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ON THE STRESS TENSOR OF CONFORMAL FIELD THEORIES IN HIGHER DIMENSIONS

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The behaviour of the stress tensor under conformal transformations of both flat and curved spaces is investigated for free theories in a classical background metric. In flat space $\mathbb{R}^d$ it is derived by the operator product expansion of two stress tensors. For Weyl transformations of curved manifolds it is given by the effective potential for the metric. In four dimensions the general form of the potential and its consistency conditions are analysed. These issues are relevant for the possible generalizations of the central charge in higher dimensions. The related subject of the Casimir effect is studied by means of closed expressions for the bosonic partition function on the manifolds $\mathbb{T}^d$ and $S^1 \times S^{d-1}$. The general relationship between the Casimir effect on $\mathbb{R} \times S^{d-1}$ and the trace anomaly is emphasized.

1. Introduction

The study of the conformal invariant field theory in two dimensions has led to rather remarkable results [1]. The theory is labelled by a number, the central charge c of the Virasoro algebra [2]; moreover for $0 < c < 1$ the requirements of unitarity [3] and existence of the partition function on a torus [4] completely determine the scaling dimensions and the multiplicities of the fields, leading to a classification of conformal theories [5]. Explicit realizations of these theories are provided by statistical models at the critical point; in their continuum formulation, they exhibit local scale invariance. This identification yields exact critical exponents and universality classes.

In this paper we investigate some of these methods in higher dimensions, having in mind their possible applications to critical phenomena. First of all, we must

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explain what we call a conformal transformation in higher dimensions. Actually, there are two approaches which coexist in two dimensions, but are rather different in higher dimensions.

In the first, more general approach, the field theory is defined on a manifold with classical background metric $g_{\mu\nu}$, by an euclidean action $S[g_{\mu\nu}\,\phi]$ of some fields ϕ. A conformal transformation is a local scale transformation of the metric

$$g_{\mu\nu}(x) = \mathrm{e}^{2\sigma(x)} g'_{\mu\nu}(x), \tag{1.1a}$$

in which the coordinates are left fixed; this is also called a Weyl transformation [6]. The "elementary" fields transform according to their scaling dimension Δ, independently of their spin

$$\phi(x) = \mathrm{e}^{-\Delta\sigma(x)}\phi'(x), \tag{1.1b}$$

The variation of the action under an infinitesimal change of the metric defines the stress–energy tensor $\mathrm{T}_{\mu\nu}$

$$\begin{aligned}\delta S[g^{\mu\nu},\phi] &= S[g^{\mu\nu}+\delta g^{\mu\nu},\phi+\delta\phi] - S[g^{\mu\nu},\phi]\\ &= -\frac{1}{2S_d}\int \mathrm{d}^d x\sqrt{g}\,\mathrm{T}_{\mu\nu}\delta g^{\mu\nu},\end{aligned} \tag{1.2}$$

which we normalize by including the factor $S_d = 2\pi^{d/2}/\Gamma(d/2)$, the area of the unit sphere S^{d-1} in $\mathbb{R}^d$. It follows that theories invariant under local transformations (1.1) have a traceless stress–energy tensor at the classical level. Moreover, reparametrization invariance,

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{1.3a}$$

$$\delta g^{\mu\nu} = \mathrm{D}^\mu\xi^\nu + \mathrm{D}^\nu\xi^\mu, \tag{1.3b}$$

leads to covariant conservation of $\mathrm{T}_{\mu\nu}$, $\mathrm{D}^\mu\mathrm{T}_{\mu\nu} = 0$. The quantization of the field ϕ must preserve this property in order to have consistent results in different coordinate patches. Then, the former invariance is in general broken, leading to the conformal (trace) anomaly [7]. This is, in two dimensions

$$\langle \mathrm{T}_\mu^{\ \mu}\rangle = -\tfrac{1}{12}cR, \tag{1.4}$$

where R is the scalar curvature*.

* In eq. (1.4) we use the notation of ref. [8] for R and we omit a constant term, because we set $\langle \mathrm{T}_{\mu\nu}\rangle \equiv 0$ in flat space.

The second, more familiar approach, sets the theory in flat space and consider local coordinate transformations (1.3), possibly singular at some points, which scale the metric as in eq. (1.1a). This implies the conditions

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{d}(\partial \cdot \xi)\,\delta_{\mu\nu}\,. \tag{1.5}$$

The infinitesimal decomposition

$$\begin{aligned}\frac{\mathrm{d}x'^\nu}{\mathrm{d}x^\mu} &= \left[1 + \frac{\partial \cdot \xi}{d}\right]\left[\delta^\nu_\alpha + \tfrac{1}{2}(\partial_\alpha \xi^\nu - \partial^\nu \xi_\alpha)\right]\left[\left(1 - \frac{\partial \cdot \xi}{d}\right)\delta^\alpha_\mu + \tfrac{1}{2}(\partial^\alpha \xi_\mu + \partial_\mu \xi^\alpha)\right]\\ &= (1 - \sigma(x))R^\nu_\alpha(x) S^\alpha_\mu(x)\,,\end{aligned} \tag{1.6}$$

shows that these transformations are made of a local dilatation $\sigma(x)$ and rotation $R^\nu_\alpha(x)$, the shear part being absent, $S^\alpha_\mu = \delta^\alpha_\mu$.

As a consequence, the fields are scaled and also displaced and rotated. In any dimension d there are fields which transform homogeneously, for example

$$\begin{aligned}\varphi(x) &\to \exp(-\Delta\sigma(x))\varphi(x')\,,\\ A_\mu(x) &\to \exp(-(\Delta - 1)\sigma(x))\frac{\mathrm{d}x'^\nu}{\mathrm{d}x^\mu}A_\nu(x')\,,\\ B_{\mu\nu}(x) &\to \exp(-(\Delta - 2)\sigma(x))\frac{\mathrm{d}x'^\alpha}{\mathrm{d}x^\mu}\frac{\mathrm{d}x'^\beta}{\mathrm{d}x^\nu}B_{\alpha\beta}(x')\,,\end{aligned} \tag{1.7}$$

for dimension Δ and spin zero, one and two, respectively; $\exp(d\sigma) = \det(\partial x/\partial x')$ is the Jacobian. In the Weyl-invariant theory this corresponds to a Weyl transformation composed with a reparametrization giving back the flat metric. The infinitesimal form is

$$\begin{aligned}\varphi(x) &\to \varphi(x) + \delta\varphi(x) = \varphi(x) + \left[\frac{\Delta}{d}(\partial \cdot \xi) + (\xi \cdot \partial)\right]\varphi(x)\,,\\ A_\mu(x) &\to A_\mu(x) + \delta A_\mu(x) = A_\mu(x) + \left[\frac{\Delta}{d}(\partial \cdot \xi) + (\xi \cdot \partial)\right]A_\mu(x)\\ &\qquad + \tfrac{1}{2}(\partial_\mu \xi^\nu - \partial^\nu \xi_\mu)A_\nu(x)\,,\\ B_{\mu\nu}(x) &\to B_{\mu\nu}(x) + \delta B_{\mu\nu}(x) = B_{\mu\nu}(x) + \left[\frac{\Delta}{d}(\partial \cdot \xi) + (\xi \cdot \partial)\right]B_{\mu\nu}(x)\\ &\qquad + \tfrac{1}{2}(\partial_\mu \xi^\alpha - \partial^\alpha \xi_\mu)B_{\alpha\nu}(x) + \tfrac{1}{2}(\partial_\nu \xi^\beta - \partial^\beta \xi_\nu)B_{\mu\beta}(x)\,.\end{aligned} \tag{1.8}$$

The Ward identities follow by inserting the two sides of eq. (1.8) in correlation functions averaged with action S and $S+\delta S$, respectively.

In two dimensions, the conditions (1.5) are solved by analytic reparametrizations $\xi(z)$ of $z=x^1+ix^2$ and antianalytic $\bar{\xi}(\bar{z})$ of $\bar{z}=x^1-ix^2$; they form an infinite-dimensional algebra and given local Ward identities, which put stringent conditions on the dynamics of the theory [1]. The fields transforming according to eqs. (1.7) and (1.8) are called primary fields. Since the coordinate transformations yield an infinite subset of Weyl transformations of the metric, these two approaches are, from the physical point of view, very similar, because both test the theory for local scale transformations.

In higher dimension d, the conditions (1.5) leave only the finite algebra $\mathrm{so}(d+1,1)$ of translations, dilatations, rotations, and special conformal transformations

$$\xi^{\mu}(x)=b^{\mu}+\lambda x^{\mu}+\omega^{\mu}{}_{\nu}x^{\nu}+a^{\mu}x^2-2(a\cdot x)x^{\mu}, \tag{1.9}$$

where b^{μ}, λ, $\omega_{\mu\nu}=-\omega_{\nu\mu}$, a^{μ} are the respective parameters. The fields transforming homogeneously, eqs. (1.7) and (1.8), under transformations (1.9) are called quasi-primary. Conformal invariance in four-dimensional flat space was analyzed in detail in the literature [9–13], especially in the hamiltonian operatorial formalism. It gives less conditions than in the two-dimensional case.

Therefore, we are led to follow the first, more general, approach in higher dimension, and to consider theories classically invariant under the infinite algebra (1.1) in a curved space; in the following we shall speak of conformal invariance in this sense. These theories are invariant under coordinate transformations (1.9) in flat space, that we shall refer to as "regular" conformal transformations. In this paper we shall work out some explicit examples for the free bosonic and fermionic field theories, which already display non-trivial properties under conformal transformations. We shall also present results which are valid for a larger class of theories.

An important issue of conformal theories is the transformation of the stress tensor: in two dimensions it is, for $z\to z'=z+\xi(z)$, $|\xi|\ll 1$,

$$\begin{aligned}\mathrm{T}_{zz}(z)\to\mathrm{T}_{zz}(z)+\delta_{\xi}\mathrm{T}_{zz}(z)&=\left(1+2\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)\mathrm{T}_{zz}(z+\xi)+\tfrac{1}{12}c\frac{\mathrm{d}^3\xi}{\mathrm{d}z^3}\\ &=\mathrm{T}_{zz}+2\left(\frac{\mathrm{d}\xi}{\mathrm{d}z}\right)\mathrm{T}_{zz}+\xi\left(\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{T}_{zz}\right)+\tfrac{1}{12}c\frac{\mathrm{d}^3\xi}{\mathrm{d}z^3}.\end{aligned} \tag{1.10}$$

We see that eq. (1.10) contains the homogeneous terms for a primary field of spin and dimension two, eq. (1.8), and an anomalous term which gives the central extension in the Virasoro algebra. There is a one-to-one correspondence between the transformation (1.10) and the operator product expansion (OPE) [1]

$$T_{zz}(z)T_{zz}(w) \sim \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T_{zz}(w) + \frac{1}{z-w}\frac{\partial}{\partial w}T_{zz}(w)$$
$$+ \text{regular terms for } z \to w. \tag{1.11}$$

It is given by the Ward identity

$$\delta_\xi T_{zz}(w) = \oint_C \frac{dz}{2\pi i}\xi(z)T_{zz}(z)T_{zz}(w), \tag{1.12}$$

where C is a contour around the point w: the singular terms of eq. (1.11) determine the contour integral and give eq. (1.10); conversely one represents eq. (1.10) as a Cauchy integral and determines the singular terms of eq. (1.11) because $\xi(z)$ is an arbitrary analytic function. Actually, eq. (1.11) is a local form of the Ward identity (1.12).

In sect. 2, after introducing the free bosonic and fermionic theories invariant under eq. (1.1) and their stress tensors, we compute the OPE $T_{\mu\nu}(x)T_{\rho\sigma}(0)$ in flat space $\mathbb{R}^d$ by the Wick theorem. The $O(|x-y|^{-d})$ and $O(|x-y|^{-d+1})$ singularities determine the transformation of $T_{\mu\nu}$ under eq. (1.9) by means of an analogous Ward identity derived in ref. [13]. One verifies that $T_{\mu\nu}$ is a quasi-primary field, i.e. it transforms homogeneously as in eq. (1.8). However, in higher dimensions many different forms of the OPE are compatible with the transformation law, because $\xi^\mu(x)$ in eq. (1.9) is not an arbitrary function.

The leading singularity $O(|x-y|^{-2d})$ is a unique function determined up to a constant [10, 13], which can be considered as a generalization of the central charge; we shall give its value for the free field theories. One expects that $T_{\mu\nu}$ develops anomalous terms for more general transformations (1.1a), as in two dimensions, eq. (1.10); their relation to the OPE $T_{\mu\nu}T_{\rho\sigma}$, in particular to the leading singularity, is presently unrevealed.

On the other hand, these anomalous terms are determined from the trace anomaly. The effective potential for the conformal factor $\sigma(x)$ in eq. (1.1a) is considered

$$\Gamma\left[g_{\mu\nu}, g'_{\mu\nu} = e^{-2\sigma}g_{\mu\nu}\right] = -\log Z[g_{\mu\nu}] + \log Z[g'_{\mu\nu}], \tag{1.13}$$

where $Z[g_{\mu\nu}]$ is the partition function. This is obtained by integration of eq. (1.2) along a conformal transformation (1.1) and it involves the trace anomaly [1]. The variation of Γ with respect to $g_{\mu\nu}$ gives the expectation value $\langle T_{\mu\nu}\rangle$. In two dimensions, one gets for infinitesimal transformations

$$\langle T_{zz}(z)\rangle_{g_{\mu\nu}} = \langle T'_{zz}(z)\rangle_{g'_{\mu\nu}} - \tfrac{1}{6}c\frac{d^2\delta\sigma}{dz^2}. \tag{1.14}$$

The relation with eq. (1.10) follows by composing the conformal transformation with a reparametrization $z \to z' = z + \xi(z)$ giving back the flat metric; then $\delta\sigma = -\frac{1}{2}(d\xi/dz + d\bar{\xi}/d\bar{z})$ and $T'_{zz}(z) = (dz'/dz)^2 T'_{zz}(z')$. Under reparametrizations, $T_{\mu\nu}$ transforms without anomalous terms because we assumed the general covariance of the quantum theory. The method of the effective potential extends to higher dimension for conformal invariant theories; the first non-trivial case – four dimensions – will be discussed in detail by developing refs. [14, 15].

Besides their theoretical interest, the anomalous term in the transformation of $T_{\mu\nu}$ determines in two dimensions the vacuum energy E_0 – the Casimir effect – in the finite geometry of the strip $[0, T] \times S^1$, with S^1 of radius $\mathscr{R}$. In the limit $T \to \infty$, it is a cylinder and one has [16]

$$\lim_{T\to\infty} \frac{1}{T}\log Z = -E_0 = \frac{c}{12\mathscr{R}}. \tag{1.15}$$

(The volume and surface terms are subtracted in $\log Z$, in order to have $E_0 = 0$ for $\mathscr{R} \to \infty$.) Since the cylinder is related to flat space by a conformal transformation (1.1), eqs. (1.13) and (1.14) apply for $\langle T_{\mu\nu}\rangle$ and determine E_0.

In sect. 3, the partition function of the scalar field is obtained in closed form for two interesting manifolds which generalize the strip, the torus $\mathbb{T}^d = (S^1)^d$ and the manifold $S^1 \times S^{d-1}$. This yields the corresponding Casimir effects. The expressions of the partition functions may also be useful in the related subject of quantization of membranes [17].

In sect. 4, we apply the method of the effective potential, eqs. (1.13) and (1.14), to relate the trace anomaly to the Casimir effect on the cylinder $\mathbb{R} \times S^{d-1}$, which is conformally equivalent to flat space $\mathbb{R}^d$ [18, 19]. As a result, E_0 vanishes in odd dimensions, due to the absence of gravitational trace anomaly. In four dimensions, the effective potential is obtained for the general form of the gravitational trace anomaly. This yields the anomalous transformation of $\langle T_{\mu\nu}\rangle$ and the value of E_0 on $S^3 \times \mathbb{R}$ in terms of parameters in the trace anomaly. E_0 agrees with the direct calculation of sect. 3. Moreover, we show that the Wess–Zumino consistency condition [20] on the effective potential rules out one term in the trace anomaly for a conformal invariant theory.

In summary, three quantities are studied: the stress tensor OPE, the trace anomaly and the Casimir energy. They are related by the transformation properties of $T_{\mu\nu}$ and provide possible generalizations of the central charge in higher dimensions. The relation between the trace anomaly and the Casimir energy in any dimension is clarified; that between OPE and trace anomaly is left to future developments of this work.

2. Operator product expansion of the stress tensor

2.1. STRESS TENSOR OF FREE THEORIES

A real scalar field $\varphi(x)$ on a d-dimensional Riemannian manifold with metric $g_{\mu\nu}(x)$ has the action [6]

$$S[g_{\mu\nu},\varphi]=\tfrac{1}{2}\int \mathrm{d}^d x\sqrt{g}\left(g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi+\xi R\varphi^2\right). \tag{2.1}$$

For $\xi=(d-2)/4(d-1)$, it is invariant under conformal transformations (1.1) where $\Delta=\frac{1}{2}(d-2)$ is the scaling dimension of φ.

The stress-energy tensor is obtained by the variation of the action with respect to the metric, according to eq. (1.2). In the classical theory, the variation of the matter field φ can be neglected because it gives a contribution proportional to the equations of motion, while in the quantum theory we are not allowed to use them in general. Then, the correct expression of $\mathrm{T}_{\mu\nu}$ to be quantized later is obtained by varying both the metric and the field. Away from $d=2$, φ acquires a physical dimension and then it is sensitive to the local scale variation associated with $\delta g^{\mu\nu}$. By requiring the invariance of the dimensionless scalar $\varphi(\det g_{\mu\nu})^{\Delta/2d}$ we obtain the transformation law

$$\delta\varphi(x)=\frac{\Delta}{2d}g_{\mu\nu}(x)\,\delta g^{\mu\nu}(x)\varphi(x), \tag{2.2}$$

which generalizes eq. (1.1b). We use the variational formulae

$$\begin{aligned}\delta\sqrt{g}&=-\tfrac{1}{2}\sqrt{g}\,g_{\mu\nu}\,\delta g^{\mu\nu},\\ \delta R&=R_{\mu\nu}\,\delta g^{\mu\nu}+\left(g^{\lambda\sigma}g_{\mu\nu}-\delta^\lambda_\mu\delta^\sigma_\nu\right)\mathrm{D}_\lambda\,\mathrm{D}_\sigma\,\delta g^{\mu\nu},\end{aligned} \tag{2.3}$$

where D_λ is the covariant derivative, and we discard boundary terms in the integration by parts. The result is

$$\begin{aligned}\frac{\mathrm{T}_{\mu\nu}}{S_d}=&-\frac{d-2}{4(d-1)}\left(R_{\mu\nu}-\frac{1}{d}g_{\mu\nu}R\right)\varphi^2\\ &-\frac{1}{2(d-1)}\Bigg(d\,\partial_\mu\varphi\,\partial_\nu\varphi-g_{\mu\nu}g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi\\ &\qquad-(d-2)\varphi\,\mathrm{D}_\mu\,\mathrm{D}_\nu\varphi+\frac{d-2}{d}g_{\mu\nu}\varphi\,\mathrm{D}^2\varphi\Bigg),\end{aligned} \tag{2.4}$$

which agrees with ref. [6]. This tensor is symmetric, traceless without the use of the equations of motion and covariantly conserved with the use of them, as it should be. In the flat space $\mathbb{R}^d$ it reduces to

$$\frac{\mathrm{T}_{\mu\nu}}{S_d} = -\frac{d}{2(d-1)} P_{\mu\nu}^{\alpha\beta}\left(\partial_\alpha \varphi \, \partial_\beta \varphi - \frac{d-2}{d} \varphi \, \partial_\alpha \, \partial_\beta \varphi \right), \tag{2.5}$$

where $P_{\mu\nu}^{\alpha\beta} = \frac{1}{2}\delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{2}\delta_\mu^\beta \delta_\nu^\alpha - (1/d) g_{\mu\nu} g^{\alpha\beta}$ is the projector on the traceless symmetric part. Notice that the canonical stress tensor for the theory in flat space, defined by eq. (1.2), with eqs. (1.3) for $\delta g^{\mu\nu}$, is not necessarily traceless at the classical level. The invariance of the action under regular conformal transformations (1.9) requires only its trace to be a total divergence. An "improved" traceless tensor can be found and has to be used [21, 22]. On the contrary, the canonical $\mathrm{T}_{\mu\nu}$ for the conformal theory in curved space (2.1) is traceless and gives, in the flat-space limit, the improved one of ref. [22], modulo terms proportional to the equation of motion.

Similarly, we consider the euclidean Dirac fermions with action

$$S\left[e_\mu^a, \psi\right] = \tfrac{1}{2} \int \mathrm{d}^d x \sqrt{g} \left(\bar{\psi}(x) \gamma^\mu(x) \vec{\mathrm{D}}_\mu \psi(x) - \bar{\psi}(x) \overleftarrow{\mathrm{D}}_\mu \gamma^\mu(x) \psi(x) \right), \tag{2.6}$$

which is conformal invariant with scaling dimension $\Delta = \frac{1}{2}(d-1)$ of ψ. The fermionic action is a functional of the moving frame $e_\mu^a(x)$

$$E_a^\mu(x) e_\mu^b(x) = \delta_a^b ,$$

$$e_\mu^a(x) \eta_{ab} e_\nu^b(x) = g_{\mu\nu}(x) . \tag{2.7}$$

The covariant derivative acting on spinors contains the spin connection and the gamma matrices are $\gamma^\mu(x) = E_a^\mu(x)\gamma^a$ (see ref. [23] for details)*. $\mathrm{T}_{\mu\nu}$ is now defined by

$$\delta S = -\frac{1}{S_d} \int \sqrt{g} \, \mathrm{T}_{\mu\nu} E_a^\nu \, \delta E_b^\mu \eta^{ab} \tag{2.8}$$

As before, we vary the moving frame and the matter field, which scales and rotates, and obtain

$$\frac{\mathrm{T}_{\mu\nu}}{S_d} = -\tfrac{1}{2} P_{\mu\nu}^{\alpha\beta} \left(\bar{\psi} \gamma_\alpha \vec{\mathrm{D}}_\beta \psi - \bar{\psi} \gamma_\alpha \overleftarrow{\mathrm{D}}_\beta \psi \right), \tag{2.9}$$

* In odd dimension $d = 2k+1$ we take the Clifford algebra of $d = 2k$ and add $\gamma^{2k+1} = i^k \gamma^1 \dots \gamma^{2k}$ for the additional coordinate.

which in flat space reduces to

$$\frac{\mathrm{T}_{\mu\nu}}{S_d} = -\tfrac{1}{2}P^{\alpha\beta}_{\mu\nu}\left(\bar{\psi}\gamma_\alpha \overrightarrow{\partial}_\beta\psi - \bar{\psi}\gamma_\alpha \overleftarrow{\partial}_\beta\psi\right). \tag{2.10}$$

2.2. OPERATOR PRODUCT EXPANSIONS IN FLAT SPACE

Wick theorem allows to compute the operator product expansion (OPE) of two stress tensors for the previous theories. The classical expression for $\mathrm{T}_{\mu\nu}$ is replaced by the normal-ordered operator $:\mathrm{T}_{\mu\nu}:$, bilinear in the fields [24]. The contractions are

$$\langle\varphi(x)\varphi(y)\rangle = \frac{1}{S_d(d-2)}\frac{1}{|x-y|^{d-2}},$$

$$\langle\psi_a(x)\bar{\psi}_b(y)\rangle = (\gamma_\mu)_{ab}\frac{1}{S_d}\frac{(x^\mu-y^\mu)}{|x-y|^d}. \tag{2.11}$$

The calculations in this section are all lengthy but straightforward. The OPE is of the form

$$:\mathrm{T}_{\mu'\nu'}(x)::\mathrm{T}_{\rho'\sigma'}(0): = P^{\mu\nu}_{\mu'\nu'}P^{\rho\sigma}_{\rho'\sigma'}\left\{\frac{2c_d}{|x|^{2d}}(\delta_{\mu\rho}-2\hat{x}_\mu\hat{x}_\rho)(\delta_{\nu\sigma}-2\hat{x}_\nu\hat{x}_\sigma)\right.$$
$$\left. + \sum_{k=1}^{2(d-\Delta)}\frac{1}{|x|^k}\mathscr{A}^{(k)}_{\mu\nu\rho\sigma}(\hat{x}^\alpha;:\phi(0)\phi(0):,\dots) + \text{regular terms}\right\}, \tag{2.12}$$

where $\hat{x}^\mu = x^\mu/|x|$, $\mathscr{A}^{(k)}_{\mu\nu\rho\sigma}$ is a tensor of $\hat{x}^\alpha$ times bilinears of the field in the theory $:\phi(0)\phi(0):$ or derivatives of them evaluated at the point $x=0$, and the sum over k extends to $d+2$ $(d+1)$ for the scalar (fermion) theory.

The leading singularity is proportional to the identity operator and it is the unique contribution to the correlation $\langle\mathrm{T}_{\mu\nu}(x)\mathrm{T}_{\rho\sigma}(0)\rangle$ in flat space. Its form is completely determined by symmetry, tracelessness and conservation of $\mathrm{T}_{\mu\nu}$ [10,13]. The constant c_d is not arbitrary because the normalization of $\mathrm{T}_{\mu\nu}$ is fixed by the definitions eqs. (1.2) and (2.1,6). One finds

$$c_d = \frac{d}{2(d-1)}, \qquad \text{scalar field}, \tag{2.13a}$$

$$c_d = \frac{d}{4}2^{[d/2]}, \qquad \text{Dirac fermion}. \tag{2.13b}$$

The constant $2^{[d/2]}$ is the trace of unity in the Clifford space and $[d/2]$ is the integer part of $d/2$. Notice that c_d is a non-trivial constant times the number of field components. $c_2 = 1$ for a scalar field and a pair of Majorana fermions is recovered in two dimensions.

The form of the other terms in eq. (2.12) depends on the theory. Let us present them in the following form. For the scalar theory

$$
\begin{aligned}
&:\mathrm{T}_{\mu'\nu'}(x)::\mathrm{T}_{\rho'\sigma'}(0): \\
&\quad= \left\langle \mathrm{T}_{\mu'\nu'}(x)\mathrm{T}_{\rho'\sigma'}(0)\right\rangle + \left(\frac{d}{2(d-1)}\right)^2 S_d \frac{P^{\mu\nu}_{\mu'\nu'}P^{\rho\sigma}_{\rho'\sigma'}}{|x|^{d+2}} \\
&\qquad\times\Bigg\{\frac{(d-2)^2}{d}\left[2\delta_{\mu\rho}\delta_{\nu\sigma} + (d+2)\left(-4\,\delta_{\mu\rho}\hat{x}_\nu\hat{x}_\sigma + (d+4)\hat{x}_\mu\hat{x}_\nu\hat{x}_\rho\hat{x}_\sigma\right)\right] :\varphi(x)\varphi(0): \\
&\qquad\quad + |x| 2(d-2)\Big[\left(2\delta_{\mu\rho}\hat{x}_\nu - (d+2)\hat{x}_\rho\hat{x}_\mu\hat{x}_\nu\right) :\varphi(x)\varphi_\sigma(0): \\
&\qquad\qquad - \left(2\hat{x}_\rho\delta_{\mu\sigma} - (d+2)\hat{x}_\mu\hat{x}_\rho\hat{x}_\sigma\right) :\varphi_\nu(x)\varphi(0):\Big] \\
&\qquad\quad + |x|^2\Bigg[4\left(\delta_{\mu\rho} - d\hat{x}_\mu\hat{x}_\rho\right) :\varphi_\nu(x)\varphi_\sigma(0): \\
&\qquad\qquad + \frac{(d-2)^2}{d}\left(\hat{x}_\mu\hat{x}_\nu :(\varphi(x)\varphi_{\rho\sigma}(0): + \hat{x}_\rho\hat{x}_\sigma :\varphi_{\mu\nu}(x)\varphi(0):\right)\Bigg] \\
&\qquad\quad + |x|^3\frac{2(d-2)}{d}\left[\hat{x}_\mu :\varphi_\nu(x)\varphi_{\rho\sigma}(0): - \hat{x}_\rho :\varphi_{\mu\nu}(x)\varphi_\sigma(0):\right] \\
&\qquad\quad + |x|^4\frac{d-2}{d^2} :\varphi_{\mu\nu}(x)\varphi_{\rho\sigma}(0):\Bigg\},
\end{aligned}
\tag{2.14}
$$

where $\varphi_\nu = \partial_\nu\varphi$ and so on. The terms $\mathscr{A}^{(k)}_{\mu\nu\rho\sigma}$ of eq. (2.12) are obtained by expanding $\varphi(x)$ in powers of x. In the fermionic theory one has

$$
\begin{aligned}
&:\mathrm{T}_{\mu'\nu'}(x)::\mathrm{T}_{\rho'\sigma'}(0): = \left\langle \mathrm{T}_{\mu'\nu'}(x)\mathrm{T}_{\rho'\sigma'}(0)\right\rangle + \tfrac{1}{4}dS_d\frac{P^{\mu\nu}_{\mu'\nu'}P^{\rho\sigma}_{\rho'\sigma'}}{|x|^{d+1}} \\
&\qquad\Bigg\{\Bigg[-\left(\delta_{\nu\sigma}\hat{x}_\beta - (d+2)\hat{x}_\nu\hat{x}_\sigma\hat{x}_\beta\right) :\bar{\psi}(x)\gamma_\mu\gamma_\beta\gamma_\rho\psi(0): \\
&\qquad\quad + |x|\left(\hat{x}_\sigma\hat{x}_\beta :\partial_\nu\bar{\psi}(x)\gamma_\mu\gamma_\beta\gamma_\rho\psi(0): - \hat{x}_\nu\hat{x}_\beta :\bar{\psi}(x)\gamma_\mu\gamma_\beta\gamma_\rho\,\partial_\sigma\psi(0):\right) \\
&\qquad\quad - |x|^2\frac{1}{d}\hat{x}_\beta :\partial_\nu\bar{\psi}(x)\gamma_\mu\gamma_\beta\gamma_\rho\,\partial_\sigma\psi(0):\Bigg] + \text{conjugate}\Bigg\},
\end{aligned}
\tag{2.15}
$$

where the conjugation is, for example, $(:\bar{\psi}(x_1)\gamma_\mu\psi(x_2):)^\dagger = -:\bar{\psi}(x_2)\gamma_\mu\psi(x_1):$. Some terms in the expressions (2.14) and (2.15) combine to give $T_{\mu\nu}(0)$ and its derivatives, both others do not, that is the OPE $T_{\mu\nu}T_{\rho\sigma}$ does not close on $T_{\mu\nu}$ in more than two dimensions. The tensor coefficients multiplying the additional operators vanish for $d=2$ and eq. (1.11) is recovered; notice the identity

$$P^{\mu\nu}_{\mu'\nu'}P^{\rho\sigma}_{\rho'\sigma'}\left(\delta_{\mu\rho}-2\hat{x}_\mu\hat{x}_\rho\right)\delta_{\nu\sigma}=0 \qquad (d=2). \tag{2.16}$$

In refs. [13, 19], the following Ward identity was derived for regular conformal transformations (1.9) of an arbitrary operator $\mathcal{O}(x)$

$$\delta_\xi\mathcal{O}(x)=\int_{\Sigma_x}\mathrm{d}S^\mu(y)\xi^\nu(y)T_{\mu\nu}(y)\mathcal{O}(x). \tag{2.17}$$

The integral is done on a surface Σ_x surrounding the point x and it does not depend on its size due to conservation of $T_{\mu\nu}$ and eq. (1.5). For a sphere centered at x one has $\mathrm{d}S^\mu(y)=\mathrm{d}\Omega(y)(x-y)^\mu|x-y|^{d-2}$, normalized by $\int\mathrm{d}\Omega=1$.

The Ward identity gives us a relation between OPE (2.12) and the transformation $\delta_\xi T_{\mu\nu}$. One inserts eq. (2.12) into eq. (2.17) and observes that only terms which give an adimensional integrand can contribute: they are $\mathcal{A}^{(d-1)}$ for translations, $\mathcal{A}^{(d)}$ for dilatations and rotations, and $\mathcal{A}^{(d+1)}$, $\mathcal{A}^{(d)}$ and $\mathcal{A}^{(d-1)}$ for special conformal transformations. For example, consider the fermionic theory and a dilatation $\xi^\mu(y)=\lambda y^\mu$. Eq. (2.17) reads

$$\begin{aligned}
\delta_\lambda T_{\rho'\sigma'}(0) &= \int_{\Sigma_0}\mathrm{d}S^{\mu'}(y)\lambda y^{\nu'}T_{\mu'\nu'}(y)T_{\rho'\sigma'}(0)\\
&=\lambda P^{\rho\sigma}_{\rho'\sigma'}\int_{\Omega_0}\mathrm{d}\Omega\,\hat{y}^{\mu'}\hat{y}^{\nu'}P^{\mu\nu}_{\mu'\nu'}\mathcal{A}^{(d)}_{\mu\nu\rho\sigma}(\hat{y};:\bar{\psi}(0)\psi(0):)+\text{vanishing integrals}\\
&=\tfrac{1}{4}\lambda dS_d P^{\rho\sigma}_{\rho'\sigma'}\int_{\Omega_0}\mathrm{d}\Omega\left(\hat{y}^\mu\hat{y}^\nu-\frac{1}{d}\delta^{\mu\nu}\right)\\
&\quad\times\Big\{\Big[\hat{y}^\beta\hat{y}_\sigma:\partial_\nu\bar{\psi}(0)\gamma_\mu\gamma_\beta\gamma_\rho\psi(0):-\hat{y}^\beta\hat{y}_\nu:\bar{\psi}(0)\gamma_\mu\gamma_\beta\gamma_\rho\,\partial_\sigma\psi(0):\\
&\qquad-\left(\delta_{\nu\sigma}-(d+2)\hat{y}_\nu\hat{y}_\sigma\right)\hat{y}^\alpha\hat{y}^\beta:\partial_\alpha\bar{\psi}(0)\gamma_\mu\gamma_\beta\gamma_\rho\psi(0):\Big]+\text{conjugate}\Big\}\\
&=\lambda d\,T_{\rho'\sigma'}(0).
\end{aligned} \tag{2.18}$$

We see that we cannot understand the rôle of the other singular terms by looking only at conformal transformations of the flat space. The same happens in two dimensions if we limit to projective transformations of the plane : the leading singularity in eq. (1.11), which gives the anomalous term in the transformation (1.10), does not contribute. In sect. 4 we shall obtain the anomalous terms in higher dimensions by other methods. Their relation to the leading singularity in eq. (2.12) remains to be understood.

We continue the analysis of regular conformal transformations and the terms $\mathscr{A}^{(d+1)}$, $\mathscr{A}^{(d)}$ and $\mathscr{A}^{(d-1)}$. We verified that the terms $\mathscr{A}^{(d)}$ and $\mathscr{A}^{(d-1)}$ gives the correct transformation of $T_{\mu\nu}$ as a quasi-primary field, of dimension d and spin 2, eq. (1.8), for both theories (2.14) and (2.15). Notice that the term $\mathscr{A}^{(d+1)}$ has the dimension for contributing to special conformal transformations $\xi(y) \sim O((x-y)^2)$, but we checked it has a vanishing integral in our cases. This singularity must never contribute to the transformation of a quasi-primary field, which only feels the local rotation, dilatation and translation in a special conformal transformation by the terms $\mathscr{A}^{(d)}$ and $\mathscr{A}^{(d-1)}$ [13].

More generally, $\delta_\xi T_{\mu\nu}$ is homogeneous in $T_{\mu\nu}$, while the corresponding OPE terms contain additional operators whose tensor coefficients vanish in the integration (2.17). The possibility of such terms not contributing to $\delta_\xi T_{\mu\nu}$ implies that this part of the OPE depends on the details of the theory; at variance with two dimensions, eqs. (1.10) and (1.11), the OPE determines $\delta_\xi T_{\mu\nu}$ but the converse is not true.

Let us discuss in more detail the freedom in these OPE terms. For a scalar quasi-primary field φ, of dimension Δ, they have a unique form [13] : $\mathscr{A}_{\mu\nu}^{(d+1)} = 0$, $\mathscr{A}_{\mu\nu}^{(d-1)}$ and $\mathscr{A}_{\mu\nu}^{(d)}$ read

$$T_{\mu'\nu'}(x)\varphi(0) = \frac{d}{d-1} P^{\mu\nu}_{\mu'\nu'} \left\{ \Delta \frac{\hat{x}_\mu \hat{x}_\nu}{|x|^d} \varphi(0) + \frac{1}{|x|^{d-1}} \left[\hat{x}_\mu \delta_{\nu\lambda} + \tfrac{1}{2}(d-2) \hat{x}_\mu \hat{x}_\nu \hat{x}_\lambda \right] \partial_\lambda \varphi(0) \right\}$$

$$+ \text{less singular terms}. \tag{2.19}$$

They are determined a priori from the general form of tensors in $\hat{x}_\mu$ and $\delta_{\alpha\beta}$ by the constraints of symmetry, tracelessness and conservation of $T_{\mu\nu}$, and the compatibility with the transformation law eq. (1.8). In the case of the bosonic theory eq. (2.1) and φ the scalar field, eq. (2.19) is indeed verified.

The OPE $T_{\mu\nu}(x)A_\rho(0)$ with a quasi-primary vector field A_ρ is already non-unique. Let us suppose that it closes on A_μ for the same terms

$$T_{\mu\nu}(x)A_\rho(0) = \frac{1}{|x|^d} \mathscr{B}^{(d)}_{\mu\nu\rho\sigma}(\hat{x}) A_\sigma(0) + \frac{1}{|x|^{d-1}} \mathscr{B}^{(d-1)}_{\mu\nu\rho\sigma\lambda}(\hat{x}) \partial_\lambda A_\sigma(0)$$

$$+ \text{less and more singular terms}. \tag{2.20}$$

The previous constraints now leave one free parameter a for $\mathscr{B}^{(d)}$

$$\mathscr{B}^{(d)}_{\mu'\nu'\rho\sigma} = P^{\mu\nu}_{\mu'\nu'}\Big\{ d\big(\delta_{\mu\sigma}\hat{x}_\nu\hat{x}_\rho - \delta_{\mu\rho}\hat{x}_\nu\hat{x}_\sigma\big) + \frac{d\Delta}{d-1}\delta_{\rho\sigma}\hat{x}_\mu\hat{x}_\nu$$
$$+ a\Big[\tfrac{1}{2}(d^2-4)\hat{x}_\mu\hat{x}_\nu\hat{x}_\rho\hat{x}_\sigma + \delta_{\mu\rho}\hat{x}_\nu\hat{x}_\sigma + \delta_{\mu\sigma}\hat{x}_\nu\hat{x}_\rho - \tfrac{1}{2}d\,\delta_{\rho\sigma}\hat{x}_\mu\hat{x}_\nu - \delta_{\mu\rho}\,\delta_{\nu\sigma}\Big]\Big\}, \tag{2.21}$$

and three for $\mathscr{B}^{(d-1)}$. Moreover, if there are operators of different spin but same dimension as A_ρ, they may also appear in the OPE. For example, the fermionic current $A_\rho = \bar{\psi}\gamma_\rho\psi$ is a quasi-primary operator and it has the OPE

$$\mathrm{T}_{\mu'\nu'}(x)A_\rho(0) = \big\langle \mathrm{T}_{\mu'\nu'}(x)A_\rho(0)\big\rangle + \frac{1}{|x|^d}\mathscr{C}_{\mu\nu\rho\sigma}(\hat{x})A_\sigma(0)$$
$$+ \frac{1}{|x|^{d-1}}P^{\mu\nu}_{\mu'\nu'}\Big\{\Big[\tfrac{1}{2}(\mathrm{d}\hat{x}_\nu\hat{x}_\alpha + \delta_{\nu\alpha})\big(\hat{x}_\rho : \partial_\alpha\bar{\psi}(0)\gamma_\mu(0):$$
$$+ \tfrac{1}{2}\hat{x}_\beta : \partial_\alpha\bar{\psi}(0)\gamma_\mu[\gamma_\beta,\gamma_\rho]\psi(0):\big)\Big] - \text{conjugate}\Big\}$$
$$+ \text{less singular terms}, \tag{2.22}$$

where $\mathscr{C}_{\mu\nu\rho\sigma} = \mathscr{B}^{(d)}_{\mu\nu\rho\sigma}$ in eq. (2.21) with $a = 0$ and $\Delta = d-1$. This freedom in the OPE $\mathrm{T}_{\mu\nu}\mathscr{O}$ is reflected in the three-point correlation function $\langle \mathrm{T}_{\mu\nu}\mathscr{O}\mathscr{O}\rangle$, which is not completely fixed by so$(d+1,1)$ invariance [25], unless, for $\mathscr{O}$, a scalar field [11]. Ref. [26] contains examples of these correlations and also stronger statements on the OPE $\mathrm{T}_{\mu\nu}\mathrm{T}_{\rho\sigma}$ in four dimensions.

3. Partition functions and the Casimir effect

In this section we specialize to the scalar field (eq. (2.1)) and derive its partition function on two compact manifolds, the torus $\mathbb{T}^d \equiv (\mathrm{S}^1)^d$ and the manifold $\mathrm{S}^1 \times \mathrm{S}^{d-1}$. These two manifolds generalize the two-dimensional torus. In this geometry conformal invariance predicts the form of finite-size corrections [16, 27] and, in combination with modular invariance of the torus parametrization, it yields powerful means of investigation of conformal theories [4, 5]. In higher dimensions the torus maintains the property of modular invariance, while finite-size effects are predictable on the cylinder $\mathbb{R} \times \mathrm{S}^{d-1}$. We first present our results and postpone this discussion to sect. 4.

The partition function is given by a determinant which is regularized by the method of the zeta function [28]. The zeta function of a Laplace-type operator K with non-negative eigenvalues on a compact manifold M is defined as

$$\zeta_{\mathrm{M}}(s) = \sum_{\lambda>0} \frac{1}{\lambda^s}, \tag{3.1}$$

where the sum is over the positive eigenvalues of K (with their multiplicites). It is analytically continued to the complex s plane from a region $\mathrm{Re}(s) > \gamma > 0$ where it converges absolutely.

The partition function is defined by

$$\begin{aligned} Z &= \int_{\mathrm{M}} \mathscr{D}\varphi\, \mathrm{e}^{-\int \varphi K \varphi} \\ &= \mathscr{N}\Big(\prod_{\lambda>0} \lambda\Big)^{-1/2} \\ &= \mathscr{N} \exp\Big(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\mathrm{M}}(s)|_{s=0}\Big). \end{aligned} \tag{3.2}$$

If K has zero eigenvalues, their constant (infinite) contribution to Z is factorized by the usual Faddeev–Popov trick. Then the constant $\mathscr{N} \neq 1$ appears from then normalization of the measure $\mathscr{D}\varphi$, if one insists on having Z dimensionless.

3.1. THE MANIFOLD $S^1 \times S^{d-1}$

We consider the sphere S^{d-1} of radius $\mathscr{R}$, identify S^1 with the segment $0 \le u \le T$, and impose periodic boundary conditions for the φ field: $\varphi(u+T) = \varphi(u)$. The metric is

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathscr{R}^2\, \mathrm{d}\Omega^2, \tag{3.3}$$

with $\mathrm{d}\Omega^2$ the usual metric on the unit hypersphere S^{d-1} (say, $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2$ for $d=2$; $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2$, $d=3$; $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta(\mathrm{d}\varphi^2 + \sin^2\varphi\, \mathrm{d}\chi^2)$, $d=4$). The scalar operator in eq. (2.1) is therefore

$$K = -\frac{\partial^2}{\partial u^2} - \sum_{\alpha=2}^{d} \mathrm{D}^\alpha \mathrm{D}_\alpha + \frac{d-2}{4(d-1)} R. \tag{3.4}$$

$\mathrm{D}^\alpha \mathrm{D}_\alpha$ is the Laplacian on S^{d-1} [29] and the scalar curvature $R = (d-1)(d-2)/\mathscr{R}^2$

is constant. The eigenvalues are labelled by two integers $n, l \in \mathbb{Z}$, $l \geq 0$

$$\lambda_{l,n} = \left(\frac{2\pi}{T}\right)^2 n^2 + \frac{1}{\mathscr{R}^2}\Lambda_l$$

$$\Lambda_l = l(l+d-2) + \frac{d-2}{4(d-1)} R\mathscr{R}^2 = \left(l + \tfrac{1}{2}(d-2)\right)^2, \tag{3.5}$$

and appear with multiplicity

$$\delta(l) = \frac{(l+d-3)!}{l!\,(d-2)!}(2l+d-2). \tag{3.6}$$

Notice that for $d > 2$ there is no zero mode.

The zeta function has the form, for $d > 2$

$$\zeta_{S^1 \times S^{d-1}}(s) = \sum_{l=0}^{\infty} \sum_{n \in \mathbb{Z}} \frac{\delta(l)}{(\lambda_{l,n})^s}. \tag{3.7}$$

Some manipulations which are analogous to those of two-dimensional calculations [28] – Kronecker's second limit formula [30] – allow us to continue analytically this function in the region $s \sim 0$ and compute its term $O(s)$. Then

$$Z = \exp\left(\frac{1}{2}\frac{d}{ds}\zeta_{S^1 \times S^{d-1}}(s)\bigg|_0\right)$$

$$= \exp\left(-\frac{1}{2}\frac{T}{\mathscr{R}}\zeta_{S^{d-1}}\left(-\frac{1}{2}\right)\right)\left\{\prod_{l=0}^{\infty}\left[1 - \exp\left(-\frac{T}{\mathscr{R}}\sqrt{\Lambda_l}\right)\right]^{-\delta(l)}\right\}. \tag{3.8}$$

This expression requires a further analytic continuation in the term

$$\zeta_{S^{d-1}}(s) = \sum_{l=0}^{\infty} \frac{\delta(l)}{\Lambda_l^s}. \tag{3.9}$$

This is obtained by rewriting eq. (3.9) as a finite sum of Riemann zeta functions $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$, whose continuation is known. After some calculations it reads

$$\zeta_{S^{d-1}}\left(-\frac{1}{2}\right) = \sum_{k=0}^{[(d-3)/2]} a_k \zeta_R(1-d+2k)$$

$$= \begin{cases} 0, & d = 2n+1 \geqslant 3, \\ (-)^{n+1} W_{n+1}, & d = 2n+2 \geqslant 4, \end{cases} \tag{3.10}$$

where W_n is a positive constant depending on the first n Bernoulli numbers (its explicit formula is given in the appendix A).

Eq. (3.8) displays the general structure of the bosonic partition function: it contains the product of factors for the Bose statistics $(1-e^{-\beta\epsilon_l})^{-1}$ of inverse "temperature" β, equal to the time evolution T, and elementary excitations of energy $\epsilon_l = \sqrt{\Lambda_l}/\mathscr{R}$. There also appears the non-trivial prefactor corresponding to the vacuum energy E_0; this dominates in the limit $T \to \infty$ (or $\beta \to 0$)

$$\log Z \underset{T\to\infty}{\sim} -TE_0 = -\frac{T}{\mathscr{R}}\tilde{c}_d . \tag{3.11}$$

From eq. (3.10), the vacuum energy is zero for any odd dimension: this is a general property related to the absence of gravitational trace anomaly (see sect. 4). In even dimensions it is, for example,

$$\tilde{c}_2 = -\tfrac{1}{12}, \qquad \tilde{c}_4 = \tfrac{1}{240}, \qquad \tilde{c}_6 = -\tfrac{31}{60480} \tag{3.12}$$

($\tilde{c}_2$ is computed in subsect. 3.2). Notice the puzzling sign oscillation.

On the contrary, a bosonic theory not conformally invariant, i.e. $\xi \neq (d-2)/4(d-1)$ in eq. (2.1), has a vacuum energy different from zero in any dimension. For $\xi = 0$ the analytic continuation of $\zeta_{S^{d-1}}(s) = \sum_{l=1}^{\infty}\delta(l)[l(l+d-2)]^{-s}$ in eq. (3.9) was done by Weisberger, ref. [29]; $\zeta_{S^{d-1}}(-1/2)$ is expressed as an infinite sum of Riemann zeta functions, to be estimated numerically.

3.2. THE TORUS

The torus $\mathbb{T}^d$ is the quotient of the flat space $\mathbb{R}^d$ by the lattice Λ^d generated by the vector moduli ω_i, $i = 1,\dots,d$, which are the periods of the field $\varphi(x+\omega_1) = \varphi(x)$. The eigenvalues of the Laplacian $K = -\partial^\mu\partial_\mu$ are

$$\lambda \equiv \lambda_{n_1,\dots,n_d} = (2\pi)^2|n_1\boldsymbol{K}_1 + \cdots + n_d\boldsymbol{K}_d|^2, \qquad n_1, n_2,\dots,n_d \in \mathbb{Z}, \tag{3.13}$$

in terms of the dual vectors $\boldsymbol{K}_j$, obtained by the inversion of the matrix $\omega_{ij} = (\omega_i)_j$

$$\boldsymbol{K}_j \cdot \omega_i = \delta_{ij}, \tag{3.14}$$

then $(\boldsymbol{K}_i)_j \equiv K_{ij} = (\omega^{-1})_{ji}$. As in the previous case, we suppose that $\omega_2,\dots,\omega_d$ span the spatial directions and only ω_1 has a time component, which we can choose as the first one

$$(\omega_i)_j = 0, \quad \text{for} \quad j=1, \quad i=2,\dots,d. \tag{3.15}$$

Then $\boldsymbol{K}_1$ is along the first axis

$$\omega_{11} = T > 0, \qquad K_{11} = 1/T, \tag{3.16}$$

and the eigenvalues eq. (3.13) are rewritten as

$$\lambda = \left(\frac{2\pi}{T}\right)^2 \left[(n_1 + Tq)^2 + (T\epsilon)^2\right],$$

in terms of momenta q and energies ϵ of elementary excitations

$$q \equiv q_{n_2,\dots,n_d} = n_2 K_{21} + \cdots + n_d K_{d1}$$

$$\epsilon = \epsilon_{n_2,\dots,n_d} = \left[(n_2 K_{22} + \cdots + n_d K_{d2})^2 + \cdots + (n_2 K_{2d} + \cdots + n_d K_{dd})^2\right]^{1/2}. \tag{3.17}$$

the zeta-function $\zeta_{\mathbb{T}^d}$ is defined by eq. (3.1) and the zero mode is subtracted in eq. (3.2), thus giving a factor $\mathcal{N} = V^{1/d}$, where V is the volume of the torus. The Kronecker limit formula applies again and it yields

$$Z = V^{1/d} \exp\left(\frac{1}{2}\frac{d}{ds}\zeta_{\mathbb{T}^d}(s)\bigg|_0\right) = V^{1/d}\frac{1}{T}\exp\left(-\tfrac{1}{2}T\zeta_{\mathbb{T}^{d-1}}\left(-\tfrac{1}{2}\right)\right)$$
$$\times \left\{\prod'_{n_2,\dots,n_d \in \mathbb{Z}} \left[1 - \exp(-2\pi T(\epsilon + iq))\right]\right\}^{-1}, \tag{3.18}$$

where in the product Π' the zero mode is omitted. The first factor is given by analytic continuation of

$$\zeta_{\mathbb{T}^{d-1}}(s) = \sum'_{n_2,\dots,n_d \in \mathbb{Z}} \frac{1}{\left(2\pi\epsilon_{n_2,\dots,n_d}\right)^{2s}}. \tag{3.19}$$

Eq. (3.18) has the same structure as eq. (3.8), displaying a vacuum energy and a product of Bose terms. The remaining analytic continuation of eq. (3.19) is done by Epstein's functional equation, which is a generalization of the Riemann symmetry equation for $\zeta_{\mathrm{R}}(s)$ [31]

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{m_1,\ldots,m_p\in\mathbb{Z}}{}' \frac{1}{\varphi(D)^{s/2}}$$

$$=\frac{\pi^{-(p-s)/2}\Gamma((p-s)/2)}{\sqrt{\det D}}\sum_{m_1,\ldots,m_p\in\mathbb{Z}}{}' \frac{1}{\varphi(D^{-1})^{(p-s)/2}}, \tag{3.20}$$

where $\varphi(D)=\sum_{i,j=1}^{p} m_i D_{ij} m_j$ is a positive-definite quadratic form.

The contribution of the vacuum to Z dominates when at least one side of the torus goes to infinity. By taking it as the time T we have, from eqs. (3.18)–(3.20), the Casimir effect

$$\log Z \underset{\omega_{11}=T\to\infty}{\sim} \frac{V}{S_d}\sum_{n_2,\ldots,n_d\in\mathbb{Z}}{}' \frac{1}{|n_2\omega_2+\cdots+n_d\omega_d|^d}. \tag{3.21}$$

In eq. (3.21) we reinstated the d-dimensional vector notation. This rather general formula applies to all possible shapes of the torus. For example, the geometry of a slab made of two infinite $(d-1)$-hyperplanes at (spatial) distance $|\omega_2|=L$ is obtained by letting $|\omega_i|\to\infty$ for all $i\neq 2$.

It yields

$$\frac{1}{V}\log Z\sim -\frac{1}{L^d}\tilde{c}_d,\qquad \tilde{c}_d(\mathbb{T}^d)=-\frac{2\zeta_{\mathrm{R}}(d)}{S_d}. \tag{3.22}$$

For example

$$\tilde{c}_2=-\tfrac{1}{6}\pi,\qquad \tilde{c}_3=-\frac{1}{2\pi}\zeta_{\mathrm{R}}(3),\qquad \tilde{c}_4=-\frac{1}{90}\pi^2. \tag{3.23}$$

Its qualitative behaviour with the dimension is different from the cylinder in eq. (3.12), because it is non-zero in odd dimensions and is of definite sign ($\zeta_{\mathrm{R}}(2k+1)$ is known only numerically and $\zeta_{\mathrm{R}}(2k)$ is proportional to the Bernoulli number B_k).

Our results, eqs. (3.21) and (3.22), agree with previous calculations by hamiltonian canonical quantization: the $d=3$ case was recently derived for quantization of extended objects like the three-dimensional membrane [17]. In this context there appears a factor $(D-3)$ for any transverse direction in the D-dimensional embedding space. The $d=4$ slab Casimir effect agrees with the classical result for the electromagnetic field when a factor of two is accounted for polarizations [32].

Besides the value of the vacuum energy, a closed expression of the partition function is interesting because it exhibits the invariance under the modular transformations of the torus parametrization $\{\omega_1,\ldots,\omega_d\}$. In order to test it, a fully covariant expression of Z is necessary; the three-dimensional case is

$$\log Z(\omega_1, \omega_2, \omega_3)$$
$$= \frac{1}{3}\log\frac{|\omega_2 \wedge \omega_3|^3}{V^2} + \frac{V}{4\pi}\sum_{n_2,n_3}{}' \frac{1}{|n_2\omega_2 + n_3\omega_3|^3}$$
$$- \sum_{n_2,n_3}{}' \log\left\{1 - \exp\left[\frac{-2\pi V|n_2\omega_3 - n_3\omega_2| + i2\pi\left[n_2(\omega_1\wedge\omega_3) - n_3(\omega_1\wedge\omega_2)\right]\cdot(\omega_2\wedge\omega_3)}{|\omega_2\wedge\omega_3|^2}\right]\right\}. \tag{3.24}$$

(The analogous formula in d dimensions is given in appendix A.)

The modular group in d dimensions is $SL(d,\mathbb{Z})/\mathbb{Z}_2$; it contains the transformations

$$T_{ij}:\quad \omega_i \to \omega_i + \omega_j,\quad \omega_j \to \omega_j,\quad i \neq j,$$
$$S_{ij}:\quad \omega_i \to \omega_j,\qquad \omega_j \to -\omega_i. \tag{3.25}$$

Some of them are trivially satisfied in eq. (3.24), but a complete proof of modular invariance will not be discussed here. For a non-trivial check of this formula we computed $\langle T_{\mu\nu}\rangle_{\mathbb{R}^3}$ by differentiation with respected to ω_i's and comparison to eq. (1.2) and we verified it has the correct symmetries (see the last of refs. [28] for the analogous calculation in two dimensions).

4. Trace anomaly and Casimir effect on the cylinder $S^{d-1} \times \mathbb{R}$

In this section we consider the transformation of the stress tensor under conformal transformations (1.1) in curved space time. The method is based on the derivation of the effective potential for $\sigma(x)$, the conformal factor of the metric – the Liouville action in two dimensions [1,14]. After recalling the two-dimensional case, we present the four-dimensional one in a fully renormalized form and we clarify some technical aspects.

A direct application of this transformation law is to the determination of the Casimir effect on a manifold related to flat space by a conformal transformation, like the generalized cylinder $S^{d-1} \times \mathbb{R}$ we discussed in sect. 3. We would like to stress that some finite-size effects of this geometry are predicted by conformal invariance as in the two-dimensional case. We first set the problem by recalling the properties discussed in ref. [18], then we derive the transformation of $T_{\mu\nu}$ and the Casimir effect.

4.1. FINITE-SIZE EFFECTS

The cylinder $S^{d-1}\times\mathbb{R}$ is a manifold conformally equivalent to $\mathbb{R}^d$, i.e. its metric can be put into the form $g_{\mu\nu}(x)=\exp(2\sigma(x))\hat{g}_{\mu\nu}$, $\hat{g}_{\mu\nu}=\delta_{\mu\nu}$; by eq. (3.3) it is

$$\mathrm{d}s^2=\mathrm{d}u^2+\mathscr{R}^2\,\mathrm{d}\Omega^2 \xrightarrow{\sigma=-u/\mathscr{R}} \mathrm{d}\hat{s}^2=\mathrm{e}^{-2\sigma}\mathrm{d}s^2=\mathrm{d}r^2+r^2\mathrm{d}\Omega^2. \tag{4.1}$$

By the conformal transformation and the change of variable $r=\mathscr{R}\exp(u/\mathscr{R})$ it becomes the flat metric of $\mathbb{R}^d$ in spherical coordinates (the point at the origin of $\mathbb{R}^d$ is excluded because the change of variables is singular there).

In a conformal invariant theory, the correlation functions in these two geometries are related by the transformation eqs. (1.1) and (4.1). For example, the two-point function (and the three-point one) of any scalar field has a form completely determined in flat space by its dimension Δ [9]. This implies on the cylinder $\mathbb{R}\times S^{d-1}$ the unique form

$$\begin{aligned}\langle\phi(u_1,\Omega_1)\phi(u_2,0)\rangle_{\mathrm{R}\times S^{d-1}} &= \mathrm{e}^{\Delta(u_1+u_2)/\mathscr{R}}\langle\phi(|\boldsymbol{x}_1|,\Omega_1)\phi(|\boldsymbol{x}_2|,0)\rangle_{\mathbb{R}^d}\\ &\equiv \mathrm{e}^{\Delta(u_1+u_2)/\mathscr{R}}|\boldsymbol{x}_1-\boldsymbol{x}_2|^{-2\Delta}\\ &\equiv \left\{\mathscr{R}^2\left[2\,\mathrm{ch}((u_1-u_2)/\mathscr{R})-2\cos\theta_1\right]\right\}^{-\Delta},\end{aligned} \tag{4.2}$$

where $\mathscr{R}\theta_1$ is the geodesic distance on the sphere of the first point from the second on the polar axis.

The relation between correlation functions on the two geometries implies a correspondence between the two Hilbert spaces. Time (u) translations in the cylinder corresponds to dilatations in $\mathbb{R}^d$, then eigenvalues Δ_n of the dilatation operator $\mathscr{D}$ are equal to eigenvalues E_n of the cylinder hamiltonian $\mathscr{H}$: $\mathscr{H}\,\delta u=\mathscr{D}\,\delta r/r$ leading to [18]

$$\frac{\Delta_n}{\mathscr{R}}=E_n-E_0. \tag{4.3}$$

Eq. (4.3) can be useful in numerical simulations of the transfer matrix of a statistical model on the geometry of the sphere. On the other hand, finite size effects on the torus $\mathbb{T}^d$, the traditional geometry for simulations [33], are not predicted by conformal invariance for $d>2$, because this geometry is not related to $\mathbb{R}^d$ by a conformal transformation.

The expectation value $\langle \mathrm{T}_{\mu\nu}\rangle$ on the former manifold is given by

$$\delta\log Z=\frac{1}{2S_d}\int_{S^1\times S^{d-1}}\mathrm{d}^dx\sqrt{g}\,\langle \mathrm{T}_{\mu\nu}\rangle\delta g^{\mu\nu}, \tag{4.4}$$

which is the quantum analogue of eq. (1.2). The form of $\langle T_{\mu\nu}\rangle$ is determined by its invariance under the isometries of the manifold [34]. It has a two-parameter freedom

$$\langle T_{\mu\nu}(x)\rangle = \frac{1}{\mathscr{R}^d}\left(A g_{\mu\nu}(x) + B\delta_{\mu,1}\delta_{\nu,1}\right), \tag{4.5}$$

where $x^1 = u$, S^1 is the segment $0 \le u < T$ with periodic boundary conditions and $x^2, \ldots, x^d$ are coordinates on the sphere S^{d-1}. In this form $\langle T_{\mu\nu}\rangle$ is also covariantly conserved. A non-vanishing trace $\langle T_\mu{}^\mu\rangle$ measures the lack of scale invariance of the partition function: for a dilatation $\delta g_{\mu\nu} = 2 g_{\mu\nu}\delta\lambda$, eq. (4.4) reads

$$\frac{\delta}{\delta\lambda}\log Z[g_{\mu\nu}] = -\frac{1}{S_d}\int d^d x\sqrt{g}\,\langle T_\mu{}^\mu\rangle = -\frac{T}{\mathscr{R}}(dA + B). \tag{4.6}$$

For example, the scalar field partition function obtained in sect. 3, eq. (3.8), is a function of $T/\mathscr{R}$; then we proved that in any dimension

$$\langle T_\mu{}^\mu\rangle = 0 \qquad (\text{scalar field on } S^1 \times S^{d-1}). \tag{4.7}$$

This result is not trivial because, for a general conformal theory on a curved manifold, the stress tensor defined by eq. (4.4) does have an anomalous trace coming from the lack of scale invariance of the integration measure in Z [7].

The vacuum energy E_0 is related to the full expectation values $\langle T_{\mu\nu}\rangle$. The contribution of the vacuum is singled out in the limit $T \to \infty$

$$E_0 = \lim_{T\to\infty} -\frac{1}{T}\log Z\left([0, T] \times S^{d-1}\right). \tag{4.8}$$

Let us consider eq. (4.4) for a metric transformation $\delta g^{\mu\nu} = -2\delta^{\mu,1}\delta^{\nu,1}\delta\epsilon$. It corresponds to a parametric variation in eq. (4.8) for $T \to T(1 + \delta\epsilon)$. By comparing eqs. (4.4), (4.5) and (4.8) one obtains the vacuum energy in terms of the expectation value of $\langle T_{11}\rangle$

$$E_0 \underset{T\to\infty}{\sim} \frac{1}{TS_d}\int d^d x\sqrt{g}\,\langle T_{11}\rangle = \frac{1}{\mathscr{R}}(A + B). \tag{4.9}$$

4.2. THE EFFECTIVE POTENTIAL

The expectation value $\langle T_{\mu\nu}\rangle$ can be obtained by integration along a conformal transformation into another manifold where $\langle T_{\mu\nu}\rangle$ is known, namely $\mathbb{R}^d$ where it is normalized to be zero. Let us consider two manifolds $(M, g_{\mu\nu})$ and $(\hat{M}, \hat{g}_{\mu\nu})$ with $g_{\mu\nu}(x) = e^{2\sigma(x)}\hat{g}_{\mu\nu}(x)$; let us first suppose they are compact so that there are no boundary terms. Let $\bar{g}_{\mu\nu}(t, x) = e^{2\bar{\sigma}(x,t)}\hat{g}_{\mu\nu}(x)$ be a path interpolating between $\hat{g}_{\mu\nu}$

and $g_{\mu\nu}$, with $\bar{\sigma}(x,t)$ a smooth function of $t \in [0,1]$ and x satisfying $\bar{\sigma}(x,0)=0$, $\bar{\sigma}(x,1)=\sigma(x)$; for example, one can take $\bar{\sigma}(x,t)=t\sigma(x)$. The difference between the partition functions for the two metrics is

$$\Gamma[g,\hat{g}] = -\log Z[g_{\mu\nu}] + \log Z[\hat{g}_{\mu\nu}]$$

$$= -\int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \log Z[\bar{g}_{\mu\nu}] = \int_0^1 \mathrm{d}t \int \frac{\mathrm{d}^d x}{S_d} \sqrt{\bar{g}(x)} \left\langle \bar{\mathrm{T}}^{\mu}_{\mu}(x) \right\rangle \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}t}. \quad (4.10)$$

For theories which are conformally invariant at the classical level, the trace in the last term of eq. (4.10) is completely anomalous, that is it does not contain expectation values of dynamical fields. We shall limit ourselves to purely gravitational anomalies, so that it is a local functional of the metric $\bar{g}$, whose form is well known in two and four dimensions, and it appears already in free theories. In such a case the general form of the trace anomaly can be integrated in eq. (4.10). The result has the form of an effective potential for the σ field in the reference metric $\hat{g}_{\mu\nu}$.

By definition $\Gamma[g,\hat{g}]$ must satisfy the cocycle conditions [20]

$$\Gamma[g,\hat{g}] + \Gamma[\hat{g},g] = 0,$$

$$\Gamma[g,\hat{g}] + \Gamma[\hat{g},\hat{\hat{g}}] + \Gamma[\hat{\hat{g}},g] = 0, \quad (4.11)$$

where g, $\hat{g}$ and $\hat{\hat{g}}$ are metrics related by conformal transformations. They are a check that the effective potential is independent on the path of integration.

For infinitesimal metrics, say $g_{\mu\nu} = (1+2\delta\sigma_1)\hat{g}_{\mu\nu} = (1+2\delta\sigma_1+2\delta\sigma_2)\hat{\hat{g}}_{\mu\nu}$, eq. (4.11) is the Wess–Zumino consistency condition on the trace anomaly. One has

$$\left(\delta_{\sigma_1}\delta_{\sigma_2} - \delta_{\sigma_2}\delta_{\sigma_1}\right)\Gamma\left[\mathrm{e}^{2\sigma}\hat{g}_{\mu\nu}, \hat{g}_{\mu\nu}\right] = 0, \quad (4.12)$$

where the r.h.s. is zero because we have an abelian group of transformations. By eq. (4.4), eq. (4.12) is

$$\mathscr{A}_{12} = \int \mathrm{d}^d x \left\{ \delta\sigma_1(x) \left(\frac{\delta}{\delta\sigma_1(x)} \int \mathrm{d}^d y \sqrt{g} \langle \mathrm{T}^{\lambda}_{\lambda} \rangle \delta\sigma_2(y) \right) \right.$$

$$\left. -\delta\sigma_2(x) \left(\frac{\delta}{\delta\sigma_2(x)} \int \mathrm{d}^d y \sqrt{g} \langle \mathrm{T}^{\lambda}_{\lambda} \rangle \delta\sigma_1(y) \right) \right\} = 0. \quad (4.13)$$

This is a necessary condition for eq. (4.11). This general discussion will be relevant in four dimensions.

Let us first recall the two-dimensional case. The trace (1.4) is inserted into eq. (4.10), written in terms of $\hat{g}_{\mu\nu}$ and $\bar{\sigma}$, $\bar{R} = \mathrm{e}^{-2\bar{\sigma}}(\hat{R} - 2\hat{\mathrm{D}}^2\bar{\sigma})$, where $\hat{\mathrm{D}}_\mu$ is the covariant derivative with respect to $\hat{g}_{\mu\nu}$ (some useful formulae for this section are given in appendix B). The integration gives [1]

$$\Gamma[g,\hat{g}] = -\frac{c}{12S_2}\int_{\hat{M}} d^2x\sqrt{\hat{g}}\left(\hat{R}\sigma + \hat{D}_\lambda\sigma\,\hat{D}^\lambda\sigma\right)$$

$$= \frac{c}{96\pi}\left[\int_M d^2x\sqrt{g}\,R\frac{1}{D^2}R - \int_{\hat{M}} d^2x\sqrt{\hat{g}}\,\hat{R}\frac{1}{\hat{D}^2}\hat{R}\right]. \qquad (4.14)$$

One can check that this potential satisfies the cocycle conditions eq. (4.11); this is trivial in the above non-local form.

Once we have eq. (4.14), the difference between $\sqrt{g}\langle T_{\mu\nu}\rangle$ (living on M) and $\sqrt{\hat{g}}\langle\hat{T}_{\mu\nu}\rangle$ (on $\hat{M}$) is obtained by varying with respect to $\hat{g}_{\mu\nu}$, $\sigma(x)$ being fixed, $\delta g^{\mu\nu}(x) = e^{-2\sigma(x)}\delta\hat{g}^{\mu\nu}(x)$

$$\delta\Gamma[g,\hat{g}] = \frac{1}{2S_d}\int d^dx\left(-\sqrt{g}\langle T_{\mu\nu}\rangle\delta g^{\mu\nu} + \sqrt{\hat{g}}\langle\hat{T}_{\mu\nu}\rangle\delta\hat{g}^{\mu\nu}\right)$$

$$= \frac{1}{2S_d}\int d^dx\left(-\sqrt{g}\langle T_\mu{}^\lambda\rangle + \sqrt{\hat{g}}\langle\hat{T}_\mu{}^\lambda\rangle\right)\hat{g}_{\nu\lambda}\delta\hat{g}^{\mu\nu}. \qquad (4.15)$$

For eq. (4.14), this gives

$$\sqrt{g(x)}\left\langle T_\alpha{}^\beta(x)\right\rangle - \sqrt{\hat{g}(x)}\left\langle\hat{T}_\alpha{}^\beta(x)\right\rangle$$

$$= \tfrac{1}{6}c\sqrt{\hat{g}}\left[-\hat{\sigma}_\alpha{}^\beta + \hat{\sigma}_\alpha\hat{\sigma}^\beta + \delta_\alpha^\beta\left(\hat{\sigma}_\lambda^\lambda - \tfrac{1}{2}\hat{\sigma}_\lambda\hat{\sigma}^\lambda\right)\right]$$

$$= \tfrac{1}{6}c\sqrt{g}\left[-\sigma_\alpha{}^\beta - \sigma_\alpha\sigma^\beta + \delta_\alpha^\beta\left(\sigma_\lambda^\lambda + \tfrac{1}{2}\sigma_\lambda\sigma^\lambda\right)\right], \qquad (4.16)$$

where $\hat{\sigma}_\alpha = \hat{D}_\alpha\sigma$, $\hat{\sigma}_\alpha{}^\beta = \hat{D}^\beta\hat{D}_\alpha\sigma$ and $\sigma_\alpha = D_\alpha\sigma$, $\sigma_\alpha{}^\beta = D^\beta D_\alpha\sigma$ are covariant derivatives of the corresponding metrics. Eq. (4.16) was originally derived in ref. [14] from the effective potential given by dimensional regularization; after the variation (4.15) a suitable $d \to 2$ limit was taken.

Reparametrization invariance must be preserved throughout our discussion, and then the transformation law must be compatible with the covariant conservation of the stress tensor. From eq. (4.16) one obtains $\sqrt{g}\,D^\mu\langle T_{\mu\nu}\rangle = \sqrt{\hat{g}}\,\hat{D}^\mu\langle\hat{T}_{\mu\nu}\rangle$.

Let us emphasize the relation between eq. (4.16) and the result of Belavin et al. [1]. They consider an analytic coordinate transformation $\hat{z} = \hat{z}(z)$ between flat metrics; in our approach it corresponds to a reparametrization followed by a conformal transformation of the metric

$$\hat{g}_{\hat{z}\bar{\hat{z}}}(\hat{z}) = \tfrac{1}{2} \to \hat{g}_{z\bar{z}}(z) = \tfrac{1}{2}|\mathrm{d}\hat{z}/\mathrm{d}z|^2 \to g_{z\bar{z}}(z) = \mathrm{e}^{2\sigma}\hat{g}_{z\bar{z}}(z) = \tfrac{1}{2}.$$

For the conformal transformation, eq. (4.16) gives with $\sigma = -\frac{1}{2}\log|\mathrm{d}\hat{z}/\mathrm{d}z|^2$

$$\langle \mathrm{T}_{zz}(z)\rangle - \langle \hat{\mathrm{T}}_{zz}(z)\rangle = -\frac{c}{6}\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}\sigma + \left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2\right]$$

$$= \tfrac{1}{12}c\left[\frac{\mathrm{d}^3\hat{z}}{\mathrm{d}z^3}\Big/\frac{\mathrm{d}\hat{z}}{\mathrm{d}z} - \frac{3}{2}\left(\frac{\mathrm{d}^2\hat{z}}{\mathrm{d}z^2}\Big/\frac{\mathrm{d}\hat{z}}{\mathrm{d}z}\right)^2\right] = \tfrac{1}{12}c\{\hat{z}, z\}. \quad (4.17)$$

Under the reparametrization, $\hat{\mathrm{T}}_{zz}$ transforms as a tensor because we assumed exact covariance of the theory. Eventually

$$\langle \mathrm{T}_{zz}(z)\rangle = \langle \hat{\mathrm{T}}_{zz}(\hat{z})\rangle\left(\frac{\mathrm{d}\hat{z}}{\mathrm{d}z}\right)^2 + \tfrac{1}{12}c\{\hat{z}, z\}. \quad (4.18)$$

This is the result of refs. [1], with $\{\hat{z}, z\}$ the schwartzian derivative. The transformation there established for the field T_{zz} is recovered here for the expectation value $\langle \mathrm{T}_{zz}\rangle$ (see also eq. (1.10)). Eq. (4.16) is actually a more general result which holds for non-analytic transformations, changing also $\langle \mathrm{T}_{z\bar{z}}\rangle$, but it is valid for expectation values only.

Let us rederive the vacuum energy E_0 for the two-dimensional cylinder $\mathbb{R} \times \mathrm{S}^1$, eq. (1.15) [16]. The previous derivation is not strictly valid in this case, because this is not a compact manifold. However, $\langle \mathrm{T}_{\mu\nu}\rangle$ describes the bulk properties of the system and should become insensitive to the time boundary conditions when the temporal extension goes to infinity. In this limit, eq. (4.16) is expected to hold. The conformal transformation to the flat space $\mathbb{R}^2$, eq. (4.1), is then used in eq. (4.16). One sets $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$, the normalization $\langle \hat{\mathrm{T}}_{\mu\nu}\rangle \equiv 0$ and obtains $\langle \mathrm{T}_{\mu\nu}\rangle$ on the cylinder M. By substitution of $\langle \mathrm{T}_{\mu\nu}\rangle$ in eq. (4.9), the correct result for E_0, eq. (1.15), is recovered.

The higher dimensional case is analogous. As a starting point, we need the gravitational trace anomaly; it is expressed in terms of local functions of $g_{\mu\nu}$ and its derivatives, which are Lorentz scalars of dimension d. For $d = 2$ there is only the scalar curvature R, eq. (1.4); for $d = 3$ and any odd dimensions, one cannot form a scalar with an odd number of derivatives of $g_{\mu\nu}$, then $\langle \mathrm{T}_\mu{}^\mu\rangle \equiv 0$. For $d = 4$ the possible terms are [35]

$$\langle \mathrm{T}_\mu{}^\mu\rangle = -\lambda\left\{\alpha C^2_{\mu\nu\rho\sigma} + \beta\left(R^2_{\mu\nu} - \tfrac{1}{3}R^2\right) - \gamma\,\mathrm{D}^2 R + \epsilon R^2\right\}, \quad (4.19)$$

where $\lambda = (1440)^{-1}$, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor [8], and the coefficients α, β, γ, ϵ were computed in ref. [36] for free theories of various spin values ($\alpha=\beta=1$, $\gamma=-1$, $\epsilon=0$ for spin zero).

Let us check the Wess–Zumino condition, eq. (4.13), on the trace anomaly eq. (4.19). It is satisfied by the first three terms, but the R^2 term gives by using eq. (2.3)

$$\mathscr{A}_{12} \propto \int \mathrm{d}^4x\sqrt{g}\, R\left[\left(\mathrm{D}^2\delta\sigma_1\right)\delta\sigma_2-\delta\sigma_1\left(\mathrm{D}^2\delta\sigma_2\right)\right]. \tag{4.20}$$

It does not vanish for a general transformation and this excludes such a term in a conformal invariant theory. Let us explain this point better. The transformation properties (4.12) must be always satisfied by the effective potential; if the R^2 term appears in the trace (4.19) there must also be additional terms not built from the metric, which compensate eq. (4.20). These are non-anomalous contributions, which appear already at the classical level. Therefore, the theories with an R^2 anomaly are not conformally invariant. On the other hand, consider the effective potential for a global scale transformation only; it is given by the integrated trace anomaly in eq. (4.6), which has a consistent form because eq. (4.20) vanishes in this case. It is possible that the previous non-anomalous contributions do not appear in the integrated anomaly. Therefore, theories with an R^2 anomaly may be scale invariant at most.

These observation are confirmed by the known four-dimensional theories where such a term has been reported: free theories with spin greater then one, when their lagrangian is not conformally invariant at the classical level (see refs. [36, 37]); interacting theories like φ^4 theory, and QED with fermions off the trivial fixed point [38].

Therefore, in the following we consider only theories with $\epsilon=0$ in eq. (4.19). The effective potential is again obtained by integration of eq. (4.10) and some formulas in appendix B:

$$\begin{aligned}\Gamma[g,\hat{g}]=-\frac{\lambda}{S_4}\int_{\hat{\mathrm{M}}}\mathrm{d}^4x\sqrt{\hat{g}}\Big\{&\alpha\hat{C}_{\mu\nu\rho\lambda}\hat{C}^{\mu\nu\rho\lambda}\sigma+3\gamma\left[\left(\tfrac{1}{6}\hat{R}-\hat{\sigma}^{\lambda}_{\lambda}-\hat{\sigma}_{\lambda}\hat{\sigma}^{\lambda}\right)^2-\left(\tfrac{1}{6}\hat{R}\right)^2\right]\\&+\beta\left[\left(\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}-\tfrac{1}{3}\hat{R}^2\right)\sigma+2\hat{\sigma}^{\mu}\hat{\sigma}^{\nu}\left(\hat{R}_{\mu\nu}-\tfrac{1}{2}\hat{g}_{\mu\nu}\hat{R}\right)+2\hat{\sigma}_{\lambda}\hat{\sigma}^{\lambda}\hat{\sigma}^{\mu}_{\mu}+\left(\hat{\sigma}_{\lambda}\hat{\sigma}^{\lambda}\right)^2\right]\Big\}\end{aligned} \tag{4.21}$$

($\hat{\sigma}_{\mu}=\hat{\mathrm{D}}_{\mu}\sigma$, $\sigma_{\mu}=\mathrm{D}_{\mu}\sigma$ as before). A similar expression was given in ref. [15]. Eq. (4.21) can be rewritten in the form

$$\begin{aligned}\Gamma[g,\hat{g}]=&-\frac{\lambda}{2\pi^2}\int_{\mathrm{M}}\mathrm{d}^4x\sqrt{g}\Big\{\tfrac{1}{2}\alpha C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}\sigma\\&\qquad+\tfrac{1}{2}\beta\left[\left(R_{\mu\nu}R^{\mu\nu}-\tfrac{1}{3}R^2\right)\sigma-\left(\mathrm{D}_{\mu}\sigma\right)^2\mathrm{D}^2\sigma\right]+\tfrac{1}{12}\gamma R^2\Big\}\\&-\frac{\lambda}{2\pi^2}\int_{\hat{\mathrm{M}}}\mathrm{d}^4x\sqrt{\hat{g}}\Big\{\tfrac{1}{2}\alpha\hat{C}_{\mu\nu\rho\lambda}\hat{C}^{\mu\nu\rho\lambda}\sigma\\&\qquad+\tfrac{1}{2}\beta\left[\left(\hat{R}_{\mu\nu}\hat{R}^{\mu\nu}-\tfrac{1}{3}\hat{R}^2\right)\sigma-\left(\hat{\mathrm{D}}_{\mu}\sigma\right)^2\hat{\mathrm{D}}^2\sigma\right]-\tfrac{1}{12}\gamma\hat{R}^2\Big\}.\end{aligned} \tag{4.22}$$

Eq. (4.22) satisfies the cocycle conditions eq. (4.11); the first one is obvious, the second one requires a detailed calculation for the β term.

The variation with respect to $\hat{g}^{\mu\nu}$ gives the stress tensor. The Weyl tensor vanishes for metrics of the form $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$ that we shall consider, and we set $\hat{C}_{\mu\nu\rho\sigma} = 0$ after this variation. Then, the α term drops out and the β and γ terms are given by the variational formulas in appendix B

$$
\begin{aligned}
-4\pi^2 \hat{g}^{\nu\alpha} \frac{\delta\Gamma}{\delta\hat{g}^{\mu\alpha}}\bigg|_{\sigma\text{ fixed}} &= \sqrt{g(x)}\left\langle \mathrm{T}_\mu^{\ \nu}(x)\right\rangle - \sqrt{\hat{g}(x)}\left\langle \hat{\mathrm{T}}_\mu^{\ \nu}(x)\right\rangle \\
&= -\lambda\sqrt{g}\left\{\tfrac{1}{4}\delta_\mu^\nu\left[\beta\left(R_{\rho\lambda}R^{\rho\lambda} - \tfrac{1}{3}R^2\right) - \gamma\,\mathrm{D}^2 R\right]\right. \\
&\qquad \left. + g^{\nu\gamma}P_{\mu\gamma}^{\alpha\beta}\left[\beta\left(\tfrac{2}{3}RR_{\alpha\beta} - R_{\alpha\lambda}R_\beta^{\ \lambda}\right) + \tfrac{1}{3}\gamma\left(\mathrm{D}_\alpha\mathrm{D}_\beta R - RR_{\alpha\beta}\right)\right]\right\} \\
&\quad + \lambda\sqrt{\hat{g}}\left\{g_{\mu\nu} \leftrightarrow \hat{g}_{\mu\nu}\right\},
\end{aligned} \tag{4.23}
$$

where $P_{\mu\nu}^{\alpha\beta}$ is the projector on the symmetric traceless part. Eq. (4.23) agrees with a previous calculation by dimensional regularization [14]. The covariant conservation of $\langle \mathrm{T}_{\mu\nu}\rangle$ is again compatible with eq. (4.23) for any value of β and γ, because the tensors multiplying these coefficients are both covariantly conserved [6].

A more explicit form of eq. (4.23) can be written in terms of σ for conformal deformations of flat space $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$, $\langle \hat{\mathrm{T}}_{\mu\nu}\rangle \equiv 0$

$$
\begin{aligned}
\sqrt{g}\left\langle \mathrm{T}_\mu^{\ \nu}(x)\right\rangle = &-\lambda\delta_\mu^\nu\left\{\beta\sigma^{\alpha\beta}\left[\sigma_{\alpha\beta} - 2\sigma_\alpha\sigma_\beta - \delta_{\alpha\beta}\left(\sigma_\lambda^\lambda + \sigma_\lambda\sigma^\lambda\right)\right]\right. \\
&\qquad \left. + 3\gamma\left[\left(\tfrac{1}{2}\partial^2 - \sigma^\lambda\partial_\lambda - \sigma_\lambda^\lambda\right)\left(\sigma_\lambda^\lambda + \sigma_\lambda\sigma^\lambda\right)\right]\right\} \\
&-\lambda\delta_\rho^\nu P_{\mu\rho}^{\ \ \alpha\beta}\left\{4\beta\left[\sigma_{\alpha\beta}\sigma_\lambda^\lambda - \sigma_\alpha\sigma_\beta\left(\sigma_\lambda^\lambda + \sigma_\lambda\sigma^\lambda\right) - \sigma_{\alpha\lambda}\sigma_\beta^\lambda + 2\sigma_\alpha\sigma_{\beta\lambda}\sigma^\lambda\right]\right. \\
&\qquad \left. - 2\gamma\left[\left(\partial_\alpha\partial_\beta - 6\sigma_\alpha\partial_\beta + 6\sigma_\alpha\sigma_\beta\right)\left(\sigma_\lambda^\lambda + \sigma_\lambda\sigma^\lambda\right)\right]\right\}
\end{aligned} \tag{4.24}
$$

($\sigma_\alpha = \partial_\alpha\sigma$, $\sigma_{\alpha\beta} = \partial_\alpha\partial_\beta\sigma$). In sects. 1 and 2, we showed that $\mathrm{T}_{\mu\nu}$ is a quasi-primary field, i.e. it transforms homogeneously under coordinate transformations belonging to the group $\mathrm{SO}(d+1,1)$. Then eq. (4.24) vanishes for the corresponding finite conformal transformations of the metric. This is manifest for translations, rotations ($\sigma = 0$) and dilatations ($\sigma = \text{cost}$); special conformal transformations are composed of one translation and two inversions $x'^\mu = -x^\mu/x^2$, then it is sufficient to check these last ones ($\sigma = -\log x^2$) in $\mathbb{R}^4\setminus\{\mathbf{0}\}$. In the same way, the schwartzian derivative, eq. (4.18), vanishes for projective transformations of the complex plane.

Let us return to the determination of the Casimir effect on the cylinder $\mathbb{R} \times S^3$. We use eq. (4.23) by assuming again that boundary terms do not affect $\langle T_{\mu\nu} \rangle$. By taking $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$, $\langle \hat{T}_{\mu\nu} \rangle = 0$ in $\mathbb{R}^4$ and $g_{\mu\nu}$ on the cylinder, eqs. (4.1) and (4.23) give

$$\langle T_{\mu\nu} \rangle = \frac{\lambda(\gamma - \beta)}{\mathscr{R}^4} \left(g_{\mu\nu} - 4\delta_{\mu,1}\delta_{\nu,1} \right), \tag{4.25}$$

By eq. (4.9), it yields the relation between the trace anomaly and the Casimir effect on the manifold $\mathbb{R} \times S^3$

$$E_0 = \frac{\tilde{c}_4}{\mathscr{R}}, \qquad \tilde{c}_4 = \frac{\beta - \gamma}{480}. \tag{4.26}$$

For scalar fields this agrees with the result in sect. 3, eq. (3.12). By inserting the β and γ values of ref. [36] this gives

$$\tilde{c}_4 = \tfrac{17}{480}, \qquad \text{Dirac fermion},$$

$$\tilde{c}_4 = \tfrac{11}{120}, \qquad \text{pure QED}. \tag{4.27}$$

Let us conclude this section with some comments. In three dimensions and other odd dimensions there is no gravitational trace anomaly, the effective potential is conformal invariant, $\Gamma[g, \hat{g}] = 0$; then $\langle T_{\mu\nu} \rangle$ and E_0 vanish on the manifolds $\mathbb{R} \times S^{2k}$ as well as in $\mathbb{R}^{2k+1}$. This agrees with the scalar field calculation of sect. 3, eqs. (3.10) and (3.11). Let us observe some simple consequences of this phenomenon.

(i) By numerical simulations of the hamiltonian of a statistical model on S^2, $E_0 = 0 + o(1/\mathscr{R})$ is a non-trivial check for conformal invariance (1.1) of the corresponding field theory at the critical point. It is also a necessary condition for the analysis of the finite-size effects we mentioned at the beginning of this section, eqs. (4.2) and (4.3) [18, 39].

(ii) An anisotropic correction to the correlator $\langle \phi(x)\phi(0) \rangle$ in finite geometries was computed [13], which is due to the insertion of $T_{\mu\nu}$ and therefore of order $O(E_0 |x|^d / \mathbb{R}^d)$. This correction has the unpleasant feature of being non-analytic in odd dimensions, but actually it vanishes on the manifold $\mathbb{R} \times S^{2k}$, leaving only analytic terms in eq. (4.2). On the contrary, the torus geometry does show this correction because $\langle T_{\mu\nu} \rangle \neq 0$ in any dimension. Moreover the effective potential cannot be simply obtained by integration of the trace anomaly, it depends on the details of the theory and general results like eq. (4.26) cannot be stated on finite-size corrections.

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Note added in proof

The term $\gamma D^2 R$ in the four-dimensional trace anomaly eq. (4.19) comes from a term $\gamma \int \sqrt{g}\, R^2$ in the effective potential eq. (4.22) which is local in the metric $g_{\mu\nu}$ and covariant. Then the value of γ can be changed by adding a local counterterm in the action, while keeping the stress tensor conserved. The choice $\gamma = 0$ corresponds to the customary renormalization scheme, in which the anomaly takes a minimal form. Other choices of scheme are presently allowed, because the trace anomaly does not spoil the consistency of the theory of matter on a curved manifold and additional constraints were not found; eventually γ could be determined experimentally.

Therefore in this paper we considered the general case $\gamma \neq 0$. The relation in eq. (4.26) between the trace anomaly and the Casimir energy E_0 on the manifold $\mathbb{R} \times S^3$ clearly holds when both quantities are computed in the same scheme of renormalization. Here we checked eqs. (4.26) and (4.27) in the zeta-function regularization by the computations in sect. 3 and the literature. We thank A. Schwimmer and J.L. Cardy for discussions on this subject.

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Appendix A

We detail here the analytic continuation and evaluation at the point $s = -\frac{1}{2}$ of the conformal zeta function on the sphere (eq. (3.9) in the text), and we display the covariant formula of the partition function on the torus $\mathbb{T}^d$.

A.1. MANIFOLD $S^1 \times S^{d-1}$, d ODD $\geqslant 3$

If we set $d = 2n + 3$, the degeneracy of the eigenvalue Λ_l in eq. (3.6) can be rewritten as

$$\begin{aligned}\delta(l) &= \frac{(2(l+n)+1)}{(2n+1)!} \prod_{i=0}^{n-1} (l+n+1+i)(l+n-i) \\ &= \frac{(2(l+n)+1)}{(2n+1)!} \frac{1}{4^n} \prod_{i=0}^{n-1} \left[(2(l+n)+1)^2 - (2i+1)^2\right] \\ &= \frac{1}{(2n+1)!} \frac{1}{4^n} \sum_{k=0}^{n} (-1)^k \alpha_k(n-1)(2(l+n)+1)^{1+2(n-k)}, \qquad \text{(A.1)}\end{aligned}$$

with

$$\alpha_k(N) = \sum_{0 \neq i_1 < \cdots < i_k \leq N} (2i_1+1)^2 \ldots (2i_k+1)^2, \qquad \alpha_0(N) = 1. \tag{A.2}$$

Therefore, eq. (3.9) reads

$$\begin{aligned}
\zeta_{S^{2n+2}}(s) &= \frac{1}{(2n+1)!} \frac{1}{4^{n-s}} \sum_{k=0}^{n} (-1)^k \alpha_k(n-1) \sum_{l=0}^{\infty} (2(l+n)+1)^{1+2(n-k-s)} \\
&= \frac{1}{(2n+1)!} \frac{1}{4^{n-s}} \left\{ \sum_{k=0}^{n} (-1)^k \alpha_k(n-1) \right. \\
&\qquad \times (1 - 2^{2(n-k-s)-1}) \zeta_R(1 - 2(n-k-s)) \\
&\qquad \left. - \sum_{l=0}^{n-1} \sum_{k=0}^{n} (-1)^k \alpha_k(n-1)(2l+1)^{1+2(n-k-s)} \right\}.
\end{aligned} \tag{A.3}$$

This formula allows us to deduce the analytic continuation of $\zeta_{S^{2n+2}}$ from that of the Riemann zeta function $\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s}$. For $s = -\frac{1}{2}$ one obtains the values $\zeta_R(-2(n-k+1))$, $0 \leq k \leq n$, which are zero [31], as well as

$$\begin{aligned}
&\sum_{l=0}^{n-1} \sum_{k=0}^{n} (-1)^k \alpha_k(n-1)(2l+1)^{2+2(n-k)} \\
&\quad = \sum_{l=0}^{n-1} (2l+1)^2 \prod_{i=0}^{n-1} \left[(2l+1)^2 - (2i+1)^2 \right] = 0.
\end{aligned}$$

Therefore

$$\zeta_{S^{2n+2}}\left(-\tfrac{1}{2}\right) = 0, \qquad d = 2n+3 \geqslant 3. \tag{A.4}$$

A.2. d EVEN ≥ 4

We set $d = 2n+2$. The degeneracy at level l is

$$\begin{aligned}
\delta(l) &= \frac{2(l+n)^2}{(2n)!} \prod_{i=1}^{n-1} \left[(l+n)^2 - i^2 \right] \\
&= \frac{2}{(2n)!} \sum_{k=0}^{n-1} (-1)^k \beta_k(n-1)(l+n)^{2(n-k)},
\end{aligned} \tag{A.5}$$

(for $d=4$ i.e. $n=1$, the value 1 is given to the product $\Pi_{i=1}^{n-1}$) with

$$\beta_k(N)=\sum_{1\leq i_1<\cdots<i_k\leq N} i_1^2\ldots i_k^2, \qquad \beta_0(N)=1. \tag{A.6}$$

Therefore

$$\zeta_{S^{2n+1}}(s)=\frac{2}{(2n)!}\sum_{k=0}^{n-1}(-1)^k\beta_k(n-1)\sum_{l=0}^{\infty}(l+n)^{2(n-k-s)}$$

$$=\frac{2}{(2n)!}\sum_{k=0}^{n-1}(-1)^k\beta_k(n-1)\left[\zeta_R(-2(n-k-s))-\sum_{l=1}^{n-1}l^{2(n-k-s)}\right]. \tag{A.7}$$

Again one notices the simplification

$$\sum_{k=0}^{n-1}(-1)^k\beta_k(n-1)\sum_{l=1}^{n-1}l^{2(n-k-s)}=\sum_{l=1}^{n-1}l^{2(1-s)}\prod_{i=1}^{n-1}(l^2-i^2)=0.$$

For $s=-\frac{1}{2}$, the values of the Riemann zeta function appearing in eq. (A.7) are

$$\zeta_R(-1-2m)=\frac{(-1)^{m+1}B_{m+1}}{2(m+1)}, \tag{A.8}$$

with B_m, the Bernoulli numbers, given by

$$\frac{x}{e^x-1}=1-\tfrac{1}{2}x+\sum_{m=1}^{\infty}(-1)^{m+1}\frac{B_m x^{2m}}{(2m)!},$$

$$B_1=\tfrac{1}{6}, \qquad B_2=\tfrac{1}{30}, \qquad B_3=\tfrac{1}{42}. \tag{A.9}$$

Therefore

$$\zeta_{S^{2n+1}}(-\tfrac{1}{2})=(-1)^{n+1}W_{n+1}, \qquad d=2n+2\geqslant 4; \tag{A.10}$$

the positive constant which appears in eq. (3.10) is

$$W_{n+1}=\frac{1}{(2n)!}\sum_{k=0}^{n-1}\frac{\beta_k(n-1)B_{n+1-k}}{n+1-k} \tag{A.11}$$

The coefficients $\beta_k(N)$ satisfy remarkable properties. By separating the cases $i_k=N+1$ and $i_k\leq N$ one gets the functional equation

$$\beta_k(N+1)=\beta_k(N)+(N+1)^2\beta_{k-1}(N). \tag{A.12}$$

From eq. (A.12) one easily derives by induction that, for fixed k, $\beta_k(N)$ is a polynomial of degree $3k$ in N, whose highest term is

$$\beta_k(N) \underset{\substack{N\to\infty \\ k \text{ fixed}}}{\sim} \frac{N^{3k}}{3^k k!}. \tag{A.13}$$

One also has the value

$$\beta_k(k) = [k!]^2. \tag{A.14}$$

In order to identify the polynomial $\beta_k(N)$ it is sufficient to check properties (A.12) and (A.14) which determine all its integer values. One, thus, gets the following explicit expressions for the first few

$$\begin{aligned}
\beta_0(N) &= 1,\\
\beta_1(N) &= \frac{(N+1)N(2N+1)}{6},\\
\beta_2(N) &= \frac{(N+1)N(N-1)(2N+1)(2N-1)}{8\times 9}\,\frac{(5N+6)}{5},\\
\beta_3(N) &= \frac{(N+1)N(N-1)(N-2)(2N+1)(2N-1)(2N-3)}{2^4 3^4}\,\frac{(35N^2+91N+60)}{35},\\
\beta_4(N) &= \frac{(N+1)N(N-1)(N-2)(N-3)(2N+1)(2N-1)(2N-3)(2N-5)}{2^7 3^5}\\
&\quad\times\frac{(175N^3+735N^2+1046N+504)}{175}.
\end{aligned} \tag{A.15}$$

It seems plausible that all integers and half integers between -1 and $k-1$ are zeros of the polynomial $\beta_k(N)$.

A.3. TORUS $\mathbb{T}^d$

The partition function in d dimensions, eqs. (3.13)–(3.21) is in vector notation,

$$\begin{aligned}
\log Z(\omega_1,\dots,\omega_d) &= \frac{1}{d}\log\frac{(V')^d}{V^{d-1}} + \frac{V}{S_d}\sum_{n_2,\dots,n_d\in\mathbb{Z}}{}' |n_2\omega_2+\cdots+n_d\omega_d|^{-d}\\
&\quad - \sum_{n_2,\dots,n_d\in\mathbb{Z}}{}' \log\left\{1-\exp\left(-2\pi T\left[\epsilon_{n_2,\dots,n_d}+iq_{n_2,\dots,n_d}\right]\right)\right\},
\end{aligned} \tag{A.16}$$

where $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_d$ are the moduli vectors, in the notation of sect. 3. They form the matrix $\omega_{ij} = (\boldsymbol{\omega}_i)_j$; $V = |\det \boldsymbol{\omega}|$; the dual vectors $\boldsymbol{K}_i$ are rewritten as

$$(\boldsymbol{\Omega}_i)_j = (-)^{j+1}\det_{ij}(\omega) = (-)^{i+1}V(\boldsymbol{K}_i)_j, \tag{A.17a}$$

where $\det_{ij}(\omega)$ is the determinant of the minor (ij) of ω. Then

$$T\epsilon_{n_2,\ldots,n_d} = \left|\boldsymbol{\Omega}_1 \wedge \left(-n_2\boldsymbol{\Omega}_2 + n_3\boldsymbol{\Omega}_3 + \cdots + (-)^{d+1}n_d\boldsymbol{\Omega}_d\right)\right| \Big/ (V')^2,$$

$$Tq_{n_2,\ldots,n_d} = \left[\left(-n_2\boldsymbol{\Omega}_2 + n_3\boldsymbol{\Omega}_3 + \cdots + (-)^{d+1}n_d\boldsymbol{\Omega}_d\right)\cdot\boldsymbol{\Omega}_1\right] \Big/ (V')^2, \tag{A.17b}$$

where $|\boldsymbol{a} \wedge \boldsymbol{b}|^2 = (\boldsymbol{a})^2(\boldsymbol{b})^2 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$ in d dimensions and $V' = |\boldsymbol{\Omega}_1|$ is the spatial volume. In three dimensions the wedge product is used for

$$\boldsymbol{\Omega}_1 = \boldsymbol{\omega}_2 \wedge \boldsymbol{\omega}_3, \qquad \boldsymbol{\Omega}_2 = \boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_3,$$

$$\Omega_3 = \boldsymbol{\omega}_1 \wedge \boldsymbol{\omega}_2, \qquad V = \left|\boldsymbol{\omega}_1 \cdot (\boldsymbol{\omega}_2 \wedge \boldsymbol{\omega}_3)\right|, \tag{A.18}$$

and eq. (3.24) is obtained from eq. (A.16).

Appendix B

We list some useful formulas for the computations of sect. 4. The expansion of the Ricci tensor $R_{\mu\nu}$ for the metric $g_{\mu\nu} = e^{2\sigma}\hat{g}_{\mu\nu}$ in terms of σ and $\hat{R}_{\mu\nu}$ for $\hat{g}_{\mu\nu}$ is

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - \hat{g}_{\mu\nu}\left(\hat{\sigma}_\lambda{}^\lambda + (d-2)\hat{\sigma}_\lambda\hat{\sigma}^\lambda\right) - (d-2)\left(\hat{\sigma}_{\mu\nu} - \hat{\sigma}_\mu\hat{\sigma}_\nu\right), \tag{B.1}$$

where $\hat{\sigma}_\mu = \hat{\mathrm{D}}_\mu\sigma$, $\hat{\sigma}^\mu = \hat{g}^{\mu\nu}\hat{\sigma}_\nu$, $\hat{\sigma}_{\mu\nu} = \hat{\mathrm{D}}_\mu\hat{\mathrm{D}}_\nu\sigma$, and $\hat{\mathrm{D}}_\mu$ is the derivative covariant with respect to $\hat{g}_{\mu\nu}$. Moreover, the derivative covariant with respect to $g_{\mu\nu}$ is

$$\mathrm{D}_\mu A_\nu = \hat{\mathrm{D}}_\mu A_\nu - \left(\hat{\sigma}_\nu A_\mu + \hat{\sigma}_\mu A_\nu - \hat{g}_{\mu\nu}\hat{\sigma}_\lambda A^\lambda\right). \tag{B.2}$$

The useful variational formulae, in addition to eq. (2.3), are (see ref. [23])

$$\delta R_{\mu\nu} = -\tfrac{1}{2}\left(g_{\nu\sigma}\mathrm{D}_\tau\mathrm{D}_\mu + g_{\mu\sigma}\mathrm{D}_\tau\mathrm{D}_\nu - g_{\mu\tau}g_{\nu\sigma}\mathrm{D}^2 - g_{\tau\sigma}\mathrm{D}_\nu\mathrm{D}_\mu\right)\delta g^{\tau\sigma}, \tag{B.3}$$

$$\left(\delta\mathrm{D}_\mu\right)A_\nu = -\left(\delta\Gamma^\sigma_{\mu\nu}\right)A_\sigma = \tfrac{1}{2}\left(A_\tau g_{\mu\sigma}\mathrm{D}_\nu + A_\tau g_{\nu\sigma}\mathrm{D}_\mu - g_{\mu\tau}g_{\nu\sigma}A^\lambda\mathrm{D}_\lambda\right)\delta g^{\tau\sigma}. \tag{B.4}$$

In four dimensions the Bianchi identity and $C_{\mu\nu\rho\sigma} = 0$ imply the identity

$$\mathrm{D}_\gamma R_{\alpha\beta} - \mathrm{D}_\beta R_{\alpha\gamma} = \tfrac{1}{6}g_{\alpha\beta}\mathrm{D}_\gamma R - \tfrac{1}{6}g_{\alpha\gamma}\mathrm{D}_\beta R, \tag{B.5}$$

which is useful for deriving ($C_{\mu\nu\rho\sigma}=0$)

$$P^{\nu\alpha}_{\mu\beta}g^{\beta\sigma}\frac{\delta}{\delta g^{\alpha\sigma}}\bigg|_{\sigma\ \text{fixed}}\int d^4x\sqrt{g}\left(R^2_{\lambda\sigma}-\tfrac{1}{3}R^2\right)\sigma$$

$$=P^{\nu\alpha}_{\mu\beta}\sqrt{g}\left[R^{\beta}_{\alpha}\,D^2\sigma+\tfrac{2}{3}R\,D_{\alpha}D^{\beta}\sigma-2R^{\beta\lambda}D_{\lambda}D_{\alpha}\sigma\right], \tag{B.6}$$

where $P^{\nu\alpha}_{\mu\beta}=\frac{1}{2}\delta^{\alpha}_{\mu}\delta^{\nu}_{\beta}+\frac{1}{2}g_{\mu\beta}g^{\nu\alpha}-\frac{1}{4}\delta^{\nu}_{\mu}\delta^{\alpha}_{\beta}$, which is used to derive eq. (4.23). Finally, the following equation

$$e^{4\sigma}\left(R^2_{\mu\nu}-\tfrac{1}{3}R^2\right)\sigma=\left(\hat{R}^2_{\mu\nu}-\tfrac{1}{3}\hat{R}^2\right)\sigma+4\left(\hat{R}_{\mu\nu}-\tfrac{1}{2}\hat{g}_{\mu\nu}\right)\hat{\sigma}^{\mu}\hat{\sigma}^{\nu}$$

$$+2\hat{\sigma}_{\lambda}\hat{\sigma}^{\lambda}\left(3\hat{\sigma}^{\alpha}_{\alpha}+2\hat{\sigma}_{\alpha}\hat{\sigma}^{\alpha}\right)+\text{total derivatives} \tag{B.7}$$

is used for passing from eq. (4.21) to eq. (4.22).

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