mu, \nu = 0, 1, \ldots, k-1, \quad 2|l| \leq k(kM+2). \tag{2.10}$$

Here $\underline{\Lambda}_\mu^{(\alpha)}$ ($\underline{\Lambda}_\nu^{(\beta)}$) are the fundamental weights (including 0) of the first (respectively the second) $su(k)$ factor. The conformal dimension of the representation $\underline{\lambda}$ is:

$$\Delta(\underline{\lambda}) = \frac{1}{2}|\underline{\lambda}|^2 = \frac{l^2}{2k(kM+2)} + \frac{\mu(k-\mu)+\nu(k-\nu)}{2k}. \tag{2.11}$$

Proposition 2.1 *A vector $\underline{\lambda}$ of L^* belongs to the sublattice Γ^* iff $(\underline{\lambda}|\underline{\mathrm{q}}) \in \mathbb{Z}$. For $\underline{\lambda}$ given by (2.10), this is equivalent to the relation:*

$$l + k(\underline{\omega}|\underline{\Lambda}_\mu^{(\alpha)} - \underline{\Lambda}_\nu^{(\beta)}) \in k\mathbb{Z} \quad \Leftrightarrow \quad l - \mu - \nu = 0 \mod k. \tag{2.12}$$

Proof. Since $\underline{\mathrm{q}} \in \Gamma$, if $\underline{\lambda} \in \Gamma^*$ the inner product $(\underline{\lambda}|\underline{\mathrm{q}})$ should be integer. For $\underline{\lambda} \in L^*$ the converse is also true in view of (2.4) and (2.7). Eq. (2.12) is then a consequence of (2.2) (2.4) and (2.7) and of the definition of fundamental weights as a dual basis for $\{\underline{\alpha}_i\}$ and $\{\underline{\beta}_i\}$: $(\underline{\alpha}_i|\underline{\Lambda}_j^{(\alpha)}) = \delta_{ij} = (\underline{\beta}_i|\underline{\Lambda}_j^{(\beta)})$, $i, j = 1, \ldots, k-1$. As a result, the inner products $(\underline{\Lambda}_i|\underline{\Lambda}_j)$ (for $\underline{\Lambda}$ belonging to the weight space of the same $su(k)$ factor, α or β) are expressed in terms of the inverse A_{k-1} Cartan matrix.

Eq. (2.12) provides the explicit form of the $\mathbb{Z}_k$ parity rule which tells us when an excitation in the bigger lattice L^* actually belongs to the sublattice Γ^* of physical excitations of the Hall fluid.

We shall label the irreducible representations of $\mathcal{A}(\Gamma)$ (i.e., the elements of Γ^*/Γ) by a pair (m, ν) where m measures the minimal charge of each irreducible representation so that $2|m| \leq (kM+2)$, while $\nu \bmod k$ characterizes the neutral part. To each pair (m, ν) there corresponds a set of k weight vectors of type (2.10), which satisfy (2.12):

$$\begin{aligned}(m, \nu) \leftrightarrow \underline{\lambda}_l(m, \nu) &= \frac{m + l(kM+2)}{k}\underline{Q} + \underline{\Lambda}_{m+l-\nu}^{(\alpha)} - \underline{\Lambda}_{l+\nu}^{(\beta)} \,, \\ &\quad (\underline{\Lambda}_\mu \equiv \underline{\Lambda}_{\mu \bmod k}, l \bmod k).\end{aligned} \tag{2.13}$$

The length squares of these vectors differ by integers [13].

Using the representation of each vector $\underline{\gamma} \in \Gamma$ as a sum of a $\underline{\lambda} \in L$ and a *gluing vector* [10] $l\underline{\mathrm{q}} \in \Gamma$ (see (2.8)) we can write the representation space $\mathcal{H}^{\Gamma}_{\underline{\gamma}^*}$ of $\mathcal{A}(\Gamma)$ ($\underline{\gamma}^* \in \Gamma^*/\Gamma$) as direct sums of representation spaces of $\mathcal{A}(L)$,

$$\mathcal{H}^{\Gamma}_{\underline{\gamma}^*} = \bigoplus_{l=0}^{k-1} \mathcal{H}^{L}_{\underline{\gamma}^*+l\underline{\mathrm{q}}}. \tag{2.14}$$

This form of $\mathcal{H}^{\Gamma}_{\underline{\gamma}^*}$ allows to write its characters as a sum of characters of $\mathcal{A}(L)$ modules. Denote by $\chi_\nu(\tau)$ the (restricted) $\widehat{su(k)}_1$ characters [15],

$$\chi_\nu(\tau, A_{k-1}) = \frac{1}{(\eta(\tau))^{k-1}} \sum_{\underline{\gamma} \in A_{k-1}} q^{\frac{1}{2}|\underline{\Lambda}_\nu + \underline{\gamma}|^2}, \quad q = \mathrm{e}^{2\pi i \tau}, \quad \nu = 0, 1, \ldots, k-1, \tag{2.15}$$

(η is the Dedekind η-function). Taking the standard notation (also used in [11]) for the 1-dimensional lattice characters,

$$K_l(\tau, \zeta; m) = \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{\frac{m}{2}(n+\frac{l}{m})^2} \mathrm{e}^{2\pi i \zeta (n+\frac{l}{m})}, \tag{2.16}$$

we can write the characters $\chi_{\underline{\lambda}}(\tau, \zeta)$ of the representation $\underline{\lambda}$ (2.10) in terms of the labels (2.13) as a sum of factorized L-characters,

$$\chi^{\Gamma}_{m\nu}(\tau, \zeta) = \sum_{s=0}^{k-1} K_{m+s(kM+2)}\left(\tau, k\zeta; k(kM+2)\right) \chi^{(\alpha)}_{m+s-\nu}(\tau) \chi^{(\beta)}_{s+\nu}(\tau), \tag{2.17}$$

with $m \bmod (kM+2)$ and $\nu \mod k$. The resulting set of $k(kM+2)$ functions is covariant under (weak) modular transformations generated by $T^2 : (\tau, \zeta) \to (\tau+2, \zeta)$ and $S : (\tau, \zeta) \to (-1/\tau, \zeta/\tau)$, which are the modular properties suitable for quantum Hall systems [16].

3 The $\mathbb{Z}_k$-parafermion coset and its representations

In this Section, we describe the coset construction (1.2); this can be done in each of the k sectors (2.14) of the abelian theory according to the $\mathbb{Z}_k$ parity rule.

3.1 $\mathbb{Z}_k$ selection rule for triples of su(k) weights. Conformal dimensions of coset representations

The PF_k coset module (1.2) is labeled, in principle, by a triple $(\underline{\Lambda}_\alpha, \underline{\Lambda}_\beta; \underline{\Lambda})$; the pair $(\underline{\Lambda}_\alpha, \underline{\Lambda}_\beta)$ of fundamental su(k) weights and the level 2 weight $\underline{\Lambda}$ fix an irreducible unitary representation of the numerator and of the denominator current algebra, respectively. The tensor product of irreducible $\widehat{su(k)}_1$-modules corresponding to the numerator in the right hand side of Eq. (1.2) splits into a direct sum of PF_k and $\widehat{su(k)}_2$-modules:

$$\mathcal{H}^{(1)}_{\underline{\Lambda}_\alpha} \otimes \mathcal{H}^{(1)}_{\underline{\Lambda}_\beta} = \bigoplus_{\underline{\Lambda}} \mathcal{H}(\underline{\Lambda}_\alpha, \underline{\Lambda}_\beta; \underline{\Lambda}) \otimes \mathcal{H}^{(2)}_{\underline{\Lambda}} \qquad (\alpha, \beta = 0, \ldots, k-1,\ \underline{\Lambda}_0 = 0)\,. \tag{3.1}$$

Not all triples $(\underline{\Lambda}_\alpha, \underline{\Lambda}_\beta; \underline{\Lambda})$ are admissible (i.e., correspond to non-empty coset modules) and different *admissible triples* may refer to *equivalent representations.* The following statement is a specialization of results on field identification (based on the use of simple currents) obtained in Ref.[17].

Proposition 3.1 *Admissible triples are characterized by the conservation of the $\mathbb{Z}_k$ charge (the k-ality):*

$$[\underline{\Lambda}] = \sum_{i=1}^{k-1} i\ \lambda_i \quad \text{for} \quad \underline{\Lambda} = \sum_{i=1}^{k-1} \lambda_i \underline{\Lambda}_i; \tag{3.2}$$

more precisely, the triple $(\underline{\Lambda}_\alpha, \underline{\Lambda}_\beta; \underline{\Lambda})$ is admissible iff:

$$[\underline{\Lambda}_\alpha] + [\underline{\Lambda}_\beta] = [\underline{\Lambda}] \mod k, \qquad \text{i.e.,} \quad \alpha + \beta = [\underline{\Lambda}] \mod k. \tag{3.3}$$

There are thus $k\binom{k+1}{2}$ admissible triples of the form $(\underline{\Lambda}_\alpha, \underline{\Lambda}_\beta; \underline{\Lambda}_{\alpha+\kappa} + \underline{\Lambda}_{\beta-\kappa})$ where all indices are taken $\mathrm{mod}\, k$. They split into $\binom{k+1}{2}$ families of equivalent triples of the form:

$$(\underline{\Lambda}_{\alpha+\sigma}, \underline{\Lambda}_{\beta+\sigma}; \underline{\Lambda}_{\alpha+\kappa+\sigma} + \underline{\Lambda}_{\beta-\kappa+\sigma}), \quad \sigma = 0, \ldots, k-1. \tag{3.4}$$

As a result, the number $N(\mathrm{PF}_k)$ of parafermionic coset sectors coincides with the number of unitary irreducible representation $N(\widehat{su(k)}_2)$ of the level 2 current algebra: $N(\mathrm{PF}_k) = N(\widehat{su(k)}_2) = \binom{k+1}{2}$.

We can define a representative for each family in Eq.(3.4) by choosing a value for σ: we set $\beta+\sigma = 0 \,\mathrm{mod}\, k$, thus normalizing the second fundamental weight to zero. Therefore, the equivalent classes of triples are labeled by the level 2 weight, say $\underline{\Lambda}_\mu + \underline{\Lambda}_\nu$, $\mu \le \nu$; we have:

$$(\underline{\Lambda}_{\mu+\nu \bmod k}, 0; \underline{\Lambda}_\mu + \underline{\Lambda}_\nu) \quad \Longleftrightarrow \underline{\Lambda}_\mu + \underline{\Lambda}_\nu \quad (\mu \le \nu). \tag{3.5}$$

Ultimately, these labels characterize the parafermion representations. We end up with the following characterization of PF_k coset modules which appears to be new [13].

Proposition 3.2 *The parafermionic coset modules are in one to one correspondence with sums* $\left(\underline{\Lambda}_\mu + \underline{\Lambda}_\nu\right)$ *of* $su(k)$ *fundamental weights* ($0 \le \mu \le \nu \le k-1$). *Their conformal weights are given by:*

$$\begin{aligned} \Delta^{(k)}_{\mu\nu} &= \frac{1}{2}\left|\underline{\Lambda}_\mu + \underline{\Lambda}_\nu\right|^2 - \Delta_2\left(\underline{\Lambda}_\mu + \underline{\Lambda}_\nu\right) \\ &= \frac{\mu(k-\nu)}{k} + \frac{(\nu-\mu)(k+\mu-\nu)}{2k(k+2)} \quad \text{for} \quad (0 \le \mu \le \nu < k)\,, \end{aligned} \tag{3.6}$$

where $\Delta_2(\underline{\Lambda})$ *is the dimension of the representation of the level 2 weight* $\underline{\Lambda}$ *of* $\widehat{su(k)}_2$. *The decomposition of a product of level-1 characters corresponding to the tensor product expansion (3.1) has the form:*

$$\chi^{(1)}(\underline{\Lambda}_\alpha)\chi^{(1)}(\underline{\Lambda}_\beta) = \sum_\gamma \mathrm{Ch}\left(\underline{\Lambda}_{\alpha-\beta+\gamma} + \underline{\Lambda}_{k-\gamma}\right)\chi^{(2)}\left(\underline{\Lambda}_{\alpha-\beta+\gamma} + \underline{\Lambda}_{k-\gamma}\right)\ , \tag{3.7}$$

where $\mathrm{Ch}(\underline{\Lambda})$ *is the coset character and* γ *takes* $\mathrm{IP}(k/2)+1$ *values corresponding to distinct sums of pairs of weights* ($\mathrm{IP}(x)$ *stands for the integer part of the real number* x).

Proof. Eq. (3.6) can be verified by using the identity $\Delta_1(\underline{\Lambda}_\mu) + \Delta_1(\underline{\Lambda}_{k-\nu}) - \Delta_2(\underline{\Lambda}_{\mu+k-\nu}) = \Delta^{(k)}_{\mu\nu}$ and observing that the triple $(\underline{\Lambda}_\mu, \underline{\Lambda}_{k-\nu}; \underline{\Lambda}_{\mu+k-\nu})$ is equivalent to $(\underline{\Lambda}_{\mu+\nu}, 0; \underline{\Lambda}_\mu + \underline{\Lambda}_\nu)$. The triples appearing in Eq. (3.7) (as arguments of the pair of $\chi^{(1)}$ and $\chi^{(2)}$) are, clearly, admissible. It is straightforward to verify that the difference of conformal weights of the two sides is an integer. The number of terms in the expansion (3.7) is independent of α and β. For the vacuum representation we have:

$$\chi^{(1)}(\underline{\Lambda}_0)\chi^{(1)}(\underline{\Lambda}_0) = \mathrm{Ch}(2\underline{\Lambda}_0)\chi^{(2)}(2\underline{\Lambda}_0) + \sum_{\gamma=1}^{\mathrm{IP}(\frac{k}{2})} \mathrm{Ch}\left(\underline{\Lambda}_\gamma + \underline{\Lambda}_{k-\gamma}\right)\chi^{(2)}\left(\underline{\Lambda}_\gamma + \underline{\Lambda}_{k-\gamma}\right)\ ,$$

where $\Delta^{(k)}_{\gamma,k-\gamma} + \Delta_2(\underline{\Lambda}_\gamma + \underline{\Lambda}_{k-\gamma}) = \gamma$ for $\gamma \le k-\gamma$. ∎

3.2 Parafermionic Hall fluids. The parafermion $\mathbb{Z}_k$ charge

The chiral algebra $\mathcal{A}_k$ of the $\mathbb{Z}_k$-parafermion Hall fluids, which reproduces the filling factor (1.1), is determined from:

$$\mathcal{A}_k \otimes \mathcal{A}(\widehat{su(k)_2}) = \mathcal{A}(\Gamma). \tag{3.8}$$

This is obtained by a standard coset projection from the lattice theory of Section 2. In particular, the lattice characters (2.17) are projected into:

$$\chi_{m\nu}(\tau,\zeta) = \sum_{s \bmod k} K_{m+s(kM+2)}(\tau, k\zeta; k(kM+2))\mathrm{Ch}(\tau, \underline{\Lambda}_{s+m-\nu} + \underline{\Lambda}_{s+\nu}). \tag{3.9}$$

The coset projection only preserves a single $\mathbb{Z}_k$ symmetry of the original product $\mathbb{Z}_k \times \mathbb{Z}_k$ of the centres of the two $SU(k)$ groups; it is the difference:

$$[\underline{\Lambda}_{\alpha+\sigma}] - [\underline{\Lambda}_{\beta+\sigma}] = (\alpha-\beta) \mod k\ , \tag{3.10}$$

which defines the $\mathbb{Z}_k$ charge (or "number") of parafermions.

Moreover, the $\mathbb{Z}_k$ parity rule of the parafermion Hall fluids is inherited from the parent abelian fluids, namely Eq.(2.10): it states that the physical Hall excitations possess parafermion number (3.10) equal (*mod k*) to the number of "fractional units" of electric charge $l \in \mathbb{Z}_{kM+2}$ in Eq.(2.10):

$$\alpha - \beta = l \mod k\ . \tag{3.11}$$

The coset representation $2\underline{\Lambda}_\nu$, corresponding to $\mu = \nu$, are the *"parafermionic currents"* of Fateev and Zamolodchikov [7]. They all have quantum dimension 1 and obey $\mathbb{Z}_k$ fusion rules:

$$2\underline{\Lambda}_\mu \times 2\underline{\Lambda}_\nu \sim 2\underline{\Lambda}_{(\mu+\nu) \bmod k}. \tag{3.12}$$

The (non-local) parafermionic currents give rise to an "anyonic" chiral algebra, say, $\mathcal{PF}_k$whose bosonic (integer dimension fields') subalgebra can be identified with the coset chiral algebra PF_k (1.2). The parafermionic algebra $\mathcal{PF}_k$admits k unitary irreducible representations, labeled by an integer $\rho \bmod k$, with conformal weights,

$$\Delta_\rho = \frac{\rho(k-\rho)}{2k(k+2)}, \quad \rho = 0, 1, ..., k-1. \tag{3.13}$$

Each of these splits into $(k-\rho)$ unitary irreducible representation of the bosonic subalgebra PF_k whose conformal weights exceed (3.13) by an integer multiple of $1/k$. Comparing (3.13) with (3.6) we see that ρ can be identified with $(\nu-\mu)$.

For each ρ in the range (3.13) there are exactly $(k-\rho)$ pairs (μ,ν) satisfying $0\le\mu\le\nu\le k$, $\rho=\nu-\mu$; they generate all different conformal weights (3.6). According to [18] the characters of the resulting coset modules are given by $\mathrm{Ch}(\tau,\underline{\Lambda}_\mu+\underline{\Lambda}_\nu)=\mathrm{Ch}_{\rho l}(\tau)$, where:

$$\mathrm{Ch}_{\rho l}(\tau)=q^{\Delta_\rho-\frac{c_k-1}{24}}\sum_{\underline{n}\in\mathbb{N}_l}\frac{q^{\underline{n}\cdot C^{-1}\cdot(\underline{n}-\underline{\Lambda}_\rho)}}{(q)_{n_1}\cdots(q)_{n_{k-1}}},\quad l\ge\rho. \tag{3.14}$$

In this equation, Δ_ρ is the PF_k weight (3.13), (c_k-1) is the parafermion central charge (cf. (1.2)), $(q)_n=\prod_{j=1}^{n}(1-q^j)$, $\mathbb{N}_l=\{\underline{n}=(n_1,\ldots,n_{k-1});\ n_i\in\mathbb{Z}_+;\ n_1+2n_2+\cdots+(k-1)n_{k-1}=l \bmod k\}$, and C^{-1} is the inverse of the $su(k)$ Cartan matrix.

The expression (3.14) corresponds to a PF_k irreducible component of the representation ρ of the non-local parafermionic algebra $\mathcal{PF}_k$. For fixed ρ, the values of l yielding inequivalent PF_k modules are:

$$l=\rho,\rho+1,\ldots,k-1 \text{ (for } \rho=0,1,\ldots,k-1). \tag{3.15}$$

The pair (ρ,l) is related to (μ,ν) of (3.6) by:

$$\rho=\nu-\mu,\quad l=\nu \text{ or } \mu=l-\rho,\nu=l\Rightarrow 2l-\rho\equiv m \bmod k. \tag{3.16}$$

The *topological order* N_k of the $\mathbb{Z}_k$-parafermion Hall fluid is equal [16] to the number of independent characters (3.9) of the corresponding conformal theory; it is given by the product of the range of m, i.e. $(kM+2)$, times the number $\binom{k+1}{2}$ of $\mathbb{Z}_k$-parafermion sectors and divided by k, due to the parity rule (3.11):

$$N_k=\frac{1}{k}(kM+2)\binom{k+1}{2}=\frac{k+1}{2}(kM+2)\ . \tag{3.17}$$

We thus recover the value obtained in Ref.[6] by other means.

4 The $k=3$ case

Here we spell out our construction in the case of $k=3$ (the analysis of the Pfaffian state ($k=2$) can be found in the Refs.[11] [13]). Consider the lattice model of Section 2 for $k=3$, $M=1$: the coset (1.2) coincides in this case with the $\mathbb{Z}_3$ Potts model (with $c_{\mathrm{PF}}=4/5$). We shall label its 6 sectors by an integer λ, $-2\le\lambda\le3$, related to the pair (μ,ν) $(0\le\mu,\nu\le2)$ of Eq. (3.6) as follows: $1 = (1,0) = (0,1)$, $-1 = (2,0) = (0,2)$, $2 = (1,1)$, $-2 = (2,2)$, $(\pm)3 = (1,2) = (2,1)$. The conformal dimensions Δ_λ are given by:

$$\Delta_0 = 0, \quad \Delta_{\pm 1} = \frac{1}{15}, \quad \Delta_{\pm 2} = \frac{2}{3}, \quad \Delta_{\pm 3} = \frac{2}{5}. \tag{4.1}$$

The 6 coset characters (3.14) for $k = 3$ can be written as,

$$\begin{aligned} ch_{2l}(\tau) &= \mathrm{Ch}_{0,l}(\tau) = q^{-\frac{1}{30}} \sum_{n_1+2n_2 = l \bmod 3, \quad (n_i \geq 0)} \frac{q^{\frac{2}{3}\left(n_1^2+n_2n_2+n_2^2\right)}}{(q)_{n_1}(q)_{n_2}}, \quad l = 0, \pm 1 (\bmod 3)\ , \\ ch_{2l-1}(\tau) &= \mathrm{Ch}_{l+1,l+1}(\tau) = q^{\frac{1}{30}} \sum_{n_1+2n_2 = 0 \bmod 3} \frac{q^{\frac{2}{3}\left(n_1^2+n_2n_2+n_2^2\right)-\frac{2n_{l+1}+n_{2-l}}{3}}}{(q)_{n_1}(q)_{n_2}}, \quad l = 0, 1\ , \\ ch_3(\tau) &= \mathrm{Ch}_{1,2}(\tau) = q^{\frac{1}{30}} \sum_{n_1+2n_2 = 2 \bmod 3} \frac{q^{\frac{1}{3}[2n_1(n_1-1)+2n_1n_2+n_2(2n_2-1)]}}{(q)_{n_1}(q)_{n_2}}. \end{aligned} \tag{4.2}$$

Their modular transformations will be omitted here and can be found in Ref.[13].

The $\mathbb{Z}_3$ parafermion Hall fluid with chiral algebra $\mathcal{A}_3$ defined by (3.8), filling factor 3/5 and central charge $c = 9/5$ has topological order 10 (according to (3.17)) and characters:

$$\begin{aligned} \chi_{m\mu}(\tau,\zeta) &= \sum_{l=-1}^{1} K_{m+5l}(\tau, 3\zeta; 15) ch_{m+3\mu+2l}(\tau) \qquad (ch_\lambda(\tau) = ch_{\lambda+6}(\tau)), \\ & \quad m = 0, \pm 1, \pm 2, \quad \mu = 0, 1. \end{aligned} \tag{4.3}$$

Note that the charge (K) and neutral (ch) parts on the Hall fluid characters in Eq.(4.3) combine according to the $\mathbb{Z}_3$ parity rule (3.11).

The minimal charges $Q_{m\mu}$ and minimal conformal dimensions $\Delta_{m\mu}$ in the sectors (4.3) are:

$$\begin{aligned} Q_{m\mu} = \frac{m}{5}, \qquad & \Delta_{m0} = \frac{1}{10}\binom{|m|+1}{2}, \\ & \Delta_{01} = \frac{2}{5} = 2\Delta_{21}, \quad \Delta_{11} = \frac{7}{10}. \end{aligned} \tag{4.4}$$

The full characters (4.3) of the parafermion Hall fluid transform linearly under the S modular inversion $\tau \to -1/\tau$; the associated matrix is:

$$S_{m\mu, m'\mu'} = \frac{\sqrt{3-\delta}}{5}(-1)^{\mu\mu'+m\mu'+m'\mu} \mathrm{e}^{i\frac{\pi}{5}mm'} \delta^{P_{\mu+m+\mu'+m'}}\ , \tag{4.5}$$

with $m, m' = 0, \pm 1, \pm 2$, $\mu, \mu' = 0, 1$, $\delta = 2\cos(\pi/5)$ the golden ratio and $P_\lambda = \left(1 - (-1)^\lambda\right)/2$. In particular, the quantum dimension d_q of the representation (m, μ) is:

$$d_q(m, \mu) = \begin{cases} 1 & \text{for } \mu + m \quad \text{even,} \\ \delta & \text{for } \mu + m \quad \text{odd .} \end{cases} \tag{4.6}$$

The presence of a non-integer (in fact, irrational) quantum dimension δ signals the non-abelian statistics of the quasi-particle excitations in this Hall fluid. Actually, some fusion rules yields more than one field in the r.h.s., for example:

$$\begin{aligned} (1, 0) \times (1, 0) &= (2, 0) + (2, 1) \\ (0, 1) \times (0, 1) &= (0, 0) + (0, 1). \end{aligned} \tag{4.7}$$

Further properties of the coset construction of $\mathbb{Z}_k$-parafermion Hall fluids, such as the $\mathcal{W}_k$ symmetry [19], can be found in Ref.[13].

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