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Hamiltonian Formulation of the $W_{1+\infty}$ Minimal Models

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Abstract

The $W_{1+\infty}$ minimal models are conformal field theories which can describe the edge excitations of the hierarchical plateaus in the quantum Hall effect. In this paper, these models are described in very explicit terms by using a bosonic Fock space with constraints, or, equivalently, with a non-trivial Hamiltonian. The Fock space is that of the multi-component Abelian conformal theories, which provide another possible description of the hierarchical plateaus; in this space, the minimal models are shown to correspond to the sub-set of states which satisfy the constraints. This reduction of degrees of freedom can also be implemented by adding a relevant interaction to the Hamiltonian, leading to a renormalization-group flow between the two theories. Next, a physical interpretation of the constraints is obtained by representing the quantum incompressible Hall fluids as generalized Fermi seas. Finally, the non-Abelian statistics of the quasi-particles in the $W_{1+\infty}$ minimal models is described by computing their correlation functions in the Coulomb Gas approach.

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1 Introduction

The dynamics of the electrons at the hierarchical Hall plateaus [1] has been described by several theoretical methods, like the Jain theory of the composite fermion [2], the effective conformal field theories [3] and the Chern-Simons gauge theories [4]. The filling fractions of these plateaus are nicely given by the Jain formula [2]:

$$\nu = \frac{m}{mp \pm 1} , \qquad m = 2, 3, \ldots, \quad p = 2, 4, 6, \ldots \ , \tag{1.1}$$

which generalizes the Laughlin series [5], $\nu = 1, 1/3, 1/5, \ldots$, corresponding to $m = 1$. In this paper we would like to discuss the approach of conformal field theory (CFT) [6]; this describes the low-energy edge excitations, which are one-dimensional (chiral) waves propagating on the boundary of the sample. The conformal field theory for the simpler Laughlin plateaus has been clearly identified as that of the Abelian chiral boson [3], which is also called the chiral Luttinger liquid. This theory describes the edge waves as well as the universal properties of the quasi-particle excitations: in particular, the latter possess fractional electric charge Q and fractional statistics θ/π, in agreement with the Laughlin theory [5]. Some of these properties have been recently measured [7] and have been found to agree with the conformal theory.

On the other hand, the edge excitations of the hierarchical plateaus (1.1) are less understood at present. Two classes of CFTs have been proposed: the first is the multi-component bosonic theory, which is characterized by the Abelian current algebra $\widehat{U(1)}^m$ (more precisely, $\widehat{U(1)} \times \widehat{SU(m)}_1$) [8]; these theories generalize the well-established one-component theory describing the Laughlin plateaus [3]. The other proposed theories are given by the minimal models of the $W_{1+\infty}$ algebra [9]. According to the Laughlin physical picture [5], the electrons form a quantum incompressible fluid at the Hall plateaus. This fluid can be characterized by the w_∞ symmetry of area-preserving diffeomorphisms of the plane [10][11]; moreover, the corresponding low-energy edge excitations are described by the conformal theories with $W_{1+\infty}$ symmetry [12], which include the specific class of the $W_{1+\infty}$ minimal models [9].

The two proposed theories are identical for the Laughlin Hall states; however, they describe the hierarchical plateaus according to different generalizations of the Laughlin theory. They possess the same spectrum of quasi-particles, but differ in their specific properties. So far, the experiments have measured the quasi-particle charges and the two-point correlation functions, which are not enough for distinguishing between the two classes of theories.

Let us recall here the main features of the $W_{1+\infty}$ minimal models:

- They exist for the hierarchical plateaus (1.1) only.
- They exhibit a smaller number of edge excitations than the corresponding multi-component Abelian theories.
- They possess a single Abelian charge for the quasi-particles, rather than the m charges of the multi-component theories; this unique charge is clearly identified with Q.
- They have neutral quasi-particles which are labelled by the weights of the $SU(m)$ Lie algebra, rather than by the $(m-1)$ Abelian charges in the multi-component theories. As a consequence, the quasi-particles of the minimal theories have non-Abelian statistics.

The $W_{1+\infty}$ minimal models have been introduced in the previous work [9]: they were obtained by the typical algebraic construction of CFT, which builds their Hilbert space from a consistent collection of representations of the symmetry algebra. Actually, the minimal models are made by the degenerate $W_{1+\infty}$ representations [13], which possess Virasoro central charge $c = m = 2, 3, \ldots$, and are equivalent to the representations of the $\widehat{U(1)} \times \mathcal{W}_m$ algebra [14]. This construction showed that the minimal theories are obtained from the multi-component Abelian ones by performing a reduction of degrees of freedom, which keeps the incompressibility of the electron fluid; specifically, the reduction of the $\widehat{SU(m)}_1$ symmetry down to the $\mathcal{W}_m$ one.

Afterwards, the exact energy spectrum was numerically computed for the Hall states of a system of ten electrons [15]; this study clearly indicated that the previous reduction of degrees of freedom is realized in the spectrum, and that the $W_{1+\infty}$ minimal models may have a chance to be the correct conformal theories of the hierarchical Hall plateaus. Therefore, it is important to describe this reduction in great detail and to understand its dynamical origin.

In this paper, we explicitly formulate the $W_{1+\infty}$ minimal models by quantizing a Hamiltonian of the multi-component bosonic fields. In Section 2, this Hamiltonian is shown to be relevant in the renormalization-group sense: the corresponding flow interpolates between the multi-component Abelian and the minimal theories, and performs the reduction of degrees of freedom for the latter theory in the infrared fixed point. This infrared limit can also be obtained by imposing a set of constraints on the states of the Abelian Fock space; this procedure defines the $\mathcal{W}_m$ algebra by the so-called Hamiltonian reduction of $\widehat{SU(m)}_1$ [16]. This rather simple description of the $W_{1+\infty}$ minimal models is completely equivalent to the previous algebraic construction

[9]: moreover, the Hamiltonian formulation may be useful for understanding the microscopic dynamics of the reduction and for comparing with other approaches and the experimental results.

In Section 3, we present a simple physical picture for the electron ground state of the $W_{1+\infty}$ minimal models: this is a "minimal" incompressible Hall fluid, which can be described as a chiral Fermi sea, as done earlier for the Laughlin fluids [10][11]; in this picture, the minimal-model constraints can be interpreted as further conditions for incompressibility. In this Section, we also describe the $W_{1+\infty}$ minimal models in terms of spinon excitations, another basis for the $\widehat{SU(2)}_1$ states introduced in Ref.[17].

In Section 4, we compute the four-point correlation functions of the $W_{1+\infty}$ minimal models by using the Coulomb Gas method [18]. We show that the quasi-particles as well as the electrons possess the non-Abelian statistics; namely, the many-particle states have additional multiplicities coming from the isospin addition. In the case of the electrons, this multiplicity is simply related the degeneracy of the unobservable fluid components. All results are shown in the simplest case of $m = 2$ in (1.1), which is relevant for the plateaus $\nu = 2/5, 2/9, \ldots$ The generalization of this analysis to $m \geq 3$ is described in the Appendix.

Let us finally stress that this Hamiltonian formulation makes it clear that the $W_{1+\infty}$ minimal models are fully consistent theories, although they are not Rational CFTs [6]; namely, their partition function is not modular invariant [19] (see the discussion in Section 2).

2 The $c = 2$ $W_{1+\infty}$ Minimal Models as Reduction of the Two-component Abelian Conformal Theories

We first describe the Hilbert spaces of the two classes of theories by recalling the results of the algebraic constructions in Ref.[9] (see Section 3.2). Next, we discuss the equivalent Hamiltonian description of the minimal models.

2.1 The $\widehat{U(1)} \times \widehat{SU(2)}_1$ Theories

The two-component Abelian theories are built out of two chiral bosonic fields $\varphi^{(i)}(R\theta - v_i t)$, $i = 1, 2$, which are defined on the edge of the Hall sample [3]: here it is chosen to be a disk, parametrized by an angle θ and a radius R. The conformal field theory is defined on the space-time cylinder made by the disk boundary and time t. As usual, the CFT operators are defined on the complex plane $z = \exp(\tau + i\theta)$, where $\tau = it$ is the Euclidean time. The Fermi velocities v_i as well as the chiralities of the two bosonic fields are taken positive*: this case is relevant for the filling fractions,

$$\nu = \frac{2}{2p+1} = \frac{2}{5}, \frac{2}{9}, \ldots \tag{2.1}$$

The edge excitations are described by the chiral currents:

$$J^{(i)} = -\frac{1}{2\pi}\frac{\partial \varphi^{(i)}}{\partial \theta} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \alpha_n^{(i)} e^{in(\theta - v_i t/R)} , \qquad i = 1, 2 , \tag{2.2}$$

whose Fourier modes $\alpha_n^{(i)}$ satisfy the two-component Abelian current algebra,

$$\left[\alpha_n^{(i)} , \alpha_m^{(j)} \right] = \delta^{ij} \; n \; \delta_{n+m,0} \; . \tag{2.3}$$

The corresponding generators of the Virasoro algebra are obtained by the usual Sugawara construction $L^{(i)} =: \left(J^{(i)}\right)^2 : /2$, and read:

$$L_n^{(i)} = \frac{1}{2}\alpha_0^{(i)} + \sum_{n=0}^{\infty} \alpha_{-n}^{(i)}\alpha_n^{(i)} , \qquad\qquad i = 1, 2 \; . \tag{2.4}$$

They satisfy

$$\left[L_n^{(i)} , L_m^{(j)} \right] = \delta^{ij} \left\{ (n-m) L_{n+m}^{(i)} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \right\} , \qquad c = 1 \; . \tag{2.5}$$

The generators of conformal transformations are thus given by $L_n = L_n^{(1)} + L_n^{(2)}$, and obey the same algebra with $c = 2$. The highest weight representations of the Abelian algebra (2.3) are characterized by the eigenvalues of $J_0^{(i)}$ and $L_0^{(i)}$; the eigenvalue of $\left(J_0^{(1)} + J_0^{(2)}\right)$ is proportional to the quasi-particle charge Q and $2\left(L_0^{(1)} + L_0^{(2)}\right)$ is their fractional statistics θ/π.

The Hilbert space of the two-component Abelian conformal theory is made of a consistent set of these representations, which is complete with respect to the fusion rules: these are the selection rules for the composition of quasi-particle excitations in the theory (the so-called bootstrap self-consistent conditions) [6]. In the Abelian theory, these rules simply require the addition of the two-component charges of the quasi-particles; as a consequence, the allowed charge values should fill the points of a two-dimensional lattice. Each lattice specifies a theory: the adjacency matrix of the

*Namely, we choose the values $\kappa_i = 1$ in Section 2 of Ref.[9]; the case of mixed chiralities (some $\kappa_i = -1$) are analogous.

lattice contains some parameters which are partially determined by the physical conditions, like the matching of the filling fraction. Special lattices allow for an extended symmetry [8]: the extension from $\widehat{U(1)}\times\widehat{U(1)}$ to $\widehat{U(1)}_{\rm diagonal}\times\widehat{SU(2)}_1$ has been chosen because it reproduces the filling fractions (2.1) uniquely (the parameter $p=2,4,\ldots$ is related to the compactification radius of the field $(\varphi^{(1)}+\varphi^{(2)})$ representing the $\widehat{U(1)}_{\rm diagonal}$ part).

The corresponding spectrum of quasi-particles was found to be [8]:

$$\mathrm{I}:\left\{\begin{array}{ll} Q & =\frac{2\ell}{2p+1}\ , \\ \frac{1}{2}\frac{\theta}{\pi} & =\frac{1}{2p+1}\ell^2+n^2\ , \end{array}\right. \qquad \mathrm{II}:\left\{\begin{array}{ll} Q & =\frac{2}{2p+1}\left(\ell+\frac{1}{2}\right)\ , \\ \frac{1}{2}\frac{\theta}{\pi} & =\frac{1}{2p+1}\left(\ell+\frac{1}{2}\right)^2+\frac{(2n+1)^2}{4}\ , \end{array}\right. \qquad \ell,n\in\mathbf{Z}\ . \tag{2.6}$$

In the literature, one often finds a unique formula for the two cases (I) and (II) above, which can be obtained by substituting $(\ell,n)\to(n_1,n_2)$, with $(2\ell=n_1+n_2, 2n=n_1-n_2)$ and $(2\ell+1=n_1+n_2, 2n+1=n_1-n_2)$, respectively (see Section 4.2 of Ref. [9]). The spectrum (2.6) is clearly factorized in charged and neutral excitations, with ℓ counting the units of fractional charge, and n labelling the neutral quasi-particles of the two-component fluids; the latter are a new feature of the two-component fluids with respect to the Laughlin ones. Note that the neutral spectrum is independent of the filling fraction, i.e. of the value of p.

Let us also introduce the current and Virasoro generators in the basis corresponding to the factorization into charged and neutral sectors of the spectrum (2.6) (case (I)):

$$\begin{array}{lll} J=J^{(1)}+J^{(2)}\ , & \frac{J_0}{\sqrt{2p+1}} \longrightarrow & Q=\frac{2\ell}{2p+1}\ ; \\ J^3=\frac{1}{2}\left(J^{(1)}-J^{(2)}\right)\ , & J^3_0 \longrightarrow & n\ ; \\ L=L^{(1)}+L^{(2)}=L^Q+L^S\ , & L_0 \longrightarrow & \frac{\ell^2}{2p+1}+n^2\ ; \\ L^Q=\frac{1}{4}\ :\ (J)^2\ :\ , \quad L^S=:\ (J^3)^2\ :\ . & & \end{array} \tag{2.7}$$

(the eigenvalues of case (II) are found for $n\to n+1/2$ and $\ell\to\ell+1/2$).

Each value in the spectrum (2.6) is the highest weight of a pair of representations of the Abelian current algebra; as is well known [6], these representations describe an infinite tower of edge excitations, generated by the bosonic Fock-space operators $\alpha_n^{(i)}$, $n<0$, $i=1,2$, which correspond to Fermionic particle-hole transitions, or to their anyonic generalizations [10][11][12]. These excitations are characterized by their angular momentum value, given by the total Virasoro operator L_0; moreover, the multiplicities of excitations are counted by the characters of the representations†:

$$\chi_{L_0}^{\widehat{U(1)}\times\widehat{U(1)}} \equiv \mathrm{Tr}_{\mathrm{Rep}\left(\widehat{U(1)}\times\widehat{U(1)}\right)} \left(q^{L_0-1/12}\right) = \chi_{\ell^2/(2p+1)}^{\widehat{U(1)}}\ \chi_{n^2}^{\widehat{U(1)}} \ , \tag{2.8}$$

e.g. in the case (I) above; the characters of the Abelian representations are [6]:

$$\chi_{L_0=h}^{\widehat{U(1)}} = \frac{q^h}{\eta(q)} \ , \qquad \eta(q) = q^{1/24} \prod_{k=1}^{\infty} \left(1-q^k\right) \ . \tag{2.9}$$

Finally, the Hamiltonian of the Abelian theory which assigns a linear spectrum to the edge excitations can be written in terms of the currents, as follows‡:

$$H = \pi \int_0^{2\pi R} dx \ :\left(v_1 J^{(1)}J^{(1)} + v_2 J^{(2)}J^{(2)}\right): = \frac{1}{R}\left[v_1 L_0^{(1)} + v_2 L_0^{(2)} - \frac{1}{12}\right] \ . \tag{2.10}$$

Let us now discuss the extended symmetry $\widehat{SU(2)}_1$ of the neutral sector of the spectrum (2.6). First, we observe that there are two highest weights with dimension one, i.e. $n = \pm 1$, which correspond to the additional chiral currents:

$$J^{\pm} = :\exp\left(\pm i\sqrt{2}\varphi\right): \ , \qquad \varphi = \frac{1}{2}\left(\varphi^{(1)} - \varphi^{(2)}\right) \ . \tag{2.11}$$

The two fields $J^{\pm}$, together with J^3 in (2.7), form the $\widehat{SU(2)}_1$ current algebra of level $k=1$; their Fourier modes satisfy:

$$\begin{aligned} \left[\ J_n^a \ , \ J_m^b \ \right] &= i\epsilon^{abc} J_{n+m}^c + \frac{k}{2}\delta^{ab}\delta_{n+m,0} \ , \qquad k=1, \ a,b,c = 1,2,3, \\ \left[\ L_n^S \ , \ J_m^a \ \right] &= -m \ J_{n+m}^a \ ; \end{aligned} \tag{2.12}$$

of course, the J_n^a commute with the generators of the charged sector $\left(L_n^Q, J_m\right)$. As is well known, there are two highest-weight representations of the $\widehat{SU(2)}_1$ algebra, which are labelled by the "spin" $\sigma = 0, 1/2$. Their (specialized) characters are [6][21]:

$$\begin{aligned} \chi_{\sigma=0}^{\widehat{SU(2)}_1} &= \frac{1}{\eta(q)^3} \sum_{k\in\mathbf{Z}} (6k+1) \ q^{(6k+1)^2/12} &= \frac{1}{\eta(q)} \sum_{k\in\mathbf{Z}} q^{k^2} \ , \\ \chi_{\sigma=1/2}^{\widehat{SU(2)}_1} &= \frac{1}{\eta(q)^3} \sum_{k\in\mathbf{Z}} (6k+2) \ q^{(6k+2)^2/12} &= \frac{1}{\eta(q)} \sum_{k\in\mathbf{Z}} q^{(2k+1)^2/4} \ . \end{aligned} \tag{2.13}$$

The second expression for these characters shows that the two $\widehat{SU(2)}_1$ representations sum up the neutral spectrum of the theory (2.6) in the two sectors (I) and (II),

†We neglect the charge dependence in the characters, which is immaterial for the following discussion. See Ref. [19] for the complete characters.

‡More general dispersion relations can also be accounted for; see Ref. [20].

respectively. Note also that the partition function of the complete theory $\widehat{U(1)} \times \widehat{SU(2)}_1$ can be found in Ref. [19] (for the annulus geometry): it includes all the states in the spectrum (2.6) with multiplicity one and it is invariant under the modular transformations. Therefore, the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theory is a rational CFT [6].

The $\widehat{SU(2)}_1$ representations can be further decomposed into the $c = 1$ Virasoro representations of weight $h = n^2/4$, $n \in \mathbf{Z}$. These are characterized by the property of being *degenerate*: they contain null-states, i.e. zero-norm states in their tower of excitations, which should be removed from the Hilbert space. This degeneracy is displayed by the corresponding characters [6],

$$\chi^{\rm Vir}_{L_0^S = n^2/4} = \frac{q^{n^2/4}\left(1 - q^{n+1}\right)}{\eta(q)} = \frac{q^{n^2/4} - q^{(n+2)^2/4}}{\eta(q)} \ ; \tag{2.14}$$

their numerator contains a negative term which cancels part of the power expansion of $\eta(q)$, and reduces the multiplicities of states with respect to the corresponding Abelian representation (2.9). Actually, the last part of equation (2.14) shows that each $\widehat{U(1)}$ representation in the neutral spectrum decomposes into an infinity of Virasoro representations ,

$$\chi^{\widehat{U(1)}}_{J_0^3 = n/2} = \sum_{\ell=0}^{\infty} \chi^{\rm Vir}_{L_0^S = (n+2\ell)^2/4} \ . \tag{2.15}$$

In summary, the decomposition of representations is given by:

$$\begin{aligned}
\left\{\sigma = 0\right\}_{\widehat{SU(2)}_1} &= \sum_{k \in \mathbf{Z}\ \rm even} \left\{ J_0^3 = \frac{k}{2} \right\}_{\widehat{U(1)}} = \sum_{s=0}^{\infty} \ (2s+1) \left\{ L_0^S = s^2 \right\}_{\rm Vir} \ , \\
\left\{\sigma = \frac{1}{2}\right\}_{\widehat{SU(2)}_1} &= \sum_{k \in \mathbf{Z}\ \rm odd} \left\{ J_0^3 = \frac{k}{2} \right\}_{\widehat{U(1)}} = \sum_{s=1/2,\ s \in \mathbf{Z}^+ + 1/2}^{\infty} (2s+1) \left\{ L_0^S = s^2 \right\}_{\rm Vir} \ .
\end{aligned} \tag{2.16}$$

These correspond to the following inclusions of $c = 1$ algebras:

$$\widehat{SU(2)}_1 \ \supset \ \widehat{U(1)} \ \supset \ {\rm Vir} \ . \tag{2.17}$$

Next, we observe that the decompositions (2.16) show the familiar multiplicities $(2s + 1)$ of the $SU(2)$ Lie-algebra representations with $s = n/2$ being related to the Virasoro dimension. Actually, s is a true isospin, because the fusion rules of the degenerate Virasoro representations are equivalent to the addition of spins: $\{s + s'\} \oplus \{s + s' - 1\} \oplus \ldots \oplus \{|s - s'|\}$. Therefore, the $\widehat{SU(2)}_1$ "spin" σ is only the parity of s (which is additive modulo two); moreover, the $\widehat{U(1)}$ charge J_0^3 is also additive and corresponds to the projection of the isospin on one axis, i.e. to m.

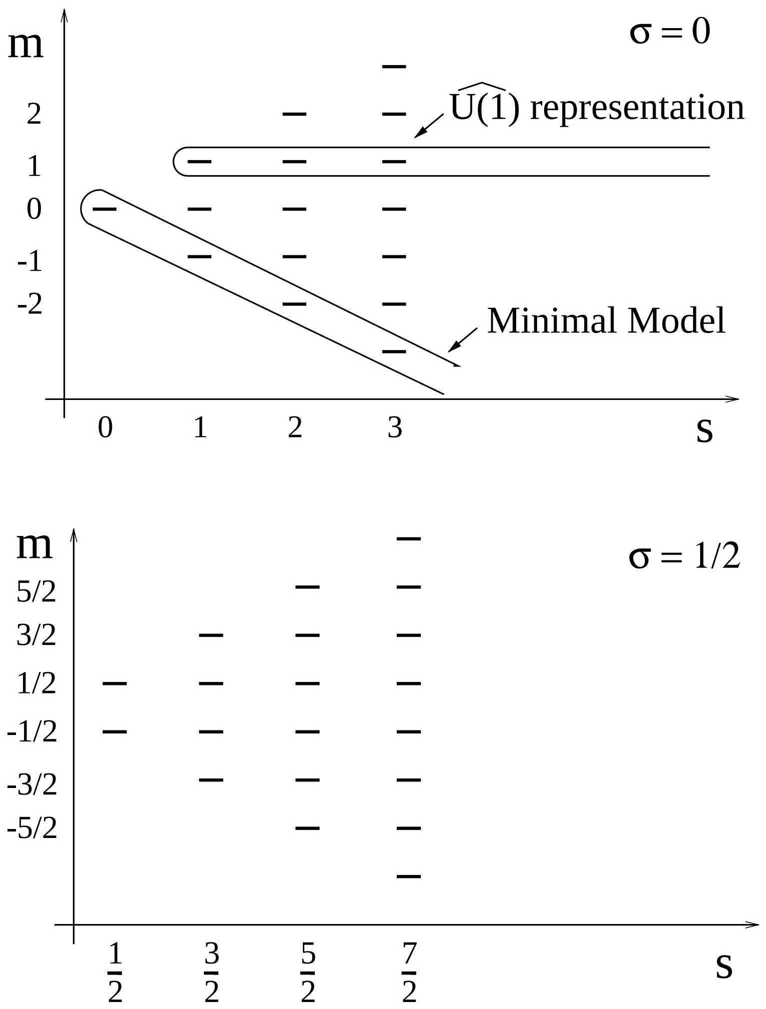

Figure 1: Decomposition of the two $\widehat{SU(2)}_1$ representations in terms of the Virasoro ones: the horizontal axis is the total isospin s (Virasoro dimension $L_0^S = s^2$); the vertical axis is the isospin component m (eigenvalue of J_0^3).

The generators of this $SU(2)$ algebra inside the $\widehat{SU(2)}_1$ representations are the zero modes of the currents $\{J_0^{\pm}, J_0^3\}$; actually, these commute with the L_n^S (see Eq.(2.12)), and thus act within the $(2s+1)$ Virasoro representations of same weight. The decomposition of the two $\widehat{SU(2)}_1$ representations is drawn schematically in Fig. (1): each dash in the (s, m) plane correspond to a Virasoro representation with $h = s^2$; the $\widehat{U(1)}$ representations are horizontal arrays of dashes corresponding to a given m value (see Eq. (2.15)).

2.2 The $c=2$ $W_{1+\infty}$ Minimal Models

According to the Laughlin theory [5], the ground state at the Hall plateaus can be described as a two-dimensional incompressible fluid, which is characterized by constant electron density ρ_o and number N. Its low-energy excitations are deformations of the fluid which have the same area $\mathcal{A}$:

$$N = \int d^2x \ \rho(\mathbf{x},t) \ \approx \ \rho_o \mathcal{A} \ . \tag{2.18}$$

These configurations can be mapped into each other by area-preserving transformations of the coordinates of the plane, whose infinitesimal generators obey the w_∞ algebra. Therefore, the (semiclassical) chiral incompressible fluids possess the w_∞ *dynamical* symmetry [10] [11].

The edge excitations are identified with the infinitesimal area-preserving deformations of a droplet of fluid; their quantization [12] leads to a conformal field theory with the symmetry of the $W_{1+\infty}$ algebra, which is the quantum analog of w_∞. Note that the $W_{1+\infty}$ algebra contains the $\widehat{U(1)}$ and Virasoro algebras. We refer to our previous works for the description of the incompressible fluids by the w_∞ transformations [10] and for the definition of the $W_{1+\infty}$ algebra [20]; here, we recall the main properties of the $W_{1+\infty}$ conformal theories.

In Ref. [9], the $W_{1+\infty}$ theories have been constructed by the algebraic method of assembling the representations of the $W_{1+\infty}$ algebra and checking the closure under their fusion rules. For $c=2$, the $W_{1+\infty}$ unitary representations are of two types: *generic* or *degenerate* [13]. The generic representations are one-to-one equivalent to those of the Abelian algebra $\widehat{U(1)} \times \widehat{U(1)}$ with charges $(r^{(1)}, r^{(2)})$ (eigenvalues of $J_0^{(1)}$ and $J_0^{(2)}$) satisfying $(r^{(1)} - r^{(2)}) \notin \mathbf{Z}$. Clearly, the $W_{1+\infty}$ theories made of generic representations correspond to the generic two-component Abelian theories. On the other hand, the degenerate $W_{1+\infty}$ representations are not equivalent to the Abelian ones, but are contained into them; their charges satisfy the condition $(r^{(1)} - r^{(2)}) \in \mathbf{Z}$. The conformal theories made of the degenerate representations (only) have been called the $W_{1+\infty}$ *minimal models* [9]. The main result is that these models are in one-to-one relation with the hierarchical filling fractions (1.1) and yield the same spectrum of charge and fractional statistics of the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theories (2.6); however, the detailed properties of the two spectra are different, as anticipated in the Introduction.

The $SU(2)$ "symmetry" present in the $c=2$ degenerate representations of $W_{1+\infty}$ is a consequence of some properties discussed in Ref.[13]: in general, the $c=m$ degenerate $W_{1+\infty}$ representations are equivalent to the representations of the $\widehat{U(1)} \times \mathcal{W}_m$ algebra, where $\mathcal{W}_m$ is the Zamolodchikov-Fateev-Lukyanov algebra at $c=m-1$

[14]. The $SU(m)$ Lie algebra is used in the construction of the $\mathcal{W}_m$ algebra, and its fusion rules are isomorphic to the tensor product of $SU(m)$ representations. In the case of interest here, $c = 2$, we have:

$$\text{degenerate } W_{1+\infty} \text{ reps } = \widehat{U(1)} \times \text{Vir reps} , \qquad (c = 2) , \tag{2.19}$$

where the Abelian algebra describes the charged sector, as usual, and Virasoro $\sim \mathcal{W}_2$ accounts for the neutral sector.

After establishing the nature of the degenerate $W_{1+\infty}$ representations, in Ref. [9] the spectrum of the corresponding minimal models was obtained by checking the fusion rules: these are again satisfied by a two-dimensional lattice of representations. Their charge and Virasoro weights are still given by the Abelian spectrum (2.6), but the integer n is restricted to $n = 0, 1, 2, \dots$ (a wedge of the lattice), and the meaning of each point in the lattice is different: in the $W_{1+\infty}$ minimal models, it represents one $\widehat{U(1)} \times$ Vir representation, while in the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theories this is a $\widehat{U(1)} \times \widehat{U(1)}$ representation of the same Virasoro weight.

Our next task is to obtain the minimal models as explicit projections of the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theories. Of course, this discussion concerns the neutral sector only. According to the previous discussion, the minimal model contains each $h = s^2$ Virasoro representation with multiplicity one, while the $\widehat{SU(2)}_1$ theories contain $(2s + 1)$ copies of it (see Eq.(2.16)). Therefore, we should choose one state per $SU(2)$ multiplet in a consistent way; this is achieved by imposing a constraint on the states of the form:

$$\mathcal{Q} \mid \text{minimal state} \rangle = 0 , \tag{2.20}$$

for some operator $\mathcal{Q}$. This constraint is satisfied by the states of the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theory which also belong to the minimal model; the other states are projected out.

There are two natural choices for the operator $\mathcal{Q}$, namely J_0^+ and J_0^-; these select the highest (resp. lowest) state in each $SU(2)$ multiplet. The previous decomposition of the $\widehat{SU(2)}_1$ representations into Virasoro ones shows that there are no other choices for $\mathcal{Q}$ in (2.20) (see Fig.(1), a more formal derivation will be given in Section 4).

Therefore, the $W_{1+\infty}$ minimal models can be defined by the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theory plus the constraint:

$$J_0^- \mid \text{minimal state} \rangle = 0 . \tag{2.21}$$

This condition can be completely solved in the basis $\{L_0^S, J_0^3\}$ already discussed at length: the $SU(2)$ symmetry is clearly killed, because it does not commute with the constraint (2.21); the $U(1)$ quantum number m, is also fixed to $m = -s$; the remaining good quantum number is the Virasoro weight $h = s^2$. We can say that $\mathcal{W}_2$, i.e. Virasoro, is the Casimir sub-algebra of $\widehat{SU(2)}_1$; in general, we have that $\widehat{SU(m)}_1 \sim SU(m) \times \mathcal{W}_m$ (see also the Appendix). Note that the total isospin s continues to compose as before in the fusion of two Virasoro representations: namely, in the minimal models, the composite quasi-particles form $SU(2)$ tensor products, but there is no $SU(2)$ symmetry. Some examples of the minimal states satisfying (2.21) will be given in Section 3. The projection (2.21) relating the Abelian and minimal models is a simple case of the general mechanism of Hamiltonian reduction of $\widehat{SU(m)}_k$, which is discussed in the Refs. [16].

2.3 The Hamiltonian of the $c = 2$ Minimal Models

A first physical interpretation of the constraint (2.21) is that of a strong polarization of the isospin in the $W_{1+\infty}$ minimal models as opposed to the unpolarized $\widehat{SU(2)}_1$ system. Note, however, that a Zeeman term would give lower energy to the larger s values, which is not true in the present case. In order to be more precise, we should introduce the Hamiltonian which enforces this constraint. This is given by the following expression ($c = 2$):

$$H = \frac{1}{R}\left(v\ L_0^Q\ +\ v'\ L_0^S\ -\ \frac{1}{12}\right)\ +\ \gamma\ J_0^+\ J_0^-\ , \qquad \gamma \in [0, \infty)\ . \tag{2.22}$$

The first term is the standard Hamiltonian for the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theory; the second term is diagonal in the (s, m) basis, and its eigenvalues are $\gamma\,(s(s+1) - m(m-1))$: for $\gamma \to \infty$, it selects the lowest weight in each $SU(2)$ multiplet, i.e. it implements the constraint (2.21).

Note that the new term in the Hamiltonian is a non-local interaction:

$$J_0^+\ J_0^-\ =\ \oint_{C_t}\ d\theta\ \oint_{C_{t-\epsilon}}\ d\theta'\ :\mathrm{e}^{i\sqrt{2}\varphi(\theta)}:\ :\mathrm{e}^{-i\sqrt{2}\varphi(\theta')}:\ ; \tag{2.23}$$

actually, this expression cannot be reduced to a single contour integral. Moreover, this term is relevant in the renormalization-group sense, because γ has the dimension of a mass. Therefore, as long as γ is switched on, the infra-red limit of the theory defined by H in (2.22) is the $c = 2$ $W_{1+\infty}$ minimal model ($\gamma_{eff} = \infty$). This infra-red fixed point is known exactly because the γ term is diagonal in the considered basis, and, moreover, it is conformally invariant because $[J_0^+\ J_0^-, L_n^S] = 0$.

Therefore, the Hamiltonian (2.22) defines a renormalization-group trajectory $\gamma \in [0, \infty)$ which interpolates between the two-component Abelian theory and the corresponding $W_{1+\infty}$ minimal model. This renormalization-group flow takes place in the same phase of the system, because the two conformal theories share the same ground-state (which obviously satisfies the constraint (2.21)). Let us remark that this Hamiltonian naturally suggests the physical relevance of the minimal models: since the conformal field theories are effective low-energy, long-distance descriptions of the edge excitations, the farthest infra-red fixed-point is physically relevant; in other words, $\gamma \to \infty$ is naturally reached without fine-tuning.

In Ref.[15], the exact low-lying spectrum has been computed numerically for ten electrons with repulsive short-range interactions on the disk geometry. Two new criteria have been introduced for the analysis of the low-lying exact states, which allow for their interpretation as CFT states and for their identification as either edge or bulk excitations. A clear pattern is observed: the edge excitations match the states of the $W_{1+\infty}$ minimal models, which satisfy (2.21); the remaining Abelian states correspond to bulk excitations. An energy splitting is found between these two set of states, as suggested by the Hamiltonian (2.22), but is rather weak: $\gamma \geq O(1/R)$ rather than $\gamma \to \infty$. In conclusion, the infrared limit cannot be reached in this finite-size system, but the correct edges excitations are nevertheless identified with the states of the minimal models described before.

Another virtue of the Hamiltonian (2.22) is that it shows that the $W_{1+\infty}$ minimal models are completely consistent conformal field theories. It had previously been found [19] that they are not rational CFTs [6], i.e. their partition function cannot by modular invariant[§]. This is a rather uncommon feature in the literature of CFT, which remained unexplained: here, we can trace it back to the non-locality of the γ term, which violates one of the hypothesis for modular invariance [6]. Note also that generalized modular transformations have been defined for the $c = 1$ (non-minimal) $W_{1+\infty}$ theories [23], for Hamiltonians which contain all possible *local* operators V_0^i, $i = 0, 1, 2, \ldots$, in the Cartan subalgebra of $W_{1+\infty}$ [20]:

$$H \;=\; \alpha\, J_0 \;+\; \frac{v}{R}\, L_0 \;+ \frac{\beta}{R^2}\, V_0^2 + \frac{\delta}{R^3}\, V_0^3 + \;\ldots\;, \qquad (2.24)$$

(here, $J_0 \equiv V_0^0$ and $L_0 \equiv V_0^1$). However, the proposed Hamiltonian (2.22) does not belong to this class.

Let us finally remark that the generalization of the constraint (2.21) and of the Hamiltonian (2.22) to the $W_{1+\infty}$ minimal models with higher central charges $c = 3, 4, \ldots$ is slightly more technical; the case $c = 3$ is presented in the Appendix.

[§]See Ref.[22] for the proposal of rational extensions of the $W_{1+\infty}$ models.

Figure 2: Energy versus momentum plot of two filled Landau levels on the disk geometry: the dashes represent one-particle states, whose energy grows at large M due to the confining potential; the crosses represent the electrons.

3 Fermionic Realization and the Minimal Incompressible Fluid Picture

In this section, we discuss the Fermionic Fock space realization of the $(c = 2)$ minimal models; this leads to the physical picture of the minimal incompressible Hall fluids as chiral Fermi seas. Moreover, it allows for a direct comparison with the Jain composite-fermion theory [2], which has been used in the numerical analysis of Ref.[15]. We first consider the case of two filled Landau levels $(\nu = 2)$ (see Fig.(2)); this is not one of the hierarchical plateaus; thus, this should not be literally taken as a physical realization of the minimal models. However, its Hilbert space is isomorphic to those of the hierarchical states $\nu = 2/(2p \pm 1)$, and, in particular, the neutral sector is identical because it is p-independent (see Eq.(2.6)).

The edge excitations of two filled Landau levels are described by two complex chiral (Weyl) fields Ψ_1 and Ψ_2; their mode expansion in the Neveu-Schwarz sector is (for $t = 0$):

$$\begin{aligned}
\Psi_1 &= \sum_{r\in\mathbf{Z}} \mathrm{e}^{i\,r\,\theta}\; u_r \;, \qquad \overline{\Psi}_1 = \sum_{r\in\mathbf{Z}} \mathrm{e}^{-i\,(r-1)\,\theta}\; u_r^\dagger \;, \\
\Psi_2 &= \sum_{r\in\mathbf{Z}} \mathrm{e}^{i\,r\,\theta}\; d_r \;, \qquad \overline{\Psi}_2 = \sum_{r\in\mathbf{Z}} \mathrm{e}^{-i\,(r-1)\,\theta}\; d_r^\dagger \;,
\end{aligned} \tag{3.1}$$

where u_r and d_r denote the Fermionic Fock space operators, which satisfy $\{u_k, u_l^\dagger\} = \{d_k, d_l^\dagger\} = \delta_{k,l}$. This system defines a $c = 2$ conformal field theory with current algebra $\widehat{U(1)} \times \widehat{SU(2)}_1$. The ground state of the system $|\Omega\rangle$ is the tensor product of two Dirac vacua, with corresponding Fermi surfaces (see Fig.(2)). One refers to these two vacua as an "upper" and "lower" components of $|\Omega\rangle$, which satisfies the conditions:

$$\begin{aligned} u_l|\Omega\rangle &= d_l|\Omega\rangle = 0 \ , \quad l > 0 \ , \\ u_l^\dagger|\Omega\rangle &= d_l^\dagger|\Omega\rangle = 0 \ , \quad l \leq 0 \ . \end{aligned} \tag{3.2}$$

In the Landau level picture, l measures the single-particle angular momentum with respect to the ground state.

The $\widehat{U(1)} \times \widehat{SU(2)}_1$ currents of Section 2 (see Eq.(2.7)) can be written in terms of the Fermionic bilinears, $J^{(i)} = :\overline{\Psi}_i\Psi_i: \ , i = 1, 2$, as follows:

$$\begin{aligned} J &= J^{(1)} + J^{(2)} \ , \qquad J^3 = \frac{1}{2}\left(J^{(1)} - J^{(2)}\right) \ , \\ J^+ &= \overline{\Psi}_1\Psi_2 \ , \qquad\qquad J^- = \overline{\Psi}_2\Psi_1 \ . \end{aligned} \tag{3.3}$$

Their Fourier modes are given by[¶]:

$$\begin{aligned} J_n &= \sum_{k\in\mathbf{Z}} \left(u_{k-n}^\dagger u_k + d_{k-n}^\dagger d_k \right) \ , \\ J_n^3 &= \frac{1}{2} \sum_{k\in\mathbf{Z}} \left(u_{k-n}^\dagger u_k - d_{k-n}^\dagger d_k \right) \ , \\ J_n^+ &= \sum_{k\in\mathbf{Z}} u_{k-n}^\dagger d_k \ , \\ J_n^- &= \sum_{k\in\mathbf{Z}} d_{k-n}^\dagger u_k \ . \end{aligned} \tag{3.4}$$

Using the Sugawara construction, one defines the two $c = 1$ stress-energy tensors of Section 2.1, which are associated to the charged and neutral sectors, $L^Q = :J^2:/4$ and $L^S = :\left(J^3\right)^2:$, respectively. Their Virasoro modes can be written in the Fermionic basis as follows:

$$\begin{aligned} L_n^Q &= \frac{1}{2}\sum_{l\in\mathbf{Z}}\left(l - \frac{n+1}{2}\right)\left(u_{l-n}^\dagger u_l + d_{l-n}^\dagger d_l\right) + \frac{1}{2}\sum_{l\in\mathbf{Z}}\left(\sum_{i\in\mathbf{Z}} u_{i-n+l}^\dagger u_i \sum_{j\in\mathbf{Z}} d_{j-l}^\dagger d_j\right) \ , \\ L_n^S &= \frac{1}{2}\sum_{l\in\mathbf{Z}}\left(l - \frac{n+1}{2}\right)\left(u_{l-n}^\dagger u_l + d_{l-n}^\dagger d_l\right) - \frac{1}{2}\sum_{l\in\mathbf{Z}}\left(\sum_{i\in\mathbf{Z}} u_{i-n+l}^\dagger u_i \sum_{j\in\mathbf{Z}} d_{j-l}^\dagger d_j\right) \ . \end{aligned} \tag{3.5}$$

Let us now discuss the excitations of the $\widehat{U(1)} \times \widehat{SU(2)}_1$ theory in the Fermionic basis: these are labelled by their angular momentum with respect to the ground

[¶]Of course, their zero modes should be normal-ordered with respect to the vacuum (3.2).

state ΔM, and by the two Abelian charges J_0 and J_0^3, which are symmetric and anti-symmetric with respect to the two layers, respectively. As explained in Section 2.2, the $c = 2$ $W_{1+\infty}$ minimal model is obtained from this theory by imposing the constraint (2.21):

$$J_0^- \mid \text{minimal state} \rangle = \sum_{k \in \mathbf{Z}} d_k^\dagger u_k \mid \text{minimal state} \rangle = 0 \ . \tag{3.6}$$

In this basis, we can interpret the constraint as follows: the operator J_0^- moves electrons down and holes up between the two layers (with a minus sign in the latter case), without changing their angular momentum value; it relates the edge excitations in the two layers and actually vanishes on their symmetric linear combinations. Therefore, the condition (3.6) projects out the edge excitations which are antisymmetric with respect to the two levels: this distinguishes the minimal $W_{1+\infty}$ CFTs from the Abelian ones. Note that the ground state is unique and symmetric, then it satisfies the constraint: namely, the two CFTs share the same ground state.

Let us see some examples of the allowed edge excitations in the minimal theory. There are two Abelian edge excitations at the first excited level $\Delta M = 1$, which can be written:

$$|1; \pm\rangle = \frac{1}{\sqrt{2}} \left(d_1^\dagger d_0 |\Omega\rangle \ \pm \ u_1^\dagger u_0 |\Omega\rangle \right) . \tag{3.7}$$

We find that the symmetric combination $|1;+\rangle$ satisfies the constraint (3.6) and the antisymmetric does not. At the next level $\Delta M = 2$, there are five Abelian edge states:

$$\begin{aligned} \mid 2; a\rangle &= d_1^\dagger d_0 \ u_1^\dagger u_0 |\Omega\rangle \ , \\ \mid 2; b\pm\rangle &= \frac{1}{\sqrt{2}} \left(d_2^\dagger d_0 |\Omega\rangle \ \pm \ u_2^\dagger u_0 |\Omega\rangle \right) \ , \\ \mid 2; c\pm\rangle &= \frac{1}{\sqrt{2}} \left(d_1^\dagger d_{-1} |\Omega\rangle \ \pm \ u_1^\dagger u_{-1} |\Omega\rangle \right) \ . \end{aligned} \tag{3.8}$$

Again, the antisymmetric combinations $\mid 2; b-\rangle$ and $\mid 2; c-\rangle$ do not satisfy the constraint (3.6). Therefore, the $W_{1+\infty}$ minimal models only contains the symmetric excitations, as we anticipated. In general, the number of edge excitations with given ΔM can be obtained by expanding the character of the vacuum representation in powers of q (see Section 2.2); this counting of states for the Abelian and minimal theories is reported in Table (1).

In Ref. [10], a physical picture has been developed for the ground state of the Laughlin plateaus $\nu = 1, 1/3, 1/5, \ldots$, which is based on the Laughlin incompressible fluid idea. At the semiclassical level, we said that the deformation of the incompressible fluids are described by the area-preserving transformations obeying the w_∞

	ΔM	0	1	2	3	4	5
$c=1$	One-component Abelian	1	1	2	3	5	7
$c=2$	Minimal Incompressible	1	1	3	5	10	16
	Two-component Abelian	1	2	5	10	20	36

Table 1: The number of edge excitations for the Laughlin Hall fluids $\nu = 1/(p+1), p = 2, 4, \ldots$ and its quasi-particles (first row) and for the hierarchical fluids $\nu = 2/(2p+1)$ (second and third rows), according to the two relevant conformal field theories.

algebra. In the ($\nu = 1$) quantum theory, the incompressible fluid can be identified as the filled chiral Fermi sea, which occurs in coordinate space rather than in momentum space (see Fig.(2)); moreover, the low-lying edge excitations are identified with the particle-hole transitions across the Fermi surface. The effective quantum theory for these excitations is given by a single Weyl fermion, which naturally possesses the $W_{1+\infty}$ quantum symmetry. Actually, the generators V^i_n of this algebra produce all possible particle-hole transitions with $\Delta M = -n$: those with negative ΔM are not allowed by the Pauli exclusion principle; thus the ground state is annihilated by all the generators with positive mode index [10],

$$V^i_n \left|\Omega\right\rangle \ = \ 0 \ , \qquad i = 0, 1, 2, \ldots, \qquad n > 0 \ . \tag{3.9}$$

These are the $W_{1+\infty}$ highest-weight conditions, which can be physically interpreted as the incompressibility of the ground state at the quantum level. The same incompressibility conditions are satisfied by the Laughlin ground states [10][11]; their edge excitations are described by more general $c = 1$ $W_{1+\infty}$ conformal theories[‖]. Actually, the Hilbert space of the bosonic ($\nu = 1/3, 1/5, \ldots$) and Fermionic ($\nu = 1$) theories are isomorphic and only differ for the quantum numbers of quasi-particles [3].

In the case of the hierarchical plateaus, there are, correspondingly, the present Fermionic basis of two Weyl fermions ($\nu = 2$) and the bosonic basis ($\nu = 2/5, \ldots$) of the previous Section, which are again isomorphic, and actually identical for the ground state and the neutral sector. The $W_{1+\infty}$ highest-weight conditions (3.9) on the two-layer Fermi sea can be interpreted as before as incompressibility conditions. However, the edge excitations possess an additional sector of up-down antisymmetric particle-hole transitions; these excitations are tangential to the Fermi surface and

[‖]In this case completely equivalent to the $\widehat{U(1)}$ chiral-boson theories.

cannot be interpreted as deformations of the incompressible fluid as a whole. In this picture, the minimal-model constraint (3.6) can be regarded as an additional incompressibility condition: it removes the antisymmetric excitations, which are not necessary for the "minimal" incompressible fluids.

Note that the symmetric excitations can be divided into two classes: those generated by the symmetric current J_{-k}, $k = 1, 2, \ldots$ in (3.4), which are the same as in the Laughlin fluids (corresponding to the $\widehat{U(1)}_{\rm diagonal}$ part); in addition, there are those generated by the Virasoro modes L^S_{-k}, $k = 1, 2, \ldots$ of the neutral sector, which are specific of the ($c = 2$) minimal incompressible fluid; the first example is $|2; a\rangle \sim L^S_{-2}|\Omega\rangle$ in (3.8), which occurs at level $\Delta M = 2$ (in agreement with the multiplicities of the respective theories in Tab.(1)).

This $\nu = 2$ description of the minimal models can be pictorially extended to the hierarchical states $\nu = 2/(2p \pm 1)$. The charged sector $\widehat{U(1)}_{\rm diagonal}$ is bosonized: it remains isomorphic to the Fermionic one, but the charge eigenvalues change. On the other hand, the p-independent neutral sector remains the same and can still be described in the Fermionic basis, provided one pays attention to distinguishing the two types of symmetric excitations discussed in the previous paragraph. This description of the hierarchical plateaus is analog to the composite-fermion transformation in the Jain theory of wave functions [2] and has been throughoutfully discussed in Ref.[15].

3.1 Spinon Basis

Another possible basis for the states in the $\widehat{SU(2)}_1$ CFT is given by the spinon excitations [17]: the spinon fields,

$$\Phi^\alpha(z) \ = \ :\exp\left(i \ \frac{\alpha}{\sqrt{2}} \ \varphi(z) \right): \ , \qquad \alpha = \pm 1 \ , \tag{3.10}$$

are the $\widehat{SU(2)}_1$ primary fields of conformal dimension $h = 1/4$ and isospin $s = 1/2$. The exchange of two spinons yields the fractional statistics $\theta/\pi = 1/2$, which is half of that of the fermions; moreover, these fields are not charged, since they belong to the neutral sector of $\widehat{U(1)} \times \widehat{SU(2)}_1$. They are also called semions, i.e. "half-fermion" anyons, which carry (iso)-spin 1/2 and no charge.

The Fourier modes of the spinon fields are obtained from the expansion [24]:

$$\Phi^\alpha(z) \ \chi_\sigma(0) \ = \ \sum_{n\in\mathbf{Z}} \ z^{n+\sigma} \ \Phi^\alpha_{-n-\sigma-\frac{1}{4}} \ \chi_\sigma(0) \ , \tag{3.11}$$

where $\chi_\sigma(0)$ is a generic state containing an even (resp. odd) number of spinons for $\sigma = 0$ (resp. $\sigma = 1/2$). The $\widehat{SU(2)}_1$ Hilbert space can be generated by applying these modes to the vacuum; this is an alternative to the standard basis using the J^a_n modes discussed in Section 2.1 (actually, the latter are spinon bilinears). The spinon fields satisfy generalized (non-local) commutation relations which imply relations among the states freely generated by the multinomials of their modes. These relations can be casted into the form of a generalized Pauli exclusion principle: therefore, the modes (3.11) build a pseudo-Fock space which is neither bosonic ($\alpha^{(i)}_n$) nor Fermionic (u_k, d_k). This basis has been discussed in the literature in relation to the solution of the $SU(2)$ Haldane-Shastry spin chain, whose spectrum breaks the $\widehat{SU(2)}_1$ symmetry down to those of the Yangian $Y(sl_2)$ Hopf algebra, which contains $SU(2)$.

This decomposition is rather different from the one relevant for the $W_{1+\infty}$ minimal models. Indeed, the corresponding interaction term of Section 2.3,

$$H^{(2)} \ = \ \gamma \ J^+_0 J^-_0 \ , \tag{3.12}$$

breaks the $\widehat{SU(2)}_1$ symmetry down to Virasoro, $[H^{(2)}, L_n] = 0$; moreover, the $SU(2)$ symmetry $\{J^{\pm}_0, J^3_0\}$ is also broken. In the case of the Haldane-Shastry chain, the interaction is given by :

$$H_2 \ = \ \lambda \sum_{k>0} \ k \ J^a_{-k} J^a_k \ ; \tag{3.13}$$

this term breaks Virasoro, $[H_2, L_n] \neq 0$ for $n \neq 0$, but keeps scale invariance; moreover, it preserves the $SU(2)$ symmetry $[H_2, J^a_0] \ = \ 0$ and the Yangian symmetry $[H_2, Q^a_1] = 0$ generated by the corresponding Q^a_1 operators (for their definition, see Ref. [17][24]).

Nevertheless, the two breakings of $\widehat{SU(2)}_1$ are compatible, because the interactions in the two problems commute, $[H^{(2)}, H_2] = 0$. Actually, we are going to show that the constraint $J^-_0 \,|$ minimal state $\rangle = 0$ defining the minimal models can be solved in the spinon basis. First, we remark that the spinon modes $\Phi^{\alpha}_{-n-\sigma-\frac{1}{4}}$ have isospin projection $m = \alpha/2$; this follows from analyzing the $\widehat{SU(2)}_1$ operator product expansion:

$$J^a(z) \ \Phi^{\alpha}(w) \ = \ (t^a)^{\alpha}_{\beta} \ \frac{\Phi^{\beta}(w)}{(z-w)} \ + \ \text{regular terms} \ , \tag{3.14}$$

where $(t^a)^{\alpha}_{\beta}$ are the Pauli matrices divided by 2. Next, the states of minimal isospin projection $s = -m$, selected by the constraint $J^-_0 \sim 0$, are simply generated by multilinears of the Φ^- modes, because m is additive. Therefore, the fully polarized N-spinon states (see Ref. [24]),

$$\Phi^-_{-\frac{(2N-1)}{4}-n_N} \cdots \Phi^-_{-\frac{5}{4}-n_3} \ \Phi^-_{-\frac{3}{4}-n_2} \ \Phi^-_{-\frac{1}{4}-n_1} \ |\Omega\rangle \ , \tag{3.15}$$

are explicit solutions of the constraint $J_0^- \sim 0$. The Virasoro weight of these states is given by:

$$h \;=\; \frac{N^2}{4} \;+\; \sum_{i=1}^{N} n_i \;; \tag{3.16}$$

clearly, an even (resp. odd) number of spinons gives a state in the $\sigma = 0$ (resp. $\sigma = 1/2$) $\widehat{SU(2)}_1$ representation (see Fig.(1)).

It remains to be established a basis of independent states among all possible N_{th}-plet $\{n_i\}$ in (3.15). This is a rather difficult task, because the "commutation relations" among the modes contain infinite terms; therefore, we shall use an indirect argument. We start from the basis of independent states made by N^+ up and N^- down spinons, which has been introduced in Ref. [24] (Basis II):

$$\begin{array}{l} \Phi^-_{-\frac{2(N^+ + N^-)-1}{4} - n^-_{N^-}} \cdots \Phi^-_{-\frac{2N^+ +1}{4} - n^-_1} \; \Phi^+_{-\frac{2N^+ -1}{4} - n^+_{N^+}} \cdots \Phi^+_{-\frac{1}{4} - n^+_1} \; |\Omega\rangle \;, \\ n^+_{N^+} \geq \ldots \geq n^+_2 \geq n^+_1 \geq 0 \;, \qquad n^-_{N^-} \geq \ldots \geq n^-_2 \geq n^-_1 \geq 0 \;. \end{array} \tag{3.17}$$

The Virasoro dimension is again given by (3.16) with $N = N^+ + N^-$ and $\{n_i\} = \{n_i^+, n_j^-\}$. It has been shown [24], that this basis reproduces the $\widehat{SU(2)}_1$ representations $\sigma = 0$ for $N^+ + N^-$ even (resp., $\sigma = 1/2$ for $N^+ + N^-$ odd); this can be checked by computing the corresponding characters, using the following identity for the sum of partitions:

$$\sum_{k_1 \geq \ldots k_N \geq 0} q^{\sum_{i=1}^{N} k_i} \;=\; \prod_{\ell=1}^{N} \sum_{m_\ell = 0}^{\infty} q^{\ell\, m_\ell} \;=\; \frac{1}{\prod_{k=1}^{N} (1 - q^k)} \;\equiv\; \frac{1}{(q)_N} \;. \tag{3.18}$$

One obtains:

$$\chi^{\widehat{SU(2)}_1}_{\sigma=0} \;=\; \sum_{N^+, N^- = 0,\; N^+ + N^- \mathrm{even}}^{\infty} \frac{q^{(N^+ + N^-)^2/4}}{(q)_{N^+}\;(q)_{N^-}} \;, \tag{3.19}$$

and similarly the sum extended to $(N^+ + N^-)$ odd for $\sigma = 1/2$. These are actually equivalent expressions for the characters of the $\widehat{SU(2)}_1$ representations (2.13), owing to the identities discussed in Ref. [25].

Next, we remark that the states (3.17) have isospin projection $m = (N^+ - N^-)/2$ and isospin $0 \leq s \leq (N^+ + N^-)/2$. Consider the subspace of states with $J_0^3 = m = 0$, i.e. $N^+ = N^-$: their sum reproduces the $\widehat{U(1)}$ vacuum representation, as is clear from Fig. (1). One can check the corresponding character:

$$\chi^{\widehat{U(1)}}_{h=0} \;=\; \sum_{N^+=0}^{\infty} \frac{q^{(N^+)^2}}{\left((q)_{N^+}\right)^2} \;=\; \frac{1}{\eta(q)} \;. \tag{3.20}$$

Similarly, the $m = -1/2$ subspace, $N^- = N^+ + 1$, yields:

$$\chi^{\widehat{U(1)}}_{h=1/4} \;=\; \sum_{N^+=0}^{\infty} \frac{q^{(2N^+ +1)^2/4}}{(q)_{N^+}\,(q)_{N^+ +1}} \;=\; \frac{q^{1/4}}{\eta(q)} \;. \tag{3.21}$$

From Fig.(1) and the discussion in Section 2.1, it follows that the $m = 0$ subspace is isomorphic to the Hilbert space of the minimal model, in the sector $\sigma = 0$ (the $m = -1/2$ one matches the $\sigma = 1/2$ sector, respectively). The mapping can be obtained by multiple application of J_0^-, as it follows: given that $s \leq N^+$ for the states with $N^+ = N^-$ (3.17), the action of $\left(J_0^-\right)^{N^+}$ yields the states of minimal projection $m = -s = -N^+$ which are required for the minimal models; otherwise, it vanishes. Moreover, this action replaces all the N^+ modes Φ_β^+ with Φ_β^- in Eq. (3.17). Therefore, the basis (3.17) is mapped into the one of fully polarized spinons (3.15), which satisfy the constraint $J_0^- \sim 0$.

In conclusion, the (neutral part of) Hilbert space of the $c = 2$ $W_{1+\infty}$ minimal models can be described by the fully-polarized spinon states (3.15) with the following choices of $\{n_i\}$:

$$
\begin{aligned}
\sigma = 0 \ : \quad & N \quad \text{even} \ , \ \begin{cases} n_N \geq n_{N-1} \geq \ldots \geq n_{\frac{N}{2}+1} \geq 0 \ , \\ n_{\frac{N}{2}-1} \geq n_{\frac{N}{2}-2} \geq \ldots \geq n_1 \geq 0 \ ; \end{cases} \\
\sigma = \frac{1}{2} \ : \quad & N \quad \text{odd} \ , \ \begin{cases} n_N \geq n_{N-1} \geq \ldots \geq n_{\frac{N+1}{2}} \geq 0 \ , \\ n_{\frac{N-1}{2}} \geq n_{\frac{N-1}{2}-1} \geq \ldots \geq n_1 \geq 0 \ . \end{cases}
\end{aligned}
\tag{3.22}
$$

Note that this basis is larger than the one introduced in Ref. [24] (Basis I) for building the Yangian highest-weight states (as it should). Let us finally remark that the physical interpretation of this basis for the ($c = 2$) minimal models remains to be understood, possibly by extending the picture of the Fermi sea discussed in the previous paragraph.

4 Correlation Functions and Non-Abelian Statistics

In this Section, we derive the correlation functions of the quasi-particles in the $c = 2$ $W_{1+\infty}$ minimal models, i.e. the $\widehat{U(1)} \times \mathrm{Vir}$ CFTs. These correlators factorize into charged and neutral parts; the charged N-point functions are given by the well-known vertex operator expectation values [6]:

$$
\begin{aligned}
\langle \ \Omega \mid V_{Q_1}(z_1) \ldots V_{Q_N}(z_N) \mid \Omega \ \rangle \ &= \ \prod_{i<j}^{N} \ (z_i - z_j)^{Q_i Q_j \left(p+\frac{1}{2}\right)} \ , \\
V_Q(z) \ &= \ :\exp\left(i \ Q \ \sqrt{2p+1} \ \left(\varphi^{(1)}(z) \ + \ \varphi^{(2)}(z) \right) \right): \ ,
\end{aligned}
\tag{4.1}
$$

where the Q_i are given by the spectrum (2.6).

The correlators of the neutral part can be obtained by taking the $c \to 1$ limit of the Dotsenko-Fateev Coulomb Gas construction [18], which applies to the $c \leq 1$ Virasoro minimal models. In this approach, the N-point functions are written in terms of the $\widehat{U(1)}$ vertex operators plus screening charges $Q_\pm$, which project the Virasoro null states out of the $c = 1$ bosonic Fock space. The properties of the screening charges have been investigated by Felder and other authors [26]: we first recall some relevant formulae of these works, then we properly take the $c \to 1$ limit, and compare the result with the formulation of Section 2. Next, we compute some examples for the correlators and discuss the non-Abelian statistics.

Let us start from the $\widehat{U(1)}$ theory of the neutral bosonic field $\varphi = \left(\varphi^{(1)} - \varphi^{(2)}\right)/2$ (Section 2); the screening charges are defined by:

$$Q_\pm \ = \ \oint dz \ : \mathrm{e}^{i\sqrt{2}\alpha_\pm \varphi(z)} : \ , \tag{4.2}$$

with

$$\alpha_+ = \sqrt{\frac{p'}{p}} \ , \qquad \alpha_- = -\sqrt{\frac{p}{p'}} \ , \qquad c = c(p,p') = 1 - 6\frac{(p-p')^2}{pp'} \leq 1 \ , \qquad p, p' > 0 \ . \tag{4.3}$$

These operators have vanishing conformal dimension in the $c(p,p')$ CFTs, and satisfy $[Q_\pm, L_n] = 0$. Their action in the bosonic Fock space is to relate states of same conformal dimension which belong to different $\widehat{U(1)}$ representations:

$$F_{\alpha_{m',m}} \xrightarrow{Q_+} F_{\alpha_{m',m-2}} \ , \qquad F_{\alpha_{m',m}} \xrightarrow{Q_-} F_{\alpha_{m'-2,m}} \ , \tag{4.4}$$

where we denote with $F_{\alpha_{m',m}}$ the representation with "charge" $J_0^3 = \alpha_{m',m}$. Its relevant values are:

$$\begin{aligned} \alpha_{m',m} &= \alpha_0 - m\alpha_+ - m'\alpha_- \ , \qquad 2\alpha_0 = \alpha_+ + \alpha_- \ , \\ h_{m',m} &= \alpha_{m',m}\left(\alpha_{m',m} - 2\alpha_0\right) = \frac{(m'p - mp')^2 - (p-p')^2}{4pp'} \ ; \end{aligned} \tag{4.5}$$

actually, they reproduce the dimensions $h_{m',m}$ of the $c < 1$ degenerate Virasoro representations (the Kac table) [6].

The screening charges are useful because they can map $\widehat{U(1)}$ highest-weight states into Virasoro null states, and thus describe their structure. The two lowest null states in the Virasoro representation $h = h_{m',m}$ have dimensions [6]:

$$h_{m',m} \ + \ m'm \quad , \quad h_{m',m} \ + \ (p' - m')(p - m) \ . \tag{4.6}$$

We are interested in the $c = 1$ degenerate Virasoro representations, which can be obtained by the controlled limit $c(p,p') \to 1$ within the Virasoro minimal models.

This limit can be achieved by letting,

$$p\ ,\ p' \ \to \infty\ , \qquad \frac{p}{p'} \to 1\ , \tag{4.7}$$

which implies:

$$\alpha_\pm \to 1\ ,\ \alpha_0 \to 0\ ,\ \alpha_{m',m} \to \frac{(m'-m)}{2} \equiv \frac{n}{2}\ , \quad h_{m',m} \to \frac{n^2}{4}\ . \tag{4.8}$$

The screening charges go into the $SU(2)$ operators, $Q_\pm \to J_0^\pm$, and the $c<1$ degenerate representations are mapped into the $c=1$ ones, $h=n^2/4$, with infinite degeneracy. It can be shown that all the $c<1$ null states but the lowest one decouple (their dimensions go to infinity). In Felder's notation [26], w_0 denotes the remaining null state in the $c=1$ representation with $h=n^2/4$; this state has the dimension $h=n^2/4+n+1=(n+2)^2/4$, e.g. $m=n+1$, $m'=1$ in (4.6); moreover, it can be obtained from the $\widehat{U(1)}$ highest-weight v_0 with charge $J_0^3=-(n+2)/2$ by the action of $Q_+\equiv J_0^+$:

$$(w_0)_{J_0^3=-n/2} \ = \ J_0^+ \ (v_0)_{J_0^3=-(n-2)/2}\ . \tag{4.9}$$

This result matches the description of Section 2: from Fig.(1) and the characters (2.13), it is apparent that the $h=n^2/4$ Virasoro representation with $s=-m=n/2$ (the $SU(2)$ lowest-weight state) can be obtained from the $\widehat{U(1)}$ representation with $m=-n/2$ (horizontal line in Fig.(1)) by subtracting the $\widehat{U(1)}$ representation of the null-vector with $m=-n/2-1$ (adjacent horizontal line).

In conclusion, the $c=1$ Virasoro theory is described in the Felder approach [26] as the $\widehat{U(1)}$ Fock space with all the null-vectors removed; according to Eq.(4.9), this condition can be written:

$$\Big|\ \text{physical state}\ \left(J_0^3=-m\right)\Big\rangle \ \neq\ J_0^+\ \Big|\ \text{any state}\ \left(J_0^3=-m-1\right)\Big\rangle \qquad (m\geq 0)\ . \tag{4.10}$$

This is clearly equivalent to the constraint $J_0^-|$ physical state $\rangle = 0$ introduced in Section 2; therefore, the Felder (Coulomb Gas) approach yields another description of the Hilbert space of the $W_{1+\infty}$ minimal models, which is equivalent to the previous ones.

4.1 Examples of Correlators

In the Dotsenko-Fateev Coulomb Gas approach [18],[6], the N-point function of Virasoro primary fields ϕ_h is obtained by the vertex operators with appropriate insertion of a number of screening charges; for example, the four-point functions have the form:

$$\begin{aligned} &\langle\ \phi_{s_1^2}(z_1)\phi_{s_2^2}(z_2)\phi_{s_3^2}(z_3)\phi_{s_4^2}(z_4)\ \rangle_{\rm Vir}\ = \\ &\langle\ \Omega\ |\ \left(\prod_{i=1}^{N_+} Q_+\right) :\!{\rm e}^{-im_1\sqrt{2}\varphi(z_1)}\!::\!{\rm e}^{-im_2\sqrt{2}\varphi(z_2)}\!::\!{\rm e}^{-im_3\sqrt{2}\varphi(z_3)}\!::\!{\rm e}^{-im_4\sqrt{2}\varphi(z_4)}\!:\ |\ \Omega\ \rangle\ ; \end{aligned} \tag{4.11}$$

in this expression, the number of screening charges is determined by the condition of charge neutrality:

$$J_0^3 = N_+ - m_1 - m_2 - m_3 - m_4 = 2\alpha_0 = 0\ , \qquad (c=1)\ , \tag{4.12}$$

with $m_i^2 = s_i$, $i = 1,2,3,4$. In the $c < 1$ minimal models, both screening charges Q_+ and Q_- (see Eq.(4.3)) must be used in the correlator (4.11), in order to satisfy the charge neutrality condition (see Eq.(4.5)); in the $c = 1$ case, there are several possibilities, because the two $Q_\pm$ charges are equivalent by the $m \to -m$ symmetry (absent for $c < 1$); moreover, in some correlators the charge neutrality condition can also be satisfied without screening charges, provided that certain permutation rules are employed (see later).

Let us first consider the example of the three-point function of the fields $s_1 = s_2 = k/2$ and $s_3 = n$:

$$\langle\ \Omega\ |\ \left(\prod_{i=1}^{k-n} Q_+\right) :\!{\rm e}^{-i\frac{k}{\sqrt{2}}\varphi(z_1)}\!::\!{\rm e}^{-i\frac{k}{\sqrt{2}}\varphi(z_2)}\!::\!{\rm e}^{in\sqrt{2}\varphi(z_3)}\!:\ |\ \Omega\ \rangle\ . \tag{4.13}$$

Using the reflection symmetry $m_i \to \pm m_i$ for each field, one can choose the simplest case of minimal number of screening charges. Moreover, we can take $z_1 = 0, z_2 = 1, z_3 \to \infty$ by using the $SL(2,\mathbf{C})$ invariance of the ground state. A number of properties can be derived from this correlator: as an example, we check the $SU(2)$ fusion rules $\{k/2\} \times \{k/2\} = \{0\} + \{1\} + \ldots + \{k\}$; these imply that the correlator should not vanish for $n = 0,1,\ldots,k$. For $n = k$, it obviously does not; for $n = k-1 \geq 0$, we find the expression:

$$\oint_{C_{u_0}} du\ u^{-k}\ (1-u)^{-k}\ , \tag{4.14}$$

where the contour C_{u_0} has to be taken around any of the singularities $u_0 = 0, 1, \infty$. This integral is found to vanish for C_∞ and to take opposite non-vanishing values for C_0 and C_1. Next, for $n = k+1$, the charge neutrality condition requires the use of one Q_+, leading to the same expression (4.14) with $k \to -k$; this vanishes for any choice of C_{u_0} and verifies again the fusion rules. Other cases for n can be worked out along the same lines [18].

The non-Abelian statistics of quasi-particles manifests itself in the four (and higher) point functions, as we shall now discuss. In the Coulomb Gas form of the correlator (4.11), the choices of contour C_{u_0} yield several independent expressions, which are called the conformal block $\mathcal{F}_i$ [6]: each one corresponds to an intermediate state with a given isospin value, obtained by the fusion of the isospins involved. For example, the correlation of four electron excitations ($Q = 1$ and $s_i = 1/2$ in (4.12)) can be written:

$$
\begin{aligned}
G(z_i) &= \langle\ \Omega \mid Q_+ {:}\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_1)}{::}\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_2)}{::}\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_3)}{::}\mathrm{e}^{\frac{i}{\sqrt{2}}\varphi(z_4)}{:} \mid \Omega\ \rangle \\
&= \frac{(\eta(1-\eta))^{1/2}}{(z_{13}z_{24})^{1/2}} \oint_{C_i} du\ u^{-1}\ (u-1)^{-1}\ (u-\eta)^{-1} \\
&= \frac{1}{(z_{13}z_{24})^{1/2}} \left[\ A_1 \left(\frac{1-\eta}{\eta}\right)^{1/2} + A_2 \left(\frac{\eta}{1-\eta}\right)^{1/2} + A_3 \left(\frac{1}{\eta(1-\eta)}\right)^{1/2}\ \right] .
\end{aligned}
\tag{4.15}
$$

In this expression, we have used the $SL(2,\mathbf{C})$ invariance to map the four points $(z_1, z_2, z_3, z_4) \to (0, \eta, 1, \infty)$, with $\eta = z_{12}z_{34}/(z_{13}z_{24})$, and $z_{ij} \equiv (z_i - z_j)$. The correlator (4.15) contains two independent blocks for the isospin channels $s = 0, 1$, which are added with coefficients A_1 and A_2; the third term in the last line of (4.15) is actually a linear combination of the first two. Note also that the full correlator also contains the factor $\prod_{i<j}\ z_{ij}^{p+1/2}$ arising from the charged part.

The presence of two independent terms in the correlator (4.15) should be interpreted as a "dynamical" degeneracy of the four-electron state. Under the exchange of two electrons, those two terms transform linearly into themselves. This is the notion of non-Abelian statistics: the exchange operation does not simply map each state into itself up to a phase, but acts on the set of degenerate states by a multi-dimensional unitary transformation. Let us compute it explicitly: putting all terms together, the four-point function becomes,

$$
\begin{aligned}
G(z_i) \times \prod_{i<j}\ z_{ij}^{p+1/2} &= \prod_{i<j}\ z_{ij}^{p+1}\ \left[\ A_1\ \mathcal{F}_1(z_i)\ +\ A_2\ \mathcal{F}_2(z_i)\ \right] , \\
\mathcal{F}_1(z_i) &= \frac{1}{z_{13}z_{24}z_{12}z_{34}} , \qquad \mathcal{F}_2(z_i) = \frac{1}{z_{13}z_{14}z_{24}z_{23}} .
\end{aligned}
\tag{4.16}
$$

For example, the exchange of the (1) and (2) electrons, $(z_1 - z_2) \to \mathrm{e}^{i\pi}(z_1 - z_2)$ leads to the following transformation: the prefactor yields the usual minus sign for Fermi statistics ($(p + 1)$ is odd), while the two blocks transform as follows,

$$
\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} .
\tag{4.17}
$$

The hierarchical Hall states have been commonly considered to posses electrons and quasi-particles with Abelian statistics [3]; this is in fact true if one adopts the description by the multi-component Abelian CFTs [8], where the quasi-particles are associated to $\widehat{U(1)}$ representations and have the usual vertex-operator correlations. On the other hand, the ($c=2$) $W_{1+\infty}$ minimal models associate the quasi-particles to the Virasoro representations, which have assigned an $SU(2)$ isospin quantum number, leading to non-Abelian statistics (and, correspondingly, an $SU(m)$ weight for $\nu = m/(mp \pm 1)$). These two different descriptions of the quasi-particles can be tested in future experiments involving interference and scattering processes, which measure the three and higher point functions.

Next, we remark that the non-Abelian statistics (4.17) occurring in the $W_{1+\infty}$ minimal models is rather "mild", i.e. it is not due to strong singularities of the correlators; actually, it has a simple *a-posteriori* explanation. In the two-component Abelian CFT, there are two electron fields, one for each component, which are represented by the vertex operators with $m=1/2$ and $m=-1/2$, respectively. On the contrary, in the minimal models there is a single electron excitation represented by the $s=1/2$ Virasoro primary field: in the Coulomb Gas approach, this field can be written as either vertex operators with $m=\pm 1/2$, due to the reflection symmetry $m \to -m$. Moreover, in the electron correlators the charge neutrality can be satisfied without adding any screening charge: for example, the two conformal blocks in (4.16) can also be written as follows:

$$\begin{aligned} \mathcal{F}_1 \quad &\propto \quad \langle\, \Omega \mid :\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_1)}::\mathrm{e}^{\frac{i}{\sqrt{2}}\varphi(z_2)}::\mathrm{e}^{\frac{i}{\sqrt{2}}\varphi(z_3)}::\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_4)}: \mid \Omega \,\rangle \ , \\ \mathcal{F}_2 \quad &\propto \quad \langle\, \Omega \mid :\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_1)}::\mathrm{e}^{-\frac{i}{\sqrt{2}}\varphi(z_2)}::\mathrm{e}^{\frac{i}{\sqrt{2}}\varphi(z_3)}::\mathrm{e}^{\frac{i}{\sqrt{2}}\varphi(z_4)}: \mid \Omega \,\rangle \ . \end{aligned} \qquad (4.18)$$

This alternative construction is obtained by taking all possible sign choices for $m_i = \pm 1/2$ having vanishing sum. Therefore, the electrons are represented as in the two-component Abelian theory, but their components, i.e. signs, are not observable and should be summed up. In conclusion, the non-Abelian statistics of the electrons simply amounts to this degeneracy of the many-electron states. Clearly, this property deserves further analysis before it can be physically observed: most notably, one should propose a probe which couples to the non-trivial neutral excitations.

The non-Abelian statistics has been also investigated in the Pfaffian and Haldane-Rezayi $\nu = 1/2$ Hall states of paired electrons [27]. Their quasi-particles are non-Abelian, but their electrons are still Abelian: actually, for these states there is a clear relation between the (bulk) wave-functions and the CFT correlators on the edge, which forces the electrons to obey the usual Fermionic Abelian statistics. On the contrary, in the $W_{1+\infty}$ models, the electrons are also non-Abelian ($s=1/2$); actually, the relation with wave functions is not immediate: we have compared the correlator (4.16) with the Jain composite-fermion wave functions, which have been recently written in the first Landau level [28]. We have found that these wave functions do not match any conformal correlator, because they are not $SL(2,\mathbf{C})$ covariant and then cannot be written as correlators of Virasoro primary fields. Presumably, the electrons are described by different fields in the bulk and in the edge. Therefore, the result (4.17) for the non-Abelian statistics of the electron excitations at the edge is consistent with the experimental and theoretical results known at present.

5 Conclusions

In this work, we have pursued the investigation of the $W_{1+\infty}$ minimal models, which were previously proposed as the conformal theories of the hierarchical Hall states [9]. We have explicitly formulated these theories in a bosonic Fock space with either a set of constraints or a relevant Hamiltonian. We have presented a detailed analysis of their physical properties and we have compared them with the other, multi-component Abelian theories [8]. We have relied upon several known methods in conformal field theory, like the Hamiltonian reduction, the Coulomb Gas approach and the spinon excitations. This explicit formulation of the $W_{1+\infty}$ minimal models has already been useful for interpreting the numerical analysis of the low-lying electron spectrum on a disk geometry of Ref.[15].

Moreover, the Hamiltonian formulation of Section 2 suggests the possibility of further investigations: for example, it would be interesting to describe the $W_{1+\infty}$ minimal models by means of the Chern-Simons field theory, along the lines of the well-known relation between edge degrees of freedom and bulk gauge fields [8]; this description may reveal interesting properties of the bulk states which support minimal edge excitations. Another interesting result is the computation of correlation functions in Section 3: this will turn out to be important for searching characteristic signals of the $W_{1+\infty}$ minimal models in two recently proposed experiments: the multi-point interferometer of Ref.[29] and the detection of the Andreev reflection [30].

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A Properties of the $c = 3$ $W_{1+\infty}$ Minimal Models

In this Appendix, we describe the $c = 3$ $W_{1+\infty}$ minimal models appropriate for filling fractions $\nu = 3/(3p+1)$ $(p = 2, 4, \dots)$ following closely the presentation of Sections 2 and 3. Our aim is to describe the reduction in the number of degrees of freedom from the $\widehat{U(1)} \times \widehat{SU(3)}_1$ conformal theories [8] to the $c = 3$ $W_{1+\infty}$ minimal models with symmetry algebra $\widehat{U(1)} \times \mathcal{W}_3$ [9]. We first rewrite the spectrum of quasi-particle charge Q and quantum statistics $\theta/\pi = 2h$ of the $\widehat{U(1)} \times \widehat{SU(3)}_1$ theories in a basis which factorizes the $\widehat{U(1)}$ part. Next, we analyze the neutral spectrum according to the representations of the nested algebras $\widehat{SU(3)}_1 \supset \widehat{U(1)}^2 \supset \mathcal{W}_3$.

The $\widehat{U(1)} \times \widehat{SU(3)}_1$ spectrum in the basis of Ref. [8] is:

$$
\begin{aligned}
Q &= \frac{n_1 + n_2 + n_3}{3p+1} , & & n_1, n_2, n_3 \in \mathbf{Z} , \\
h &= \frac{1}{2}\left(n_1^2 + n_2^2 + n_3^2\right) - \frac{p}{2(3p+1)}\left(n_1 + n_2 + n_3\right)^2 . & & \qquad (A.1)
\end{aligned}
$$

The integers n_i span the three-dimensional lattice Γ, those vectors have norm h, the total Virasoro dimension; each point of the lattice identifies a $\widehat{U(1)}^3$ highest-weight representation in this theory. A two-dimensional sub-lattice is the $SU(3)$ weight lattice $\mathbf{P}$; this can be generated by the fundamental weights $\vec{\Lambda}^{(i)}$, $i = 1, 2$, which are dual to the simple positive roots $\vec{\alpha}_{(j)}$. A standard basis is [31]:

$$
\begin{aligned}
\vec{\Lambda} &= \ell_1\, \vec{\Lambda}^{(1)} + \ell_2\, \vec{\Lambda}^{(2)} \quad \in \mathbf{P} , \\
\vec{\Lambda}^{(1)} &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\right) , \qquad \vec{\Lambda}^{(2)} = \left(0, \frac{2}{\sqrt{6}}\right) , \\
\vec{\alpha}_{(1)} &= \left(\sqrt{2}, 0\right) , \qquad \vec{\alpha}_{(2)} = \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right) ; \qquad (A.2)
\end{aligned}
$$

they satisfy $\vec{\alpha}_{(i)} \cdot \vec{\Lambda}^{(j)} = \delta_i^j$, $\left\|\vec{\alpha}_{(i)}\right\| = 2$ and $\left\|\vec{\Lambda}^{(i)}\right\| = 2/3$, with $i, j = 1, 2$,

We recall that each $SU(3)$ Lie-algebra representation can be characterized by a positive highest weight $\vec{\Lambda} \in \mathbf{P}^+$, i.e. by the pair (ℓ_1, ℓ_2) with $\ell_1, \ell_2 \geq 0$. Other copies of the representations have $\ell_i \in \mathbf{Z}$ and can be obtained by acting on $\mathbf{P}^+$ with the Weyl group. The triality of the (ℓ_1, ℓ_2) representation is given by:

$$
\alpha = \ell_1 - \ell_2 \mod 3 , \quad \alpha = 0, \pm 1 . \qquad (A.3)
$$

We denote with $\mathbf{P}_\alpha$ the three sub-lattices of $\mathbf{P}$ with given triality (each one is equivalent to the root lattice up to a translation).

Form the construction of the lattice Γ in Ref. [8][9], it is apparent that its factorization into $\mathbf{P}$ and the one-dimensional charge lattice can only be done within each α-sector; moreover, the integers in the two lattices are related by $\ell_1 = n_1 - n_2$ and $\ell_2 = n_2 - n_3$. Therefore, we consider the α-dependent change of basis:

$$
\begin{array}{lll}
\alpha = 0 \ : & & \\
3\ell = n_1 + n_2 + n_3 \ , & \ell_1 = n_1 - n_2 = 2k_1 - k_2 \ , & \ell_2 = n_2 - n_3 = 2k_2 - k_1 \ ; \\
\alpha = 1 \ : & & \\
3\ell + 1 = n_1 + n_2 + n_3 \ , & \ell_1 = n_1 - n_2 = 2k_1 - k_2 + 1 \ , & \ell_2 = n_2 - n_3 = 2k_2 - k_1 \ ; \\
\alpha = -1 \ : & & \\
3\ell - 1 = n_1 + n_2 + n_3 \ , & \ell_1 = n_1 - n_2 = 2k_1 - k_2 \ , & \ell_2 = n_2 - n_3 = 2k_2 - k_1 + 1 \ .
\end{array}
\tag{A.4}
$$

This rewrites the spectrum (A.1) in the form:

$$
\begin{aligned}
Q &= \frac{3\ell + \alpha}{3p + 1} \ , \\
h &= \frac{(3\ell + \alpha)^2}{6(3p+1)} \ + \ h_{\vec{\Lambda}} \ ,
\end{aligned}
\tag{A.5}
$$

where $h_{\vec{\Lambda}}$ is the dimension of the neutral $\widehat{U(1)}^2$ representations in the lattice $\mathbf{P}$:

$$
h_{\vec{\Lambda}} = \frac{1}{2} \left\| \vec{\Lambda} \right\|^2 = \frac{1}{3} \ \left(\ell_1^2 + \ell_2^2 + \ell_1 \ell_2 \right) \ = \left\{ \begin{array}{ll} k_1^2 + k_2^2 - k_1 k_2 \ , & \alpha = 0 \\ k_1^2 + k_2^2 - k_1 k_2 + k_1 + \frac{1}{3} \ , & \alpha = \pm 1 \ . \end{array} \right.
\tag{A.6}
$$

The factorization of Γ into charged and neutral orthogonal sub-lattices is achieved in (A.5): the former is parametrized by $\ell \in \mathbf{Z}$; the latter by $\ell_1, \ell_2 \in \mathbf{Z}$ (lattice $\mathbf{P}$), or alternatively by $k_1, k_2 \in \mathbf{Z}$ in each sector $\mathbf{P}_\alpha$.

Form now on, we discuss the neutral spectrum only. We want to analyze the decompositions of its representations with respect to the algebras $\widehat{SU(3)}_1 \supset \widehat{U(1)}^2 \supset \mathcal{W}_3$. The $\widehat{SU(3)}_1$ algebra has three (integrable) representations labelled by $\alpha = 0, \pm 1$ [6]; each of these decomposes: (i) into the sum of the $\widehat{U(1)}^2$ representations of the lattice $\mathbf{P}_\alpha$, Eqs.(A.5),(A.8), each with multiplicity one; (ii) into the sum of the $\mathcal{W}_3$ representations of weights $\vec{\Lambda} \in \mathbf{P}_\alpha^+$, each with multiplicity given by the dimension of the corresponding $SU(3)$ representation . The latter decomposition can be written $\widehat{SU(3)}_1 \sim SU(3) \times \mathcal{W}_3$: actually, the $SU(3)$ generators J_0^a, $a = 1, \ldots, 8$, commute with the $\mathcal{W}_3$ ones, L_n^S and $\mathbf{W}_n$, which are the modes of the Virasoro and spin-three currents, respectively; this factorization is completely analogous to the $\widehat{SU(2)}_1$ case of Section 2.

These decompositions of representations can be checked by computing the corresponding characters. The $\widehat{SU(3)}_1$ characters are found in Ref. [21]: they involve the sum over the sub-lattice $M_\alpha = \left\{\vec{\Lambda} | \vec{\Lambda} = \vec{\gamma}_o + 4\vec{\gamma}\right\}$, with $\gamma_o = \left\{0, \vec{\Lambda}^{(1)}, \vec{\Lambda}^{(2)}\right\}$, for $\alpha = \{0, 1, -1\}$, respectively, and $\vec{\gamma}$ a vector of the root lattice. Their explicit form is:

$$\chi_\alpha^{\widehat{SU(3)}_1} = \frac{1}{\eta(q)^8} \sum_{(a,b)=(i_1,i_2)+4(2k_1-k_2,2k_2-k_1)} d(a-1, b-1)\; q^{(a^2+b^2+ab)/12} , \qquad (A.7)$$

with

$$\begin{aligned} (i_1, i_2) &= \{(1,1),(2,1),(1,2)\} \ , \quad \text{resp. for } \alpha = \{0,1,-1\} \ , \qquad \text{and } \ k_1, k_2 \in \mathbf{Z}; \\ d_{\vec{\Lambda}} &\equiv d(\ell_1, \ell_2) = \frac{1}{2}(1+\ell_1)(1+\ell_2)(2+\ell_1+\ell_2) \ . \qquad (A.8) \end{aligned}$$

In the last equation, $d_{\vec{\Lambda}}$ is the dimension of the $SU(3)$ representation $(\ell_1, \ell_2) \in \mathbf{P}^+$ (here extended to all $\mathbf{P}$).

In order to verify the first of the decompositions above, these characters are compared with those obtained by summing the $\widehat{U(1)}^2$ representations in each sector $\mathbf{P}_\alpha$ of the quasi-particle spectrum (A.5), (A.6). Using the form (2.9) of the $\widehat{U(1)}$ character, we obtain the expressions:

$$\begin{aligned} \chi_{\alpha=0}^{\widehat{SU(3)}_1} &= \frac{1}{\eta(q)^2} \sum_{k_1,k_2 \in \mathbf{Z}} q^{k_1^2+k_2^2-k_1k_2} \ , \\ \chi_{\alpha=\pm 1}^{\widehat{SU(3)}_1} &= \frac{1}{\eta(q)^2} \sum_{k_1,k_2 \in \mathbf{Z}} q^{k_1^2+k_2^2-k_1k_2+k_1+1/3} \ . \qquad (A.9) \end{aligned}$$

The expressions (A.7) and (A.9) are actually equivalent, due to non-trivial Jacobi-like identities; these can be checked by expanding both expressions in series of q with the help of Mathematica [32]. This completes the analysis of the representation content of the $c = 3$ Abelian CFT with symmetry $\widehat{U(1)} \times \widehat{SU(3)}_1$.

The second, more relevant, decomposition is $\widehat{SU(3)}_1 \sim \mathcal{W}_3 \times SU(3)$: we need the $c = 2$ $\mathcal{W}_3$ characters, which are obtained from Ref.[13]. They read:

$$\chi_{\mathbf{\Lambda}}^{\mathcal{W}_3} = \chi_{\ell_1,\ell_2}^{\mathcal{W}_3} = \frac{1}{\eta(q)^2}\, q^{(\ell_1^2+\ell_2^2+\ell_1\ell_2)/3} \left(1 - q^{\ell_1+1}\right) \left(1 - q^{\ell_2+1}\right) \left(1 - q^{\ell_1+\ell_2+2}\right) . \qquad (A.10)$$

Actually, the $c = 3$ degenerate $W_{1+\infty}$ representations [13] of weights $\vec{r} = \{s + n_1, s + n_2, s + n_3\}$, with $s \in \mathbf{R}$ and $n_1 \geq n_2 \geq n_3$, are equivalent to the $\widehat{U(1)} \times \mathcal{W}_3$ representations with $\ell_1 = n_1 - n_2$ and $\ell_2 = n_2 - n_3$ [9]. Note that these $\mathcal{W}_3$ characters are similar to the degenerate Virasoro ones (2.14), in the sense that they are again linear combinations, with alternating signs, of the corresponding Abelian characters. In Eq. (A.10), there are six of them, which sit at the points of an hexagon of the root lattice: actually, we can rewrite this character, e.g. for $\alpha = 0$,

$$\chi_{2k_1-k_2,2k_2-k_1}^{\mathcal{W}_3} = \chi_{k_1,k_2}^{\widehat{U(1)}^2} - \chi_{k_1+1,k_2}^{\widehat{U(1)}^2} + \chi_{k_1+2,k_2+1}^{\widehat{U(1)}^2} - \chi_{k_1+2,k_2+2}^{\widehat{U(1)}^2} + \chi_{k_1+1,k_2+2}^{\widehat{U(1)}^2} - \chi_{k_1,k_2+1}^{\widehat{U(1)}^2} . \quad (A.11)$$

This describes the decomposition of representations for $\widehat{U(1)}^2 \supset \mathcal{W}_3$ and shows the existence of $\mathcal{W}_3$ null vectors.

According to the equivalence $\widehat{SU(3)}_1 \sim \mathcal{W}_3 \times SU(3)$, the character $\chi_\alpha^{\widehat{SU(3)}_1}$ (A.7) should be reproduced by summing over the $\mathcal{W}_3$ ones with weights $\vec{\Lambda} \in \mathbf{P}^+_\alpha$ and multiplicities equal to the dimensions of the corresponding $SU(3)$ multiplets (A.8). These sums are written:

$$
\begin{aligned}
\chi_{\alpha=0}^{\widehat{SU(3)}_1} &= \sum_{k_1=0}^{\infty} \sum_{2k_1\ge k_2\ge k_1/2} d(2k_1-k_2, 2k_2-k_1)\ \chi^{\mathcal{W}_3}_{2k_1-k_2,2k_2-k_1}\ , \\
\chi_{\alpha=1}^{\widehat{SU(3)}_1} &= \sum_{k_1=0}^{\infty} \sum_{2k_1+1\ge k_2\ge k_1/2} d(2k_1-k_2+1, 2k_2-k_1)\ \chi^{\mathcal{W}_3}_{2k_1-k_2+1,2k_2-k_1}\ , \\
\chi_{\alpha=-1}^{\widehat{SU(3)}_1} &= \sum_{k_1=0}^{\infty} \sum_{2k_1\ge k_2\ge (k_1-1)/2} d(2k_1-k_2, 2k_2-k_1+1)\ \chi^{\mathcal{W}_3}_{2k_1-k_2,2k_2-k_1+1}\ ,
\end{aligned}
\tag{A.12}
$$

where the ranges for (k_1, k_2) are obtained from $\ell_1, \ell_2 \ge 0$. The identities (A.12) are again checked by power expansion in q. Therefore, we have proven the expected decomposition of $\widehat{SU(3)}_1$ into $\mathcal{W}_3$: note the complete analogy with the $\widehat{SU(2)}_1$ case (2.16) illustrated in Fig. (1).

The Hamiltonian reduction from the $\widehat{U(1)} \times \widehat{SU(3)}_1$ Abelian theory to the $c = 3$ $W_{1+\infty}$ minimal model is then obtained by keeping one state per $SU(3)$ multiplet; this is the $SU(3)$ lowest-weight state, which is annihilated by the $SU(3)$ shift operators [31]:

$$
\begin{aligned}
E_0^{-\vec{\alpha}_{(1)}}\ |\text{minimal state}\rangle &= 0\ , \\
E_0^{-\vec{\alpha}_{(2)}}\ |\text{minimal state}\rangle &= 0\ .
\end{aligned}
\tag{A.13}
$$

These two constraints define the $c = 3$ $W_{1+\infty}$ minimal models in the Abelian Fock space, in analogy with (2.21). Note that the third negative root $\vec{\alpha}_{(1)} + \vec{\alpha}_{(2)}$ does not give an independent condition because $\left[E_0^{-\vec{\alpha}_{(1)}}, E_0^{-\vec{\alpha}_{(2)}}\right] = -E_0^{-\vec{\alpha}_{(1)}-\vec{\alpha}_{(2)}}$.

The equivalent Hamiltonian formulation of the $c = 3$ $W_{1+\infty}$ minimal models is obtained by adding the following relevant term to the Hamiltonian:

$$
H^{(2)} = \gamma \left(E_0^{\vec{\alpha}_{(1)}}\ E_0^{-\vec{\alpha}_{(1)}} + E_0^{\vec{\alpha}_{(2)}}\ E_0^{-\vec{\alpha}_{(2)}} \right)\ , \tag{A.14}
$$

and by considering the infrared limit $\gamma \to \infty$ of the corresponding renormalization-group trajectory.

Finally, we present the Fermionic realization of the $c = 3$ $W_{1+\infty}$ minimal models that parallels that of Section 3. Consider three Weyl chiral fields Ψ_i, $i = 1, 2, 3$.

We can realize the $\widehat{U(1)} \times \widehat{SU(3)}_1$ theory in terms of fermion bilinears $:\overline{\Psi}_i O^{ij} \Psi_j:$, where the matrices O^{ij} represent the fundamental of $SU(3)$. The Cartan sub-algebra contain the charge current,

$$J = \left(\overline{\Psi}_1 \Psi_1 + \overline{\Psi}_2 \Psi_2 + \overline{\Psi}_3 \Psi_3 \right) \ , \tag{A.15}$$

and the neutral currents,

$$H_1 = \frac{1}{\sqrt{2}} \left(\overline{\Psi}_1 \Psi_1 - \overline{\Psi}_2 \Psi_2 \right) \ , \qquad H_2 = \frac{1}{\sqrt{6}} \left(\overline{\Psi}_1 \Psi_1 + \overline{\Psi}_2 \Psi_2 - 2 \ \overline{\Psi}_3 \Psi_3 \right) \ . \tag{A.16}$$

The shift currents are:

$$E^{\vec{\alpha}(1)} = \overline{\Psi}_1 \Psi_2 \ , \qquad E^{\vec{\alpha}(2)} = \overline{\Psi}_2 \Psi_3 \ , \qquad E^{\vec{\alpha}(1) + \vec{\alpha}(2)} = \overline{\Psi}_1 \Psi_2 \ , \tag{A.17}$$

together with their Hermitian conjugates. By construction, the zero-modes of these currents satisfy the standard commutation relations of the $SU(3)$ Lie algebra in the Cartan basis [31]. As in Section 3, the constraints (A.13) can be written in the three-component Fermionic Fock space, and they imply that the states of the minimal-model are made by particle-hole excitations which are completely symmetric with respect to the three components. The constraints can be interpreted as further incompressibility conditions which eliminate the anti-symmetric excitations tangential to the Fermi surface.

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