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Classical scattering in 2 + 1 gravity with N point sources

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The classical dynamics of N point sources in three-dimensional gravity is considered. An explicit solution is found in the first-order formalism, where the dreibein and the spin connection correspond to a Chern–Simons gauge field on the Poincaré group. If we are not interested in a smooth metric, as in the gauge theory, such a solution exists for arbitrary trajectories and conventional scattering is not defined. Only particle exchanges are meaningful, and are described by the braid group acting on the holonomies of the gauge field. On the other hand, the metric in the Einstein theory must be smooth and invertible, outside the particle trajectories, and fulfil proper asymptotic conditions. We argue that these requirements constrain the asymptotic motion of the particles, so that the two-body scattering problem is well defined. We determine the scattering angle in some special limits and we argue its exact form in the massless case.

1. Introduction

Gravitational theories in 2+1 dimensions have received much attention in the past few years, in the hope of finding a consistent approach to quantum gravity. There are several motivations for such a hope.

On the one hand, the classical Einstein theory in 2+1 dimensions is completely flat outside the sources. Based on this simplifying feature, Deser et al. made considerable progress towards a solution of the classical dynamics [1,2], and of the one-body and perhaps two-body quantum mechanics [3–5].

On the other hand, as suggested by Witten [6] and developed by other authors [7–12], gravity in 2+1 dimensions is connected to the Chern–Simons

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theory on the Poincaré group. This observation confirms the lack of local degrees of freedom in $D = 3$ apart from the sources and constitutes also a basis for an alternative quantization method. However, it needs a deep discussion, because the Chern–Simons theory is purely topological, while gravity, requiring a physical metric, is perhaps not.

Finally, string theory itself [13], compactified to D physical dimensions, has a quantum gravity limit which has been studied at planckian energies [14] and especially for $D = 4$ [15]. This high-energy behaviour could as well be studied for $D = 3$, and compared with the two-body massless scattering results of the previous approaches.

In this paper we reconsider the classical dynamics of point sources from both the Einstein and the Chern–Simons points of view, and in particular the two-body scattering problem. Gravitational theories are thought to be different from gauge theories in this respect. In fact, as emphasized in the pioneering work of Einstein et al. [16], the equations of motion for the sources follow from the Bianchi identities for the field equations, i.e. from reparametrization invariance itself. In practice, however, the formidable non-linearity of this problem in $D = 4$ has not allowed much progress beyond perturbation theory [17], and, eventually, the geodesic approximation in an effective many particle field. Nor it is known in general to what extent the asymptotic trajectories are determined, independently of local reparametrizations at finite times.

In three dimensions the non-linearity is softened by the flatness property and also the ambiguities connected with radiation are avoided, because of the absence of physical gravitons. However, the infrared problem is more severe, and it is still not easy to set in a precise way the asymptotic scattering problem.

We approach this problem in two steps. First, by using a first-order formalism, we exhibit a class of explicit N-particle solutions for the dreibein and the spin connection and therefore for the related Chern–Simons gauge field (sects. 2 and 3). The metrics corresponding to them in the Einstein theory contain generally delta-function singularities. If we are not interested in a smooth metric, such solutions exist for *arbitrary* trajectories, which therefore turn out to be gauge degrees of freedom of the Chern–Simons theory.

Secondly, we look at the metric obtained for such solutions in the Einstein theory by setting for it smoothness (sect. 5) and asymptotic conditions (sect. 6) on physical grounds. We then find that the asymptotic trajectories are restricted in the Einstein theory and we set a method for finding, up to finite-time reparametrizations, a unique metric. The Einstein solution appears, in this respect, as a gauge fixing of the Chern–Simons ones.

In the first step our expressions for the dreibein and the spin connection yield an explicit realization, for any number of particles and any speed, of the matching conditions of ref. [1], which are here related to the Poincaré holonomies of the Chern–Simons theory, studied in sect. 4. They are also connected with recent particular solutions [18] of the Einstein equations in the radial gauge [19] and of the Chern–Simons theory [20–22].

At this level of analysis, the observables are just topological, and given by the loop variables mentioned before. These are global quantities, invariant under smooth deformations of the metric. Since the time evolution of the metric is one kind of smooth deformation, the holonomies are constants of motion of the particle plus field dynamics in the – yet unknown – Einstein space-time [8]. At this stage, the time evolution produces particle exchange, rather than scattering in a real sense. The holonomies induce a representation of the braid group which in turn satisfies the Yang–Baxter equation, as shown in sect. 4. An important feature is that these holonomies are non-abelian, except in the static limit.

In the second step of the analysis, where the Einstein space-time and particle trajectories are determined by smoothness and asymptotic conditions, other observables are to be found, like the classical "scattering matrix" relating outgoing dreibeins and momenta to the incoming ones.

In sect. 5 we give the transformation from singular to smooth coordinates for one particle, so as to yield an isotropic metric in its rest frame. We recover in this way the known conical space-times [1,2,5,23] and their geodesics [24]. The "scattering matrix" is, in this case, the parallel-transport along the infinite geodesic.

The two-body problem is considered in sect. 6. The explicit form of the mapping from singular to Einstein space-time is not easy to obtain, because the Poincaré holonomies of the two particles do not commute. We are able to determine it perturbatively in $G_N E$, for massive particles, and we argue its exact form in the massless, or high-energy, limit.

Next, we determine the scattering angle versus energy relation. We agree with previous work [1,3,7] to first-order in $G_N E$, and we give a new exact formula in the massless case. This analysis involves a choice of "centre-of-mass" frame, which turns out to be different from previous expectations [3,7].

Our results are discussed and compared with previous work in the conclusive section 7, where we also give some suggestions for the related quantum problem.

2. First-order form of Einstein equations and Chern–Simons theory

We start by recalling the problem of motion for pointlike matter sources, with the peculiar simplifications due to working in $2+1$ dimensions. In general, in D space-time dimensions one has to solve the field equations

$$\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2}\mathcal{R}g_{\mu\nu} = T_{\mu\nu} \quad (8\pi G_N = 1) \tag{2.1}$$

with an energy–momentum tensor satisfying the covariant conservation constraint

$$D^{\nu} T_{\mu\nu} = 0 . \tag{2.2}$$

For pointlike sources, $T_{\mu\nu}$ takes the form

$$\begin{aligned} T_{\mu\nu} &= \sum_{(r)} \int m_{(r)} \dot{\xi}^{(r)}_{\mu} \dot{\xi}^{(r)}_{\nu} \frac{\delta^{(D)}(x - \xi^{(r)}(\tau))}{\sqrt{g}} \mathrm{d}\tau \\ &= \sum_{(r)} \frac{p^{(r)}_{\mu} p^{(r)}_{\nu}}{m_{(r)}} \frac{\delta^{(D-1)}(\boldsymbol{x} - \boldsymbol{\xi}^{(r)}(t))}{\sqrt{g}} \frac{\mathrm{d}\tau}{\mathrm{d}t} . \end{aligned} \tag{2.3}$$

In the first line of this equation, we have introduced the trajectory three-vector $\xi^{\mu}_{(r)} = (t, \boldsymbol{\xi}_{(r)})$, and in the second line we have integrated the parameter τ as a function of t. The momenta of the various particles are

$$p^{\mu}_{(r)} = m_{(r)} \frac{\mathrm{d}\xi^{\mu}_{(r)}}{\mathrm{d}\tau} \equiv m_{(r)} \dot{\xi}^{\mu}_{(r)}, \quad v^{\mu}_{(r)} \equiv \frac{\dot{\xi}^{\mu}_{(r)}}{\dot{\xi}^{0}_{(r)}} , \tag{2.4}$$

and the non-covariant velocity v^{μ} will be useful in the following. In principle, one has to solve eq. (2.1) for the field as a functional of the trajectories $\xi^{\mu}_{(r)}(\tau)$, and then restrict the latter by the conservation constraint (2.2), which plays the role of the equations of motion. This gives rise to a formidable non-linear problem, which, so far, has been tackled only in various perturbative approaches [17], following the ideas of Einstein et al. [16].

2.1. FIRST-ORDER FORM OF THE EQUATIONS

In three dimensions most of the non-linearity of the gravitational problem can be avoided by switching to a first-order formalism involving the dreibein $(e^{a}{}_{\mu})$ and spin connection $(\omega^{a}_{\mu b})$. The reason for this simplification lies in the relation between the Riemann and the Einstein tensors, and in the absence of a physical on-shell graviton. More precisely, in $D = 3$, the kinematical relation

$$R_{\mu\nu,\alpha\beta} = -g \epsilon_{\mu\nu\lambda}\, \epsilon_{\alpha\beta\gamma}\, \mathcal{G}^{\lambda\gamma} \tag{2.5}$$

(where $\sqrt{g}\epsilon_{\mu\nu\lambda}$ is the covariant completely antisymmetric tensor) implies, by eq. (2.1), that the full Riemann tensor is determined by the sources

$$\begin{aligned} (R_{\mu\nu})^{\alpha}{}_{\beta} &= (\partial_{[\mu} \Gamma_{\nu]} + [\Gamma_{\mu}, \Gamma_{\nu}])^{\alpha}{}_{\beta} \\ &= -g\, \epsilon_{\mu\nu\lambda} \epsilon^{\alpha}{}_{\beta\gamma} T^{\lambda\gamma} . \end{aligned} \tag{2.6}$$

This equation has the important consequence that the space is flat outside the sources. Let us introduce the spin connection ω_{μ} and the dreibein e_{μ} by the decomposition

$$(\Gamma_{\mu})^{\alpha}{}_{\beta} = \Gamma^{\alpha}_{\mu\beta} = (e^{-1}(\partial_{\mu} + \omega_{\mu}) e)^{\alpha}{}_{\beta} = (e^{-1}(\partial_{\beta} + \omega_{\beta}) e)^{\alpha}{}_{\mu} , \tag{2.7}$$

$$g_{\mu\nu} = e^{a}{}_{\mu} \eta_{ab} e^{b}{}_{\nu} , \tag{2.8}$$

and let us define the Lorentz momenta $P^{a}_{(r)}$ by the relation

$$e^{a}{}_{\mu}|_{(r)} p^{\mu}_{(r)} = P^{a}_{(r)} . \tag{2.9}$$

By introducing eqs. (2.8) and (2.9) into (2.6) and (2.3), we obtain the equations for the spin connection

$$\begin{aligned} (\partial_{[\mu}\omega_{\nu]} + [\omega_{\mu}, \omega_{\nu}])^{a}{}_{b} &= -\epsilon_{\mu\nu\lambda} \epsilon^{a}{}_{bc} t^{\lambda c} , \\ t^{\lambda c} &= \sum_{(r)} v^{\lambda}_{(r)} P^{c}_{(r)} \delta^{(2)}(\boldsymbol{x} - \boldsymbol{\xi}_{(r)}(t)) , \end{aligned} \tag{2.10}$$

where the $\sqrt{g}$ factor has disappeared, while the dreibein is subject to the constraint

$$D_{[\mu} e_{\nu]} = \partial_{[\mu} e_{\nu]} + \omega_{[\mu} e_{\nu]} = 0 , \tag{2.11}$$

due to the absence of torsion ($\Gamma^{\alpha}_{\mu\beta} = \Gamma^{\alpha}_{\beta\mu}$) in our spinless case. Eqs. (2.10) and (2.11) have been derived under the assumption that $g = |e|^{2} \neq 0$ (non-singular dreibein) and do not contain any $\sqrt{g}$ factor. Similarly, by introducing (2.8) and (2.9) in the covariant conservation (2.2), we obtain

$$D_{\mu} t^{\mu a} = \sum_{(r)} \delta^{(2)}(\boldsymbol{x} - \boldsymbol{\xi}_{(r)}(t)) \left[\frac{\mathrm{d}}{\mathrm{d}t} P^{a}_{(r)} + v^{\mu}_{(r)} \omega^{a}_{\mu b} P^{b}_{(r)} \right] = 0 . \tag{2.12}$$

It can be shown that eq. (2.12) is nothing other than the geodesic equation in the first-order formalism,

$$\frac{\mathrm{d}}{\mathrm{d}t} P^{a}_{(r)} + v^{\mu}_{(r)} \omega^{a}_{\mu b} P^{b}_{(r)} = 0 , \tag{2.13}$$

but notice that ω_{μ} is itself a functional of all the trajectories.

In the following we shall attempt to solve the (18+6) equations (2.9)–(2.12) in place of the (6+3) non-linear equations (2.1)–(2.2). Our equations are at most quadratic in $e^{a}{}_{\mu}, \omega^{a}_{\mu b}$, but are still functionals of the trajectories $\xi^{\mu}(\tau)$ and of the Lorentz frame momenta $P^{a}(\tau)$.

2.2. RELATION TO THE CHERN–SIMONS THEORY

It is instructive to compare the previous problem of particles interacting gravitationally with the one of particles interacting with a gauge field with Chern–Simons action, the so-called anyons. Following the work of Witten [6], we consider the gauge field A_μ taking values in the Poincaré group ISO(2, 1),

$$A_\mu = e^a{}_\mu \mathcal{P}_a + \omega^a{}_\mu \mathcal{J}_a , \tag{2.14}$$

where $\omega^a_\mu = -\frac{1}{2}\epsilon^a{}_{bc}\omega^{bc}_\mu$, and the generators satisfy

$$[\mathcal{J}_a, \mathcal{J}_b] = \epsilon_{abc}\mathcal{J}^c , \quad [\mathcal{J}_a, \mathcal{P}_b] = \epsilon_{abc}\mathcal{P}^c , \quad [\mathcal{P}_a, \mathcal{P}_b] = 0 . \tag{2.15}$$

The Chern–Simons action is

$$\begin{aligned} S_{\text{CS}} &= \int \left\langle A, \left(\mathrm{d}A + \tfrac{2}{3}A^2\right)\right\rangle \\ &= -\frac{1}{2}\int \mathrm{d}^3x\, \epsilon^{\mu\nu\rho}\epsilon_{abc}e^a{}_\rho \left(\partial_{[\mu}\omega_{\nu]} + \omega_{[\mu}\omega_{\nu]}\right)^{bc} \end{aligned} \tag{2.16}$$

and the source term is

$$S' = -2\sum_{(r)} \int \mathrm{d}\tau \left[\dot{\xi}^\mu \left(P_a e^a{}_\mu + J_a \omega^a{}_\mu\right)\right]_{(r)} . \tag{2.17}$$

In the first line of eq. (2.16), the gauge indices are contracted by using the invariant metric of ISO(2, 1),

$$\langle \mathcal{J}_a, \mathcal{P}_b\rangle = \eta_{ab} , \quad \langle \mathcal{P}_a, \mathcal{P}_b\rangle = 0 , \quad \langle \mathcal{J}_a, \mathcal{J}_b\rangle = 0 . \tag{2.18}$$

The source action has been generalized to particles with spin $J^a(t)$ producing non-vanishing torsion in eq. (2.11).

The comparison with the first-order Einstein formalism is more transparent in the 4×4 representation of the generators $\mathcal{J}_a, \mathcal{P}_a$ [10], which yields

$$A^A_{\mu B} = \begin{pmatrix} \omega^a_{\mu b} & e^a{}_\mu \\ 0 & 0 \end{pmatrix} , \tag{2.19}$$

where the index $A = a$ for $A = 0, 1, 2$, and is omitted if it takes the value 3. The variation of $S_{\text{CS}} + S'$ with respect to the gauge field gives

$$\begin{aligned} F^a_{\mu\nu b} &\equiv \left(\partial_{[\mu}A_{\nu]} + A_{[\mu}A_{\nu]}\right)^a{}_b \\ &= \sum_{(r)} \epsilon_{\mu\nu\lambda} v^\lambda_{(r)} I^a_{(r)b}\, \delta^{(2)}\left(\boldsymbol{x} - \boldsymbol{\xi}_{(r)}(t)\right) . \end{aligned} \tag{2.20}$$

In the matrix representation one easily sees that this equation reproduces the Einstein equations for (e, ω) (2.10), (2.11), with source matrix

$$I^a_{(r)b} = \begin{pmatrix} -\epsilon^a{}_{bc}P^c_{(r)} & J^a_{(r)} \\ 0 & 0 \end{pmatrix} \tag{2.21}$$

in the spinless case $J^a = 0$. In these formulae, it is natural to allow a spin source J^a, $J^a P_a = m\sigma$ for particles with spin, but only spinless particles can be described in the Einstein metric theory. Therefore, in sect. 3 we shall present a solution to these equations in the spinless case *.

In the Einstein theory, the equation of motion for the particles eq. (2.13) comes from the requirement that they should produce a reparametrization-invariant field, since eq. (2.2) follows from reparametrization invariance of the field equations. The same property holds in the Chern–Simons gauge theory, once the particle action is correctly chosen [10]. Gauge invariance of the field equations (2.20) is expressed by the Bianchi identity

$$\epsilon^{\mu\nu\rho} D_\rho F_{\mu\nu} = 0 , \quad D_\rho \equiv \partial_\rho + [A_\rho, \;] . \tag{2.22}$$

By expanding it in components, it follows that

$$\begin{aligned} \dot{P}^a_{(r)} + \dot{\xi}^\mu_{(r)}\omega^a_{\mu b}P^b_{(r)} &= 0 , \\ \frac{\mathrm{d}J^a_{(r)}}{\mathrm{d}\tau} + \dot{\xi}^\mu_{(r)}\left(\omega^a_{\mu b}J^b_{(r)} + \epsilon^a{}_{bc}P^c_{(r)}e^b{}_\mu\right) &= 0 . \end{aligned} \tag{2.23}$$

These same equations were obtained in ref. [10], by varying a suitable action for the particle in the external Chern–Simons gauge field. The first one is the geodesic equation (2.13), the second one is not empty even in the case $J^a = 0$, in which it implies

$$P^a_{(r)} \propto \dot{\xi}^\mu_{(r)} e^a{}_\mu |_{(r)} .$$

The source of the gauge field in eq. (2.20) is indeed related to the motion of the particles, but the constant of proportionality $m_{(r)}$, i.e. the scale of the energy in the Einstein theory, eq. (2.9) must be fixed from outside the Chern–Simons theory (see later also). Note that the particle equations conserve the values of the two Casimirs of ISO(2, 1), $P^2 = m^2$ and $P^a J_a = m\sigma$, which specify the representation carried by the particle.

Finally, Witten [6] has shown that the symmetries of the Einstein theory in the first-order formalism, reparametrization and local Lorentz invariance, are reproduced on-shell by the gauge transformations of the Poincaré group,

$$\delta A_\mu = D_\mu w , \quad w = \tau^a \mathcal{J}_a + \rho^a \mathcal{P}_a . \tag{2.24}$$

*Spinning particles are discussed in ref. [31].

These relations led to the conclusion that this Chern–Simons theory is a better formulation of three-dimensional gravity, in the familiar context of gauge theories. More precisely, equivalence with the Einstein theory would be reached when the dreibein is invertible, while the gauge formulation makes sense also for configurations $e^a{}_\mu \sim 0$, which should be included as quantum fluctuations for renormalizability of the theory. Actually, Witten was able to show that the Chern–Simons quantum theory is renormalizable (and finite) without particles, when we expand around the classical field $e \sim 0$, $\omega \sim 0$, corresponding to a "short-distance limit" of gravity.

However, there is an important difference between the two approaches, which shows up clearly in the study of particle dynamics. The Chern–Simons theory possesses an additional topological invariance, absent in the Einstein theory. Note that the action (2.16) and the equations of motion (2.20) are expressed in terms of differential forms, defined on a manifold of coordinates x^μ without any need of a metric. Thus the Chern–Simons theory is invariant under smooth deformations of the metric. On the other hand, $e^a{}_\mu, \omega^a_{\mu b}$ in the Einstein theory satisfy the same equations, but it is implicit that they live on the space-time manifold with the specific metric given by the "soldering condition"

$$g_{\mu\nu} = e^a{}_\mu \eta_{ab} e^b{}_\nu . \tag{2.25}$$

This equation is outside the Chern–Simons framework, it is a sort of gauge fixing for it. Actually, it is only under this condition that $S_{CS} = -\frac{1}{2}\int \sqrt{g}R$. This *definition* of the metric gives, by eq. (2.9), the relation between the "internal" momentum P^a and the space-time momentum p^μ, thus it determines, together with the covariant conservation (2.12) the equations of motion for the trajectories $\xi^\mu_{(r)}(\tau)$ on the Einstein space-time. Moreover, a metric comes naturally with its inverse, then the soldering condition implicitly requires that the dreibein is invertible, outside trajectories. Our understanding of particle motion requires a well-specified metric! This is selected by physical requirements external to the Chern–Simons problem, like the behaviour at infinity fixing the inertial frame, and isotropy in the rest frame. They will come about when describing the asymptotic motion and the scattering process in sects. 5 and 6.

3. General form of N-particle solutions

We now describe a class of solutions to eqs. (2.9)–(2.13), which are the same in Einstein and Chern–Simons theories. Most of our discussion (subsect. 3.1) will not require the metric (according to eq. (2.25), because we do not need to raise or lower indices. Thus it applies directly to the Chern–Simons case, and indeed constitutes a general solution for it. However, from the Einstein point of view, the corresponding metrics (2.25) are generally singular, as explained in subsect. 3.2.

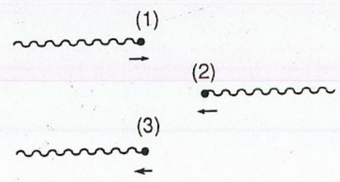


Fig. 1. A possible tail configuration in X-coordinates for the N-particle solution with parallel velocities.

3.1. FORM OF ω AND e

In this section the solutions to eqs. (2.9)–(2.13) will be characterized by constant internal momenta, $P^a_{(r)} = m_{(r)} U^a_{(r)} =$ constant, and the (non-covariant) velocities $V^i_{(r)} = U^i_{(r)}/U^0_{(r)}$ $(i = 1,2)$ will also be parallel to each other ($r = 1,\ldots,N$, for the N particles involved). The generalization to non-parallel ones and non-constant P^a will be discussed in sect. 4.

We start by parametrizing the trajectories $\{x^i = \xi^i_{(r)}(t), i = 1,2\}$ by the equations

$$\begin{aligned} X^2(\xi^\mu_{(r)}) &= B_{(r)} = \text{const.}, \\ X^1(\xi^\mu_{(r)}) &= V_{(r)} X^0(\xi^\mu_{(r)}), \end{aligned} \tag{3.1}$$

where $X^a(x^\mu)$ $(a = 0,1,2)$ are, for the time being, arbitrary functions of the coordinates x^μ. For the trajectories $\xi^i_{(r)}$ to be well defined, we also require

$$\Delta_{(r)} = \left| \frac{\partial(X^1 - V_{(r)}X^0, X^2)}{\partial(x^1, x^2)} \right|_{(r)} > 0 . \tag{3.2}$$

the positive sign being a matter of convention. Eqs. (3.1) represent in the "coordinates" X^a straight lines parallel to the X^1 axis (fig. 1).

We can now state the general form of the solutions to eqs. (2.9)–(2.13). The spin connection and the dreibein are given by

$$\omega_\mu = \sum_{(r)} \omega^{(r)}_\mu , \tag{3.3}$$

$$e^a{}_\mu = \partial_\mu X^a + \sum_{(r)} \left(\omega^{(r)}_\mu\right)^a{}_b (X^b - B^b_{(r)}) , \tag{3.4}$$

where, for each particle (r),

$$\begin{aligned} \left(\omega^{(r)}_\mu\right)^a{}_b &= \epsilon^a{}_{bc} P^c_{(r)} \left(\partial_\mu X^2\right) \delta(X^2 - B_{(r)}) \Theta_{(r)} , \\ \Theta_{(r)} &= -\Theta(X^1 - V_{(r)}X^0), \quad \text{or} \quad \Theta(V_{(r)}X^0 - X^1) . \end{aligned} \tag{3.5}$$

We first note that the spin connection has support along branch cuts, or "tails", located to the right (the left) of each particle according to the choice in eq. (3.5). The tail location can be freely rotated around the branch point by a gauge transformation, provided another tail is not met. Notice that $\omega_\mu^{(r)}$ is proportional to the same matrix $\epsilon^a{}_{bc}P^c_{(r)}$ for all μ, thus $(1+\omega_{(r)})$ is an infinitesimal rotation around the constant vector $P_{(r)}$. It follows that

(i) the ω_μ term in the geodesic equation (2.13) drops out, thus $P^a_{(r)}$ is constant, as assumed in our ansatz;

(ii) $[\omega_\mu^{(r)}, \omega_\nu^{(s)}] = 0$ because of different support ($r \neq s$) or zero commutator ($r = s$).

In order to check that ω_μ in (3.3) is indeed a solution of eq. (2.10), it is sufficient to note that $[\omega_\mu, \omega_\nu] = 0$ and that

$$\epsilon^{\mu\nu\lambda}\partial_{[\mu}\omega^a_{\nu]b} = -\sum_{(r)} \epsilon^a{}_{bc}P^c_{(r)}\epsilon^{\mu\nu\lambda}\partial_\mu(X^1 - V_{(r)}X^0)\partial_\nu(X^2) \\ \times \delta(X^1 - V_{(r)}X^0)\delta(X^2 - B^2_{(r)})\,. \tag{3.6}$$

The r.h.s. of this equation is just the energy–momentum tensor $\epsilon_{abc}t^{c\lambda}$. For $\lambda = 0$ this follows from (3.1) and the identity

$$\delta^{(2)}(\boldsymbol{x} - \boldsymbol{\xi}^{(r)}(t)) = \Delta_{(r)}\delta(X^2 - B)\,\delta(X^1 - V_{(r)}X^0)\,. \tag{3.7}$$

For the remaining values of $\lambda = 1, 2$, eq. (3.6) is consistent with the usual definition of $v^i_{(r)}$ derived from the implicit trajectory equations (3.1).

Subsequently, we check that (2.11) is satisfied. In fact, in a region surrounding the rth branch cut (and no other cut) the dreibein (3.4) takes the form

$$e^a{}_\mu = \left[(\partial_\mu + \omega_\mu^{(r)})(X - B_{(r)})\right]^a \quad \text{(at the } r\text{th branch cut)} \tag{3.8}$$

so that, in the same region,

$$D_{[\mu}e^a_{\nu]} = \left[\partial_\mu + \omega_\mu^{(r)}, \partial_\nu + \omega_\nu^{(r)}\right]^a{}_b(X - B_{(r)})^b \\ = 2\epsilon_{\mu\nu\lambda}v^\lambda_{(r)}\epsilon^a{}_{bc}P^c_{(r)}(X - B_{(r)})^b\delta^{(2)}(\boldsymbol{x} - \boldsymbol{\xi}_{(r)}) = 0\,, \tag{3.9}$$

because $(X - B_{(r)})$ is parallel to $P_{(r)}$ itself. Similarly for the regions surrounding the remaining branch cuts.

Finally, let us verify eq. (2.9). By the ansatz (3.4) we obtain simply

$$e^a{}_\mu|_{(r)} = \partial_\mu X^a|_{(r)} \tag{3.10}$$

because the $\omega^{(r)}$ contribution vanishes for $(X^b_{(r)} - B^b_{(r)}) \propto P^b_{(r)}$. Thus we obtain from (2.9)

$$\left.\frac{\mathrm{d}X^a}{\mathrm{d}\tau}\right|_{(r)} = \dot{\xi}^\mu e^a{}_\mu|_{(r)} = \frac{P^a_{(r)}}{m_{(r)}} = U^a_{(r)}\,. \tag{3.11}$$

Eq. (3.9) is consistent with eq. (2.9) by the trajectory equations (3.1), provided we set

$$P^0_{(r)} = m_{(r)}\frac{\mathrm{d}X^0_{(r)}}{\mathrm{d}\tau}\,, \quad \frac{\mathrm{d}X_{(r)}}{\mathrm{d}X^0_{(r)}} = \frac{P_{(r)}}{P^0_{(r)}} = V_{(r)}\,. \tag{3.12}$$

While the covariant conservation equation (2.2) ensures that $P^a_{(r)}$ is constant, as noted before, eq. (2.9) implies the normal velocity–momentum relations. Therefore, this pair of equations plays the role of the usual Hamilton equations for relativistic particles.

These remarks complete our proof that the ansatz (3.3)–(3.5) is a solution of the field equations *for any choice of* $X^a(x^\mu)$ consistent with eq. (3.2). However, the corresponding metric tensor $g = e^{\mathrm{T}}\eta e$ contains in general δ-function singularities due to the spin connection in (3.4), unless the latter are cancelled in the dreibein by the $\partial_\mu X^a$ term. We shall see in the following that such singular solutions correspond to cut portions of space-time and are therefore not fully acceptable.

In sects. 5 and 6 we shall give a method for constructing regular solutions. But, for the time being, let us analyze in more detail the singular ones.

3.2. MATCHING CONDITIONS

The simplest coordinate choice in eq. (3.1) is to set $x^\mu = \delta^\mu_a X^a$, so that the particle trajectories are straight lines. The dreibein is given by

$$e^a{}_\mu(X) = \delta^a{}_\mu + \sum_{(r)} (\omega_\mu^{(r)})^a{}_b(X - B_{(r)})^b \tag{3.13}$$

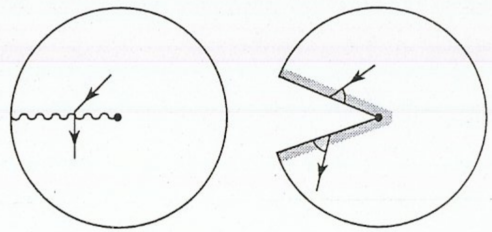
and the metric is therefore simply minkowskian outside the tails, with δ-function singularities along them.

In order to understand the geometrical meaning of such a singular metric, let us study a beam of geodesics crossing the rth tail. The geodesic equation can handle δ-function singularities in the first-order formalism and in an integrated form. As in eq. (2.13), this equation acts on the geodesic curve $x^\mu(\tau)$ as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(e^a{}_\nu\dot{x}^\nu) + \dot{x}^\mu\omega^a_{\mu b}(e^b{}_\nu\dot{x}^\nu) = 0\,. \tag{3.14}$$

By integrating once,

$$e^a{}_\nu\dot{x}^\nu(\tau) = \left(\mathrm{P}\exp - \int_{\tau_0}^{\tau} \mathrm{d}x^\mu\omega_\mu\right)^a{}_b U^b\,, \tag{3.15}$$



a) b)

Fig. 2. (a) Tail representation vs. (b) Minkowski space in singular X-coordinates.

where P is the path ordering along the geodesic, and U is a constant velocity. In the region outside tails $\omega_\mu = 0$, thus the geodesics are straight lines. Across the tail, the tangent vector makes a jump. Near the rth tail (oriented to the left of the particle) we can substitute eq. (3.8) for $e^a{}_\mu$, which contains implicitly the trajectory, and integrate again eq. (3.15). If we evaluate it for a point just above (X_+) and just below (X_-) the tail, we obtain

$$(X_+ - X(\xi_{(r)}))^a = [\exp(P^a_{(r)}\mathcal{J}_a)]^a{}_b(X_- - X(\xi_{(r)}))^b \quad (r\text{th tail}) \tag{3.16}$$

(we used the representation $(\mathcal{J}_a)^b{}_c = \epsilon^b{}_{ac}$ and $(P_{(r)}\cdot\mathcal{J})V_{(r)} = 0$).

According to this result, the loop operator corresponding to the Lorentz transformation

$$L_{(r)} = \exp(-P_{(r)}\cdot\mathcal{J}) \tag{3.17}$$

applied at the tail of particle (r), relates the values of the minkowskian coordinate above and below the tail. For example, a static particle rotates the geodesic of an angle $\delta\phi = m$. If we want to describe the space-time with a one-valued coordinate, we have to cut out an angular sector and identify the edges by a rotation or, in general, by the Lorentz transformation (3.17) (fig. 2).

This interpretation was suggested by Deser et al. [1] and discussed more in general by 't Hooft [3], as an economical way to present their results for one-particle space-times. Solving the Einstein equations in the form (2.1), they obtained for the static particle a conical space-time, represented by the metric

$$ds^2 = dt^2 - (dr^2 + r^2 d\phi^2), \quad 0 < \phi < 2\pi\alpha \tag{3.18}$$

$\alpha = 1 - m/2\pi$, i.e. by Minkowski space-time with an angular limitation.

Similarly, a static particle with spin σ and vanishing mass produces a helical space-time, with metric

$$ds^2 = \left(dt + \frac{\sigma}{2\pi}d\phi\right)^2 - dr^2 - r^2 d\phi^2, \quad -\pi < \phi < \pi \tag{3.19}$$

(the case $m \neq 0$ being a superposition of these two). By introducing the "time" variable $T = t + \sigma\phi/2\pi$, this metric too takes a minkowskian form, but a counterclockwise loop around the particle at fixed time $-t$ brings down to an earlier time

$$T_+ = T_- + \sigma. \tag{3.20}$$

This is the matching condition produced by the spin of the particle. We can also reproduce this result by an extension of our solution. For the sources $P^a = 0, J^a = \delta^a_0\sigma$ located in the origin, and the tail along the negative X^1-axis, this reads

$$\omega_\mu = 0, \quad e^a{}_\mu = \delta^a_\mu - \sigma\delta^a_0\delta^2_\mu\delta(X^2)\theta(-X^1). \tag{3.21}$$

By integrating the geodesic equation with this solution, eq. (3.20) is recovered.

By Poincaré invariance, the previous authors concluded that the general one-particle space-time is minkowskian with excised regions given by the matching condition

$$\begin{pmatrix} X_- \\ 1 \end{pmatrix} = \begin{pmatrix} L_{(r)} & -J_{(r)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X_+ \\ 1 \end{pmatrix} = U\begin{pmatrix} X_+ \\ 1 \end{pmatrix} \tag{3.22}$$

characterized by the invariants $P^2 = m_{(r)}$ and $P\cdot J = (m\sigma)_{(r)}$. They also argued that an N-particle solution could be the Minkowski space-time with an excised region for each particle.

We have just shown that our singular solution is an explicit realization of the ideas of Deser, Jackiw and 't Hooft, and it reproduces their results for one particle. In sect. 4 we shall discuss the N-particle case, by studying the composition of matchings as a composition of holonomies of the Poincaré group.

4. Poincaré holonomies and constants of motion

The observables of the ISO(2,1) Chern–Simons theory are the loop integrals

$$U_\Gamma(y,x) = \mathrm{P}\exp\left(-\int_\Gamma \omega\cdot\mathcal{J} + e\cdot\mathcal{P}\right), \tag{4.1}$$

where Γ is an oriented path from x to y. They can also be considered as a set of non-local degrees of freedom of the theory, which lacks local ones [9]. We shall now compute them by using our previous N-particle solution.

The path can be deformed as long as it does not hit any particle position and the end-points are fixed, because the field is localized at the sources. This is a consequence of the Mandelstam formula $\delta U/\delta\sigma^{\mu\nu} = F_{\mu\nu}U$, where $\delta\sigma^{\mu\nu}$ is a deformation of the path encircling a little surface.

The loops are also invariant under smooth deformations of the metric, because they are given by the integral of a one-form. Their value, computed by our singular solution, will be the same in any topologically equivalent metric, including the smooth space-time solution of the Einstein theory. The particle motion being a kind of topological deformation, it will not affect their value. Thus the holonomies are constants of motion of the particle plus field dynamics, as noticed in ref. [8] in the context of canonical quantization of pure gravity.

Gauge-invariant quantities can be obtained from closed loops, by looking for quantities invariant under $U \to gUg^{-1}$. For Poincaré holonomies, there are two such quantities. They are defined in general for any representation [9], but the spin-one 4×4 representation has physical interest for the relation with the Einstein theory, as discussed in sect. 2. By recalling the form of the generators given there,

$$(\mathcal{J}_a)^A{}_B = \begin{pmatrix} -(\epsilon_a)^b{}_c & 0 \\ 0 & 0 \end{pmatrix}, \quad (\mathcal{P}_a)^A{}_B = \begin{pmatrix} 0 & (\delta_a)^b \\ 0 & 0 \end{pmatrix} \tag{4.2}$$

and by parametrizing

$$U_\Gamma(x,x) = \begin{pmatrix} L = e^{w\cdot\mathcal{J}} & q \\ 0 & 1 \end{pmatrix} \equiv (L, q, 1)\,, \tag{4.3}$$

we find the following two invariants:

$$\sqrt{w^2} \leftrightarrow \mathrm{Tr}_{(1)}L = 1 + 2\cos(\sqrt{w^2}), \tag{4.4}$$

$$\sigma = q^a w_a/\sqrt{w^2} \tag{4.5}$$

($\sigma = q^a w_a$ for $w^2 = 0$). The first invariant is the angle of the Lorentz (pseudo)rotation $\sqrt{w^2}$, the second one is the projection of the translation q^a on the rotation axis [10].

We shall now compute some of these invariants and discuss what information they can give on the particle motion.

4.1. ELEMENTARY HOLONOMIES

Any particle (r) has associated an elementary holonomy $U_{(r)}(x_{(r)}, x_{(r)})$, for a tiny loop surrounding it counterclockwise, with basepoint $x_{(r)}$ near the particle, off the tail. Loops winding around many particles can be decomposed in the building blocks $U_{(r)}$ and the open loops $U_{(r',r)}$ going from (near) particle (r) to particle (r'). The elementary closed loop $U_{(r)}$ can be computed by using the non-abelian Stokes theorem, an integrated form of the previous Mandelstam formula [20]. For a small contour around a point-like sources, this turns out to be the same as the usual abelian one, and it gives, up to a similarity transformation which depends on the choice of basepoint,

$$U_{(r)}(x_{(r)}, x_{(r)}) = \exp(-\int_{\Sigma_{(r)}} \mathrm{d}\sigma^{\mu\nu}F_{\mu\nu}) = \exp(-I(\xi_{(r)}))$$

$$= \left(L_{(r)} = e^{-\mathcal{J}\cdot P_{(r)}},\ \left[-\int_0^1 \mathrm{d}l\, e^{-l\mathcal{J}\cdot P_{(r)}}\right]\cdot J_{(r)},\ 1\right), \tag{4.6}$$

where we used the field equations (2.20). By using the invariance of (4.5) under rotations of axis $w \sim P_{(r)}$, we see that the invariants (4.4)and (4.5) are just the Casimirs of the particle $w^2 = P_{(r)}^2 = m_{(r)}^2$, and $\sigma = (P\cdot J/m)_{(r)} = \sigma_{(r)}$, independent of time as announced.

For holonomies associated to open contours, an explicit solution is needed. We integrate our singular solution (3.3) and (3.5), with the simplest choice of minkowskian coordinates $X^a = \delta^a_\mu x^\mu$. From eq. (4.1) we obtain the equations

$$U\mathrm{d}U^{-1} = A_\mu \mathrm{d}x^\mu\,, \quad U^{-1} = U^{-1}(x, x_0) \equiv (L(x), q(x), 1) \tag{4.7}$$

which can be integrated as a function of the endpoint x. In components,

$$L^{-1}\mathrm{d}L = \omega_\mu \mathrm{d}x^\mu\,, \quad L^{-1}\mathrm{d}q = e_\mu \mathrm{d}x^\mu\,. \tag{4.8}$$

Around the rth tail, we can substitute the solution (3.8) and integrate as in the case of the geodesic equation (3.15). If the path from x_0 to x encircles counterclockwise the (r)-particle, we get

$$U_{(r)}(x, x_0) = \left(\exp(-\int_{x_0}^x \omega_{(r)}),\ x_0 - x,\ 1\right), \quad x_0 = B_{(r)}. \tag{4.9}$$

U is a multivalued function of x, so we should specify the domain of definition. Fig. 3 shows a two-particle configuration, with a particular choice of tails and domains of definition (dashed lines). We can arrange them to

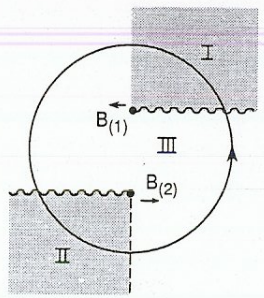


Fig. 3. Regions of definition for the open holonomies $U_{(r)}, U_{(r,r')}, r = 1,2,$and the contour for the O-loop.

have a trivial form of the holonomy in the region III between the particles, at any time

$$\begin{aligned} U_{(1)}(x, B_{(1)}) &= (L_{(1)},\ B_{(1)} - x,\ 1) \quad \mathrm{I} \\ U_{(2,1)}(x, B_{(1)}) &= \ (1,\ \ B_{(1)} - x,\ 1) \quad \mathrm{III} \\ U_{(2)}(x, B_{(2)}) &= (L_{(2)},\ B_{(2)} - x,\ 1) \quad \mathrm{II} \\ U_{(1,2)}(x, B_{(2)}) &= \ (1,\ \ B_{(2)} - x,\ 1) \quad \mathrm{III} \end{aligned} \tag{4.10}$$

For the closed loop around one particle, we recover the previous result. Notice that in this choice of solution, the basepoint can be carried arbitrarily close to the particle in region III by composing trivial holonomies.

4.2. COMPOSITE HOLONOMIES

Let us consider now two particles. The loop encircling them once counter-clockwise, called the O-loop in the following, is given by the ingredients in eq. (4.10), as follows:

$$\begin{aligned} U_O(B_{(2)}, B_{(2)}) &= U_{(2,1)}(B_{(2)}, B_{(1)})\ U_{(1)}(B_{(1)}, B_{(1)}) \\ &\quad \times U_{(1,2)}(B_{(1)}, B_{(2)})\ U_{(2)}(B_{(2)}, B_{(2)}) \\ &= \ (L_{(1)}L_{(2)},\ (L_{(1)} - 1)(B_{(2)} - B_{(1)}),\ 1). \end{aligned} \tag{4.11}$$

The invariants (4.4),(4.5) are computed as follows. The products of Lorentz matrices are computed with Pauli matrices; for the Poincaré translation q^a we use the formula in the $s = 1$ representation

$$\left(\mathrm{e}^{w\cdot\mathcal{J}}\right)^a{}_b = (1 - \cos\sqrt{w^2})\frac{w^a w_b}{w^2} - \sin\sqrt{w^2}\epsilon^a{}_{bc}\frac{w^c}{\sqrt{w^2}} + \cos\sqrt{w^2}\delta^a{}_b. \tag{4.12}$$

The Poincaré invariant σ and the Lorentz invariant $\sqrt{w^2}$ are called in this case S and $\mathcal{M}$ respectively. They read

$$\begin{aligned} S \sin\frac{\mathcal{M}}{2} &= 2s_{(1)}s_{(2)}\epsilon_{abc}(B_{(1)} - B_{(2)})^a \frac{P^b_{(1)}P^c_{(2)}}{m_{(1)}m_{(2)}} \to S = \frac{\Delta Bp}{\sqrt{1 - p^2/4}}, \\ \cos\frac{\mathcal{M}}{2} &= c_{(1)}c_{(2)} - s_{(1)}s_{(2)}\frac{P_{(1)}\cdot P_{(2)}}{m_{(1)}m_{(2)}} \to \cos\frac{\mathcal{M}}{2} = 1 - \frac{p^2}{2}, \end{aligned} \tag{4.13}$$

where $c_{(i)} = \cos(m_{(i)}/2)$, $s_{(i)} = \sin(m_{(i)}/2)$. We also showed them in the massless limit $P_{(1,2)} \to (p, \pm p, 0)$.

These two formulas agree with the result of Deser, Jackiw and 't Hooft [1], obtained by the composition of matching conditions for each particle. It was partly explained in terms of holonomies in ref. [7], which gave the Lorentz part of eq. (4.2). Let us pause to explain this relation better. In terms of invariants, we have found two of the infinite conservation laws of the Chern–Simons theory for the two-particle system. By comparison with the one-particle loop, we can interpret $\mathcal{M}$ as the total invariant mass and S as the total angular momentum. Actually, for small momenta they reduce to the usual formulas of special relativity, and for large momenta they are dressed by the gravitational field. On the other hand, Deser, Jackiw and 't Hooft made a parallel statement in the non-invariant language of minkowskian metrics with excised regions. The matching condition of the one-particle metric was associated to mass and spin, which are the invariants of the one-particle holonomy. By analogy, the O-loop suggests that there is a choice of coordinates, which at large space-like distances $|\boldsymbol{x}| \gg |\boldsymbol{B}|$ is minkowskian with a single excised region and jump in time determined by $\mathcal{M}$ and S in eq. (4.2). Therefore the rule they used for the composition of matching conditions is justified, because it is a composition of holonomies, interpreted within a specific coordinate system. Note however, that the choice of this coordinate frame is not unique, depending on the base point (e.g. $B_{(1)}$ or $B_{(2)}$), and on the tail location. Moreover, for a smooth metric a different frame will be introduced in sect. 6.

Let us show other examples of two-particle holonomies. The whole set is generated by $U_{(1)}$ and $\hat{U}_{(2)} = U_{(1,2)}U_{(2)}U_{(2,1)}$, i.e. $U_{(2)}$ parallel-transported near (1). This non-abelian group characterizes the topological class of the — yet unknown— two-particle smooth space-time, a kind of non-abelian orbifold.

The previous O-loop probed the two-particle space-time at large space-like distances. On the other hand, the loop winding around them at large time-like distances can be deformed into an 8-loop at constant time, i.e. encircling the particles with opposite orientations. Its invariants are

$$\begin{aligned}
\cosh\frac{\alpha}{2} &= c_{(1)}c_{(2)} + s_{(1)}s_{(2)}\frac{P_{(1)}\cdot P_{(2)}}{m_{(1)}m_{(2)}} \\
&\to \cosh\frac{\alpha}{2} = 1 + \frac{p^2}{2}, \\
\Sigma \sinh\frac{\alpha}{2} &= -2s_{(1)}s_{(2)}\epsilon_{abc}(B_{(1)} - B_{(2)})^a \frac{P^b_{(1)}P^c_{(2)}}{m_{(1)}m_{(2)}} \\
&\to \Sigma = -\frac{\Delta Bp}{\sqrt{1+p^2/4}} = -\cos\frac{\theta}{2}\Delta Bp.
\end{aligned} \tag{4.14}$$

It is natural to guess that a probe of large time-like distances should somehow measure the scattering of particles, but this point is unclear without a smooth metric. We only remark, in the massless limit of the Σ invariant in eq. (4.14), the occurrence of the scattering angle θ to be found in eq. (6.30).

The next example is the "commutator" loop $\Delta = U^{-1}_{(1)}\hat{U}^{-1}_{(2)}U_{(1)}\hat{U}_{(2)}$, which measures the non-abelian nature of the holonomies. Static particles do not interact gravitationally, and have abelian holonomies. Thus it is tempting to relate Δ to a measure of interaction between particles. Its invariants are

$$\begin{aligned}
\cosh\frac{\lambda}{2} &= 1 + 2(s_{(1)}s_{(2)})^2\left(\left(\frac{P_{(1)}\cdot P_{(2)}}{m_{(1)}m_{(2)}}\right)^2 - 1\right) \to \cosh\frac{\lambda}{2} = 1 + \frac{p^4}{2} \\
\Gamma \sinh\frac{\lambda}{2} &= -8(s_{(1)}s_{(2)})^2\frac{P_{(1)}\cdot P_{(2)}}{m_{(1)}m_{(2)}}\epsilon_{abc}(B_{(1)} - B_{(2)})^a\frac{P^b_{(1)}P^c_{(2)}}{m_{(1)}m_{(2)}}.
\end{aligned} \tag{4.15}$$

After these examples, we can ask how much these invariants describe the classical dynamics. Actually the latter is expressed in terms of trajectories, which are determined once a physical metric is chosen, as will be described in sects.5 and 6. Classical dynamics is seen as a gauge fixing of the Chern–Simons theory, and it has also associated non-invariant quantities, like open loops going to $\pm\infty$ for the scattering angle. Thus, the previous invariants cannot give a complete description.

Nevertheless, to pursue the topological-invariant Chern–Simons description is interesting for the quantum theory. As shown by Witten [6], exact reparametrization invariance is preserved in the pure gravity case, and it is an open problem whether this breaks (asymptotically) after the inclusion of particles. In the broken case the classical picture, described in the following sections, will go through to the quantum theory. In the unbroken case, the invariants issued from holonomies are the only observables, and particle dynamics is very limited. For example, the invariant description of two particles obtained so far is as follows:

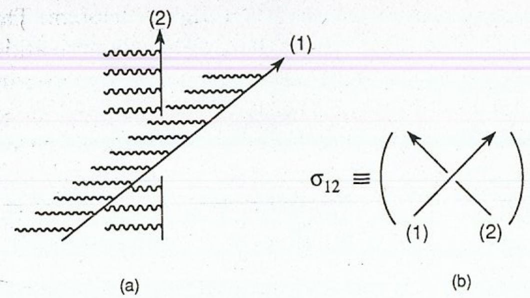


Fig. 4. (a) Possible tail crossing configuration for the solution with non-parallel velocities and (b) the exchange operation σ_{12}.

(i) elementary holonomies $U_{(r)}$ define the mass and spin of each particle;

(ii) the holonomy of the O-loop define their total mass and angular momentum, the latter in particular distinguishes between colliding and parallel moving particles.

The next invariant concept in Chern–Simons theory is that of *particle exchanges*, replacing the ill-defined one of scattering. This is the subject of subsect. 4.3.

4.3. PARTICLE EXCHANGES AND THE YANG–BAXTER EQUATION

In the solutions of sect. 3, crossings of tails were not allowed. However, for three or more particles with non-parallel velocities, they necessarily cross, thus we shall extend the solution to this case.

Without loss of generality, consider the case of particle (2) crossing the tail of particle (1) (fig. 4).

$P^a_{(2)}$ is no longer constant, and it evolves according to the integrated eq. (2.13)

$$P_{(2)}(x^+) = \mathrm{P}\exp\left(-\int_{x^-}^{x^+}\omega\right)P_{(2)}(x^-), \tag{4.16}$$

where $x^+(x^-)$ correspond to $t>0$ $(t<0)$, the collision being at $t=0$. If we ignore the feedback of (2) on (1) (test-particle limit), this equation gives the matching condition of the geodesic equation (3.16)

$$P_{(1)}(x^+) = P_{(1)}(x^-), \quad P_{(2)}(x^+) = L_{(1)}P_{(2)}(x^-). \tag{4.17}$$

However, for two interacting particles, we have to care about the feedback and we have to test this hypothesis back in the equation of motion for (e,ω).

Since the source is discontinuous, an integrated form of these equations is needed, and is provided by the non-abelian Stokes theorem. This is described in detail in ref. [20], thus we shall give here a sketchy derivation of the result.

Let us reconsider the elementary holonomy Γ around a particle, eqs. (4.1) and (4.6), but parallel-transport it to a fixed basepoint x_0. The surface integral of $F_{\mu\nu}$ is carried on a disk D with $\partial D = \Gamma$. Since this can be deformed to cut the trajectory at different times, we conclude that the holonomy around a trajectory is independent of time, when transported back to a fixed basepoint

$$U_\Gamma(x_0, x_0) = U^{-1}(x(t), x_0) U_\Gamma(x(t), x(t)) U(x(t), x_0) . \qquad (4.18)$$

This is the integrated form of the equation of motion. Let us apply it to the previous crossing in fig. 4a. The holonomy around particle (1) can be computed at any time without winding around (2), thus it is trivially parallel-transported to the basepoint $B_{(1)}$ at $t = 0^-$,

$$\begin{aligned} U_{(1)}(B_{(1)}, B_{(1)}) &= U^{-1}\, U_{(1)}(B_{(1)} + V_{(1)}t, B_{(1)} + V_{(1)}t)\, U \\ &= U_{(1)}(B_{(1)} + V_{(1)}t, B_{(1)} + V_{(1)}t) , \end{aligned} \qquad (4.19)$$

where $U \equiv U(B_{(1)} + V_{(1)}t, B_{(1)})$ is the trivial translation of region III in this case. Therefore, eq. (4.19) implies that $P_{(1)}$ is constant, and the first line of eq. (4.17) is checked. For particle (2), eq. (4.18) is applied with basepoint $B_{(2)}$, again at $t = 0^-$,

$$U_{(2)}(B_{(2)}, B_{(2)}) = U^{-1}(x_{(2)}(t), B_{(2)})\; U_{(2)}(x_{(2)}(t), x_{(2)}(t))\; U(x_{(2)}(t), B_{(2)}) . \qquad (4.20)$$

Transporting the holonomy to the trajectory $x_{(2)}(t)$ for $t > 0$ requires the contour to pierce at $t = 0$ the surface spanned by the tail (1), i.e. winding (1) first counterclockwise then clockwise. Using the two particle holonomies in eq. (4.10), we get

$$U(x_{(2)}(t), B_{(2)}) = \left(L_{(1)},\; L_{(1)}(B_{(2)} - B_{(1)}) + B_{(1)} - x_{(2)}(t),\; 1\right) , \qquad (4.21)$$

where $x_{(2)}(t)$ and $P_{(2)}(t), (t > 0)$ are unknown variables. By plugging this form of U in eq. (4.20), it follows that

$$\begin{aligned} x^+_{(2)} - B_{(1)} &= L_{(1)}(B_{(2)} - B_{(1)}) , \\ L^+_{(2)} &= L_{(1)} L^-_{(2)} L^{-1}_{(1)} . \end{aligned} \qquad (4.22)$$

This is nothing other than the usual matching condition for the test particle (2) crossing cut (1), computed at the end of sect. 3 and in eq. (4.17). We conclude that the full equations of motion imply the absence of feedback, as assumed before. Therefore, the solutions for (e, ω) in which tails are crossed can be built at any time on the basis of eqs. (4.17)–(4.22), which are boundary conditions for the local solution of sect. 3.

Since the momenta P_a change in the crossing of tails, one could think that this is a picture for scattering in the Chern–Simons theory [3,7]. However, it has nothing to do with classical scattering, which is the relation between outgoing asymptotic trajectories and incoming ones, e.g.

$$p^\mu_{(r)}(t = \infty) = R^\mu{}_\nu(\theta) p^\nu_{(r)}(t = -\infty) ,$$

where R is a rotation of scattering angle θ (see sect. 6). The similarity was only due to our choice of gauge $X^a = \delta^a_\mu x^\mu$ and tail orientations, which fixes the matching condition (4.17). It is possible to put the tail crossing operation in an invariant form, such that it holds for *arbitrary* $X^a(x^\mu)$, as the solutions of sect. 3. Thus it involves only the topology of the trajectories and manifestly gives no conditions on the $p^\mu_{(r)}$. Therefore, a correct name for the process of tail crossing during time evolution is *particle exchange*.

More precisely, let us define a particle exchange operator σ_{12} (fig. 4b), acting on the tensor space $V_{(1)} \otimes V_{(2)}$ of Poincaré holonomies of the two particles. They are generated by the elementary holonomies $U_{(1)}, \hat{U}_{(2)} = U_{(1,2)} U_{(2)} U_{(2,1)}$ with a common basepoint, say $B_{(1)}$, as seen before,

$$U_{(1)} = (L_{(1)}, 0, 1) , \quad \hat{U}_{(2)} = (L_{(2)}, (L_{(2)} - 1)(B_{(1)} - B_{(2)}), 1) . \qquad (4.23)$$

The effect of particle (2) crossing the tail of (1) is given by the action

$$\sigma_{12} : \begin{cases} L_{(1)} \to L_{(1)} & , B_{(1)} \to B_{(1)} \\ L_{(2)} \to L_{(1)} L_{(2)} L^{-1}_{(1)} & , B_{(2)} \to B_{(1)} + L_{(1)}(B_{(2)} - B_{(1)}) \end{cases} , \qquad (4.24)$$

where $L_{(r)} = \exp(-P_{(r)} \cdot \mathcal{J})$, and $B_{(r)}$ are the positions of the particles just before the collision time. Notice that this can be written in the form

$$\sigma_{12} : \begin{cases} U_{(1)} \to U_{(1)} \\ \hat{U}_{(2)} \to U_{(1)} \hat{U}_{(2)} U^{-1}_{(1)} \end{cases} \qquad (4.25)$$

which is manifestly topological invariant. Gauge transformations at the basepoint change it by an overall conjugation. Moreover, a different tail orientation, i.e. a different choice of basepoint, would have given the operator σ_{21}, thus exchanging the (asymmetric) role of the two particles. The full monodromy of the particle (2) around (1) is obtained by braiding twice, i.e. by the product $\sigma_{12}\sigma_{21}$, which is independent of tail orientations. The action of σ_{12} in eq. (4.25) was given for the Lorentz part in ref. [7], and also discussed in the context of non-abelian anyons in ref. [25].

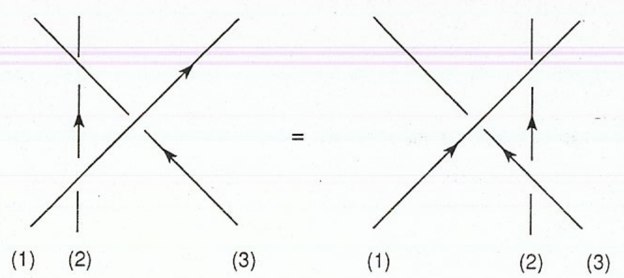


Fig. 5. Sequence of exchanges for the Yang–Baxter equation.

Next, the exchanges of N particles follow by composition of the operators $\sigma_{i,i+1}$, $i = 1, ..., N-1$, which generate the braid group $\mathcal{B}_N$. They act on tensor spaces $\mathrm{V}_{(r)}$ of particle holonomies, all with the same basepoint. It is well known that the generators $\sigma_{i,i+1}$ should satisfy the Yang–Baxter equation

$$\sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \quad \text{for} \quad k = j+1 = i+2, \tag{4.26}$$

$$\sigma_{i,i+1}\sigma_{j,j+1} = \sigma_{j,j+1}\sigma_{i,i+1} \quad \text{for} \quad |i-j| \geqslant 2, \tag{4.27}$$

where the first equation acts on $\mathrm{V}_{(i)} \otimes \mathrm{V}_{(j)} \otimes \mathrm{V}_{(k)}$ and the second one on $\mathrm{V}_{(i)} \otimes \mathrm{V}_{(i+1)} \otimes \mathrm{V}_{(j)} \otimes \mathrm{V}_{(j+1)}$. This equation expresses associativity of particle exchanges, as required by the deformability of trajectories (fig. 5).

Let us now verify the Yang–Baxter equation in our case, eq. (4.25), for a triple of particles $(1,2,3)$. The two sides of eq. (4.26) read

$$\begin{aligned}
\sigma_{23}\sigma_{13}\sigma_{12}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} &= \sigma_{23}\sigma_{13}\begin{pmatrix} U_1 \\ U_1U_2U_1^{-1} \\ U_3 \end{pmatrix} \\
&= \sigma_{23}\begin{pmatrix} U_1 \\ U_1U_2U_1^{-1} \\ U_1U_3U_1^{-1} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_1U_2U_1^{-1} \\ (U_1U_2)U_3(U_1U_2)^{-1} \end{pmatrix}, \\
\sigma_{12}\sigma_{13}\sigma_{23}\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} &= \sigma_{12}\begin{pmatrix} U_1 \\ U_2 \\ U_1U_2U_3(U_1U_2)^{-1} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_1U_2U_1^{-1} \\ (U_1U_2)U_3(U_1U_2)^{-1} \end{pmatrix},
\end{aligned} \tag{4.28}$$

Indeed, the Yang–Baxter equation should be identically satisfied at the classical level. We cannot expect conditions on the momenta, which would

mean a classical motion only for some initial condition. On the other hand, eq. (4.28) shows that this equation is not completely trivial in this non-abelian case, and it gives a consistency check of our previous analysis on crossings.

The YB equation is expected to be a powerful condition at the quantum level, and to constrain the scattering amplitudes, as in the case of purely elastic scattering of two-dimensional integrable models [26]. For non-abelian anyons, this has been recently discussed in refs. [27,28]. It is hoped that the loop algebra studied here will be useful in this respect.

5. Smooth one-particle metric and its geodesics

We take, from now on, the point of view of looking at the gravitational problem, for which the behaviour of the metric tensor $g = e^{\mathrm{T}}\eta e$ is important. We want to avoid the δ-function singularities present in the general solution (3.3)–(3.5), which lead to the cuts in Minkowski space discussed in sect. 4.

Let us recall [1] the well-known case of one static particle. For a given time the space is a cone, with vertex on the particle, which can be described in singular (X)-coordinates as the Minkowski space with an angular limitation, i.e.

$$\mathrm{d}s^2 = \mathrm{d}X_0{}^2 - (\mathrm{d}R^2 + R^2\mathrm{d}\phi^2), \quad |\phi| \leqslant \pi\alpha, \tag{5.1}$$

where $\alpha = 1 - m/2\pi$ is the deficit angle, in units $8\pi G_{\mathrm{N}} = 1$.

It can also be described by embedding the conical space in $\mathbb{R}^3$, which corresponds to rescaling the angular variable in eq. (5.1), as follows:

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}t^2 - (\mathrm{d}r^2 + r^2\alpha^2\mathrm{d}\varphi^2), \quad |\varphi| \leqslant \pi \\
&= \mathrm{d}t^2 - (\mathrm{d}x^i)^2 + (1-\alpha^2)(\delta_{ij} - \frac{x_ix_j}{r^2})\mathrm{d}x^i\mathrm{d}x^j,
\end{aligned} \tag{5.2}$$

where $x^i \equiv (x^1, x^2) \equiv (x, y)$. This time there is no cut, but the space is not minkowskian, and the metric tensor is only singular at the particle location, and smooth and single-valued elsewhere. The explicit angular deficit has disappeared, even if the circumference-to-radius ratio is still $2\pi\alpha < 2\pi$.

5.1. THE Λ-TRANSFORMATION FOR THE DREIBEIN AND THE METRIC

We want to apply systematically to our solutions of sect. 3, the singular coordinate change which transforms (5.1) into (5.2). In the case of a static particle, the general solution with tail on the left reads

$$\omega^a_{\mu b} = m\epsilon^a_{b0}\delta(X^2)(\partial_\mu X^2)\Theta(-X^1), \quad e^a{}_\mu = [(\partial_\mu + \omega_\mu)X(x)]^a. \tag{5.3}$$

If we set $x = X$, the metric becomes (5.1). The angular rescaling from (5.1) to (5.2) can be interpreted in (5.3) as the transformation

$$X = \exp\left(\frac{m}{2\pi}\varphi(x)\mathcal{J}_0\right) x \equiv \Lambda(x)x\,, \tag{5.4}$$

where $\mathcal{J}_0$ is the generator of rotations. The Λ-rotation varies from $-m/2$ to $m/2$ when φ varies from $-\pi$ to π and thus generates a solution to the matching condition for the X-variable,

$$X_+ = \exp(m\mathcal{J}_0)\, X_- \tag{5.5}$$

as in eq. (3.16). As a consequence, x is continuous, as it should be. In the continuous coordinate x, the dreibein reads

$$\begin{aligned} e^a{}_\mu &= [(\partial_\mu + \omega_\mu)\Lambda x]^a = \Lambda^a{}_b[(\partial_\mu + \Omega_\mu)x]^b\,, \\ \Omega^a_{\mu b} = (\Lambda^{-1}\partial_\mu\Lambda)_{\text{reg}} &= \frac{m}{2\pi}\partial_\mu\varphi(\mathcal{J}_0)^a{}_b \\ &= \begin{cases} 0 & (\mu = 0) \\ \frac{m}{2\pi}\epsilon_{ij}\frac{x^j}{r^2}\epsilon^a_{b0} & (\mu = i) \end{cases}\,, \end{aligned} \tag{5.6}$$

where we notice that the discontinuous behaviour of Λ at $\varphi = \pm\pi$ has cancelled the δ-function singularity present in ω_μ. Thus in more detail we have

$$e^a{}_\mu = \Lambda^a{}_b(\delta^b_\mu + \frac{m}{2\pi}n^b n_\mu)\,, \quad n_\mu = (0, \epsilon_{ij}\hat{x}^j) \tag{5.7}$$

and the metric tensor becomes the one in (5.2), as expected.

This method is trivially extended to the case of one particle with any speed V, say in the x-direction. By applying a Lorentz boost $B_1(V)$ to the transformation (5.4) we obtain

$$\begin{aligned} X &= \exp\left(\frac{\varphi^0(x)}{2\pi}P\cdot\mathcal{J}\right)x = \Lambda_P(x)x\,, \\ \tan\varphi^0 &= \frac{1}{\gamma}\frac{y}{x - Vt}\,, \quad P^\mu = m(\gamma, V\gamma, 0)\,, \end{aligned} \tag{5.8}$$

where φ^0 is the azimuthal angle in the particle rest frame, and $\gamma = 1/\sqrt{1-V^2}$. By repeating the steps in (5.6) we obtain of course the dreibein

$$\begin{aligned} e^a{}_\mu &= (\Lambda_P)^a{}_b\left(\delta^b_\mu + \frac{m}{2\pi}n^b(P)n_\mu(P)\right)\,, \\ n^a(P) &= -\left(\mathcal{J}\cdot\frac{P}{m}\right)^a{}_b x^b\left[\left(\mathcal{J}\cdot\frac{P}{m}x\right)^2\right]^{-1/2} \\ &= \gamma\,(Vy, y, -x + Vt)\left[\gamma^2(x - Vt)^2 + y^2\right]^{-1/2} \end{aligned} \tag{5.9}$$

and the metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + (1-\alpha^2)n_\mu(P)n_\nu(P)\,. \tag{5.10}$$

A particularly interesting limit is the massless case, with fixed energy E, obtained from (5.8) by setting

$$m \to 0 \quad (\gamma \to \infty)\,, \quad \text{with} \quad m\gamma = E\,. \tag{5.11}$$

From the explicit expression of n_a in (5.9) we obtain

$$ds^2 = 2du\,dv - (dy)^2 + \sqrt{2}E\delta(u)|y|(du)^2\,, \tag{5.12}$$

where we have used the representation

$$\frac{\gamma y^2}{\gamma^2 u^2 + y^2} \to \pi|y|\delta(u) \tag{5.13}$$

and the light-cone variables

$$u = \frac{t-x}{\sqrt{2}}\,, \quad v = \frac{t+x}{\sqrt{2}}\,.$$

Eq. (5.12) is nothing but the Aichelburg–Sexl metric [23] for a massless particle in $D = 3$.

In the same limit we also have the expression for the dreibein

$$\begin{aligned} e^a{}_\mu &= \left[L(P)^{-\epsilon(y)\theta(u)/2}\right]^a{}_b\left(\delta^b_\mu + \frac{E}{\sqrt{2}}|y|\delta(u)\delta^b_v\delta^u_\mu\right)\,, \\ L(P) &= \exp(-P\cdot\mathcal{J})\,, \quad \epsilon(y) = \text{sign}(y)\,, \end{aligned} \tag{5.14}$$

where we have introduced the loop matrix $L(P)$ of sect. 4. The pattern of the Λ-transformation in the massless limit is Lorentz contracted as shown in fig. 6, and in particular the Λ-transformation takes the value $L(P)^{-1/2}$ ($L(P)^{1/2}$) above (below) the branch cut. In coordinates (u, v, y), $P^a = \sqrt{2}E\delta^a_v$ and $L(P)^{1/2}$ reads

$$L(P)^{1/2} = \begin{pmatrix} 1 & 0 & 0 \\ E^2/4 & 1 & -E/\sqrt{2} \\ -E/\sqrt{2} & 0 & 1 \end{pmatrix}\,, \quad (P^2 = 0)\,. \tag{5.15}$$

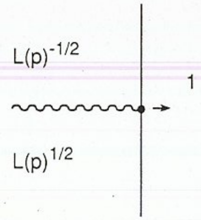


Fig. 6. Lorentz contracted configuration of the Λ-mapping in the massless limit. The full line indicates the metric's wavefront.

5.2. TEST PARTICLE SCATTERING

The Λ-mapping is particularly useful to describe the motion of a test particle in the one-particle metric described before. In fact we already know that, in X-coordinates, the geodesics are straight lines,

$$X^a(\tau) = U^a\tau + B^a = \Lambda^a{}_b(\varphi^0(\tau))x^b(\tau) \tag{5.16}$$

whenever they do not cross the tail (subsect. 3.2). Eq. (5.16) can be inverted for x, by noting that eq. (5.4) is equivalent to the angular rescaling

$$\varphi^0(\tau) = \frac{\phi^0(\tau)}{\alpha}, \tag{5.17}$$

where ϕ^0 denotes the azimuthal X-coordinate in the rest frame. We thus obtain the trajectory

$$x^a(\tau) = \Lambda^{-1}\left(\frac{\phi^0(\tau)}{\alpha}\right)^a{}_b(U^b\tau + B^b). \tag{5.18}$$

If the X-geodesic meets the tail, $\phi^0(\tau)$ will reach the value $\phi^0 = \pi\alpha$ for some τ-value. In such a case X jumps by $L(P)$, but the x-coordinate stays continuous because of the transformation $\Lambda(\mp\pi\alpha/\alpha) = L^{\pm 1/2}(P)$. Therefore the geodesic in x-space is obtained by continuing (5.18) on the Riemann sheet of the X-coordinate, i.e. by displacing the tail.

It is thus easy to discuss the asymptotic motion and scattering of the test particle. The asymptotic three-velocity becomes by eq. (5.18)

$$u_\pm(\infty) = \Lambda^{-1}(\pm\frac{\pi}{\alpha})u_\pm(-\infty) = L(P)^{\pm 1/2\alpha}u_\pm(-\infty), \tag{5.19}$$

where the $+(-)$ sign holds according to whether the geodesic runs above (below) the incident particle (fig. 7).

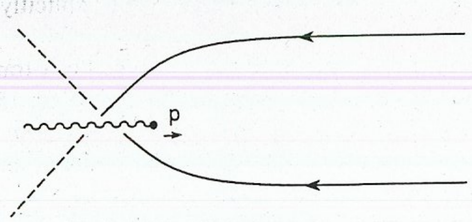


Fig. 7. Upper (lower) geodesics in the space-time of one particle with momentum P.

From eq. (5.19) we obtain the following results:

(i) The geodesic scattering angle off a particle of mass m at rest is [4,5]

$$\theta_0 = \frac{m}{2\alpha} = \frac{m}{2}\left(1 - \frac{m}{2\pi}\right)^{-1}. \tag{5.20}$$

(ii) For a particle of mass m and momentum P, the geodesic velocity undergoes the Lorentz transformation (5.19). For a geodesic moving in the same direction but opposite to P, this corresponds to the scattering angle

$$\tan\frac{\theta}{2} = \tan\frac{\theta_0}{2}\left(\frac{E+P}{m}\right). \tag{5.21}$$

(iii) In particular, for a massless particle, the geodesic scattering angle becomes

$$\tan\frac{\theta}{2} = \frac{E}{2}, \tag{5.22}$$

as is known for the case of an Aichelburg–Sexl metric [24].

The above results show that the gravitational problem for test particles is characterized by quantities which are not purely topological, and in our case are given in terms of the Christoffel connection by

$$\mathrm{P}\exp\left(-\int_{-\infty}^{\infty} \mathrm{d}x^\mu\Gamma_\mu\right)_\pm = [L(P)]^{\pm 1/2\alpha}, \tag{5.23}$$

where the path runs along the geodesic above (below) the point source of momentum P.

One may wonder to what extent this result is gauge dependent.

In principle, one may choose to change the Λ-transformation in the particle rest frame, by some reparametrization of the azimuthal variable φ^0. This, however, will make the scattering angle in general dependent on the direction of the probe in an arbitrary way. Since this violates the physical notion of isotropy of space for a spinless particle, we shall explicitly exclude it in the following.

In other words, we require on physical grounds the Λ-transformation to be given (at large distance, i.e. for $r \to \infty$) by eq. (5.4) or (5.8), which define the isotropic single-particle metric in our framework. Then the result (5.23), due to the properties of parallel-transport, will be invariant under any local reparametrization which is asymptotically consistent with our choice.

This remark shows that, due to the non-compact nature of space-time, the monodromy requirement for the metric is far-reaching and may indeed define the scattering problem in an unambiguous way.

To summarize, in the singular X-coordinates (in which particles go simply on straight lines outside the branch cuts) the geodesic scattering is ambiguous, because of the possibility that a branch cut is crossed. On the other hand, in the smooth x-coordinate, the geodesic scattering angle is uniquely determined in the rest-frame (with the isotropic choice) and therefore also in any other frame, giving rise to the more general "S-matrix" in eq. (5.23).

6. The scattering problem

We concentrate now on the problem of two (interacting) particles in the Einstein theory. In order to define the scattering problem, we look for a metric tensor given in terms of our solutions of sect. 2, $g = e^{\mathrm{T}}\eta e$, which satisfies some physical smoothness and asymptotic conditions that we discuss below. The latter give rise to a gauge fixing $X^a = \overline{X}^a(x^\mu)$ in our general parametrization of sect. 2.

6.1. SMOOTHNESS CONDITIONS

We want our metric to be singular only at the particle sites $x^\mu = \xi^\mu_{(r)}(\tau)$, $(r = 1, 2)$ and thus regular and single valued otherwise. We will look at a gauge fixing $X^A = \overline{X}^A$, $(A = 0, 1, 2, 3)$ of the form

$$\overline{X}^A = (\mathcal{T}(x))^A{}_B x^B, \tag{6.1}$$

where

$$\mathcal{T}(x) = \Lambda(x)T(x) \tag{6.2}$$

is a Poincaré transformation chosen in such a way as to build a solution of the matching conditions around each particle. In other words, we require that, for x^μ close to $\xi^\mu_{(r)}(\tau)$,

$$\mathcal{T} \simeq \mathcal{T}_{(r)}(x)S_{(r)}(x), \quad x^\mu \simeq \xi^\mu_{(r)}(\tau), \tag{6.3}$$

where

$$\mathcal{T}_{(r)} = \mathrm{e}^{B_{(r)}\cdot\mathcal{P}}\Lambda_{(r)}\left(\varphi^0_{(r)}\right)\mathrm{e}^{-B_{(r)}\cdot\mathcal{P}} \quad (r = 1, 2)$$

$$\Lambda_{(r)} = \exp\left(\frac{\varphi^0_{(r)}(x)}{2\pi}P_{(r)}\cdot\mathcal{J}\right) \tag{6.4}$$

is the (properly translated) single particle transformation discussed in sect. 5, and $S_{(r)}$ is instead regular at that point.

From the definition (6.4) it follows that, close to $\xi^\mu_{(r)}$, $\mathcal{T}$ performs the change of variables

$$X - B_{(r)} = \exp\left(\frac{\varphi^0_{(r)}(x)}{2\pi}P_{(r)}\cdot\mathcal{J}\right)\left(S_{(r)}x - B_{(r)}\right), \tag{6.5}$$

where $\varphi^0_{(r)}$ is the azimuthal variable in the c.m. frame, so as to build a solution of the matching conditions at $\varphi^0_{(r)} = \pm\pi$. In this way the discontinuity of the variable X^a is factorized in the $\Lambda_{(r)}(x)$ transformation and the x^μ variable turns out to be continuous.

It is clear at this point that the particle trajectories in (6.5) must satisfy the equation

$$\xi_{(r)}(\tau) = S^{-1}_{(r)}(\xi_{(r)})(B_{(r)} + \frac{P_{(r)}}{m_{(r)}}\tau), \tag{6.6}$$

where $P_{(r)}$ are the constant momenta of sect. 2. Note that the limit $x^\mu \to \xi^\mu_{(r)}$ in (6.6) is well defined because the singular transformation $\Lambda_{(r)}(x)$, belonging to the little group of $P_{(r)}$, acts trivially on $P_{(r)}$.

Eq. (6.6) yields the trajectories only in an implicit way, because the definition of $S_{(r)}$ (and $\mathcal{T}_{(r)}$) is itself dependent on the singularity point at $x^\mu = \xi^\mu_{(r)}(\tau)$.

A particularly simple application of this "gauge fixing" is the case of two or more static particles. Concentrating on the Lorentz part $\Lambda(x)$ of eq. (6.2), it is clear that a solution to all the conditions (6.3) is provided by the composed transformation

$$\Lambda(x) = \prod_r \Lambda_{(r)}(x), \quad \Lambda_{(r)} = \exp\left(\frac{\varphi^0_{(r)}}{2\pi}m_{(r)}\mathcal{J}_0\right) \tag{6.7}$$

because the $\Lambda_{(r)}$'s, which are just rotations, commute among themselves. Eq. (6.7) gives rise to a multi-conical space-time simply related to the conformal metric description of ref. [1].

In the case of interest for scattering, in which the particles move, it is not immediately clear how to find a solution to eq. (6.3) because the $\Lambda_{(r)}$'s do not commute. Furthermore we need to impose further conditions, which correspond to the definition of asymptotic states, as we shall now discuss.

6.2. ASYMPTOTIC CONDITIONS

Although $\Lambda_{(1)}$ and $\Lambda_{(2)}$ do not commute in general, their commutator becomes vanishingly small when the particles are far apart. Thus we can impose the initial condition

$$\mathcal{T}(x) \to \mathcal{T}_{(1)}(x)\mathcal{T}_{(2)}(x) \simeq \mathcal{T}_{(2)}(x)\mathcal{T}_{(1)}(x) T \to -\infty \tag{6.8}$$

for fixed spatial coordinates and large negative times. This ensures that the class of metrics we are interested in is asymptotically consistent with the gauge fixing for a single particle given in sect. 5.

Furthermore, the momenta of the incident particles turn out to be the constants $P_{(1)}, P_{(2)}$ of our solution. This follows from the expression of the dreibein in eq. (5.9)

$$\begin{aligned}(\Lambda^{-1}e)^a{}_\mu &= \left(\delta^a_\mu + \frac{m_{(1)}}{2\pi} n^a_{(1)} n_{(1)\mu} + \frac{m_{(2)}}{2\pi} n^a_{(2)} n_{(2)\mu}\right)\left(1 + \mathrm{O}\left(\frac{|x|}{|T|}, \frac{B}{|T|}\right)\right), \\ n^a_{(r)} &= n^a(P_{(r)})\end{aligned} \tag{6.9}$$

which is asymptotically additive. By setting $\boldsymbol{P}_{(1)}$ and $\boldsymbol{P}_{(2)}$ along the x-axis, the $n_{(r)}$'s in eq. (5.9) point asymptotically along the y-axis

$$n^a_{(1)} \simeq n^a_{(2)} \simeq -\delta^a_2 \tag{6.10}$$

and thus we obtain

$$e^a{}_\mu \simeq \delta^a{}_\mu - \frac{m_{(1)} + m_{(2)}}{2\pi} \delta^a_2 \delta^2_\mu . \tag{6.11}$$

Therefore,

$$p^\mu_{(1)}(-\infty) = \delta^\mu_a P^a_{(1)}, \quad p^\mu_{(2)}(-\infty) = \delta^\mu_a P^a_{(2)}, \tag{6.12}$$

as stated. For finite values of $|\boldsymbol{x}|$, the asymptotic metric is not completely minkowskian for $m_{(r)} \neq 0$,

$$g_{\mu\nu} \to \eta_{\mu\nu} + (1 - \alpha^2)\delta^2_\mu \delta^2_\nu, \quad \alpha \equiv 1 - \frac{m_{(1)} + m_{(2)}}{2\pi} \tag{6.13}$$

but one can check that the parallel-transport from (1) and (2) to the central region $|x| \ll T$ is instead trivial. Thus in this region we can define the total momentum in the naive way

$$P^\mu = P^\mu_{(1)} + P^\mu_{(2)}, \quad P_\mu = g_{\mu\nu}(-\infty)P^\nu = \eta_{\mu\nu}P^\nu \tag{6.14}$$

and an invariant mass squared

$$s = (P_{(1)} + P_{(2)})^2 = \eta_{\mu\nu}P^\mu P^\nu . \tag{6.15}$$

Finally, we also define a "centre-of-mass" frame in the usual way by setting

$$P_{(1)} = (E_{(1)}, p, 0) = m_{(1)}\gamma_1(1, V_{(1)}, 0), \tag{6.16}$$

$$P_{(2)} = (E_{(2)}, -p, 0) = m_{(2)}\gamma_2(1, -V_{(2)}, 0). \tag{6.17}$$

There is an additional asymptotic condition that we need, at fixed time and large $|x|$, in order to avoid rotating frames at infinity. This is obtained by requiring that the Christoffel connection Γ_μ be as small as dimensionally allowed, e.g.,

$$\Gamma_\mu = \mathrm{O}\left(\frac{1}{|\boldsymbol{x}|}\right) \quad (|\boldsymbol{x}| \to \infty, \quad T \text{ fixed}). \tag{6.18}$$

In contrast, in a rotating frame Γ_μ would be of order $(\Omega^2|\boldsymbol{x}|)$.

6.3. PERTURBATIVE SOLUTION

A simple, but non-trivial, solution of the gauge-fixing conditions for scattering is obtained in perturbation theory, i.e. for $GE_{(r)} \ll 1$ and for any speed $V_{(r)}$. This is analogous to the "fast", weak coupling, approximation used in four dimensions [17].

In fact, to first non-trivial order, the Λ-mapping can be linearized in the form

$$\begin{aligned}\Lambda(x) &= 1 + \frac{\varphi^0_{(1)}}{2\pi} p_{(1)} \cdot \mathcal{J} - \frac{\varphi^0_{(2)}}{2\pi} p_{(2)} \cdot \mathcal{J}, \\ \tan\varphi^0_{(1)} &= \frac{1}{\gamma_1}\frac{y - b/2}{x - V_{(1)}t}, \quad \tan\varphi^0_{(2)} = -\frac{1}{\gamma_2}\frac{y + b/2}{x + V_{(2)}t},\end{aligned} \tag{6.19}$$

where the particle trajectories are taken at zeroth order (straight lines), and $b \simeq B_{(1)} - B_{(2)}$ is the relative impact parameter.

It is easily seen that eq. (6.19) satisfies all conditions in subsects. 6.1 and 6.2. First, the Λ-transformation is approximately of product type

$$\Lambda \simeq \Lambda_{(1)}\Lambda_{(2)} \simeq \Lambda_{(2)}\Lambda_{(1)} \tag{6.20}$$

because the commutator terms are of second order in the $GE_{(r)}$'s. This takes care of both the polydromy and initial state conditions to this order. Finally there are no rotations at infinity, because the angles $\varphi^0_{(r)}$ are body-fixed with the straight-moving particles, so that $\Gamma_\mu = \mathrm{O}(1/|\boldsymbol{x}|)$ as it should.

The simple $\Lambda(x)$ transformation in eq. (6.19) still yields asymptotically a non-trivial scattering. In fact from eq. (6.6) we obtain in this case

$$p_{(r)}(\infty) = \Lambda^{-1}(\infty)p_{(r)}(-\infty) \quad (r = 1, 2) \tag{6.21}$$

with

$$\Lambda^{-1}(\infty) \simeq 1 - \frac{1}{2}(p_{(1)} + p_{(2)}) \cdot \mathcal{J} \simeq R(\frac{\sqrt{s}}{2}) \qquad (6.22)$$

because $\varphi^0_{(1)}(\infty) = -\varphi^0_{(2)}(\infty) \simeq \pi$. Thus, the first-order c.m. scattering angle is just given by

$$\theta = \frac{\sqrt{s}}{2}(1 + \mathrm{O}(Gm_{(r)})) \quad (8\pi G_{\mathrm{N}} = 1), \qquad (6.23)$$

as expected from previous results [3,15], to this order.

6.4. EXACT SOLUTION FOR THE MASSLESS CASE

The zero mass, or high-energy limit $m_{(r)} \ll \sqrt{s}$, has some simplifying features which allow an exact solution. In fact the Λ-transformation shows in this case a Lorentz contracted form of a shock wave, as discussed in sect. 5. Therefore, the initial state in the c.m. frame has the simple form of fig. 8a, with the following features:

(1) The mapping from X-to x-coordinates is trivial for $|x| < |t|$, $t < 0$, so that $x^\mu = \delta^\mu_a X^a$ in this region, where the metric is purely minkowskian, too.

(2) The mapping is non-trivial, but piecewise constant in the remaining regions, with the form

$$\begin{aligned}(X - b/2) &= \begin{cases} L_1^{-1/2}(x - b/2), & Y > b/2, \quad X < T \\ L_1^{+1/2}(x - b/2), & Y < b/2, \quad X < T \end{cases} \qquad (T < 0)\\ (X + b/2) &= \begin{cases} L_2^{1/2}(x + b/2), & Y > -b/2, \quad X > -T \\ L_2^{-1/2}(x + b/2), & Y < -b/2, \quad X > -T \end{cases} \end{aligned} \qquad (6.24)$$

where the L's are the loop variables, given by $L_r = L(P_{(r)})$, with $L(P_{(1)})^{1/2}$ as in eq. (5.16) and $L_2^{-1} = L_1^T$, and

$$P_{(1),(2)} = E(1, \pm 1, 0), \quad \sqrt{s} \simeq 2E, \quad b \simeq B_{(1)} - B_{(2)}. \qquad (6.25)$$

(3) The x variable is continuous by construction across the tails (unlike the X variable), but is discontinuous across the wavefronts $x = \pm t$, where the metric takes the Aichelburg–Sexl form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}y^2 + \sqrt{2}E\left(|y - b/2|\delta(u)\mathrm{d}u^2 + |y + b/2|\delta(v)\mathrm{d}v^2\right) \quad (t < 0). \qquad (6.26)$$

In the following, we shall refer to the x-variables, measured in the reference frame between the particles as "internal" variables, and to those measured in

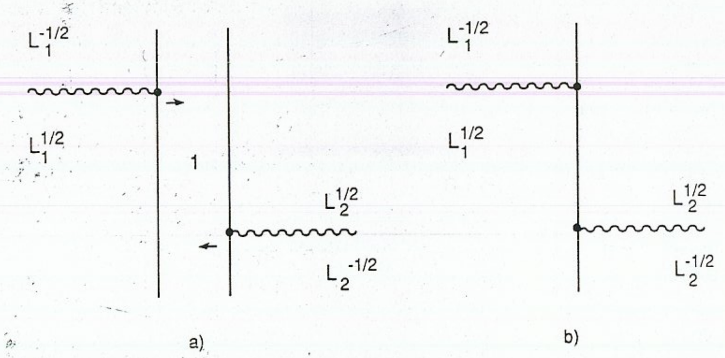


Fig. 8. (a) Initial state and (b) $t = 0^-$ pattern of the Λ-mapping for the scattering of two massless particles.

the remaining regions as "external" ones. The internal variables coincide with the X-variables for $t < 0$.

The above picture holds up to $t = 0$. Here the picture changes according to the coordinate system. In the singular X-coordinates nothing peculiar happens, the particles keep going straight, so that $X|_{(1)}$ and $X|_{(2)}$ are continuous, and $P_{(1)}$ and $P_{(2)}$ remain constant. On the other hand, in the x-coordinates, the shock waves collide at $t = 0$, by exposing the region of external coordinates in which the shock wave 1 (2) is seen by the transformation $L_2^{1/2}$ ($L_1^{1/2}$) (see fig. 8b).

In order to avoid rotating frames at infinity, one would like the shock wave location, in such external coordinates, to stay continuous at large distances during evolution to positive times. Thus we shall look for a Λ-mapping at $t > 0$ which matches that condition. The internal coordinates instead, which are trapped between the shock waves, will presumably acquire a discontinuity.

Since the $P_{(r)}$'s are constant, and therefore the discontinuities across the wavefronts and tails are fixed, the $t > 0$ mapping is still piecewise constant in the regions bounded by wavefronts and tails, and uniquely determined by the polydromy requirements of subsect. 6.1 up to a Lorentz transformation that we define as follows:

$$R^{-1} = \lim_{\epsilon \to 0} \mathrm{P}\exp\left(-\int_{t=-\epsilon}^{t=+\epsilon} \Gamma_\mu \mathrm{d}x^\mu\right)\Big|_{x=0}. \qquad (6.27)$$

Thus R^{-1} parametrizes the non-trivial evolution of the internal x-coordinates, which lie between the wavefronts.

The solution for the $t > 0$ Λ-mapping is shown in fig. 9a. The corresponding dreibein satisfies (by construction) the non-abelian Stokes theorem for the

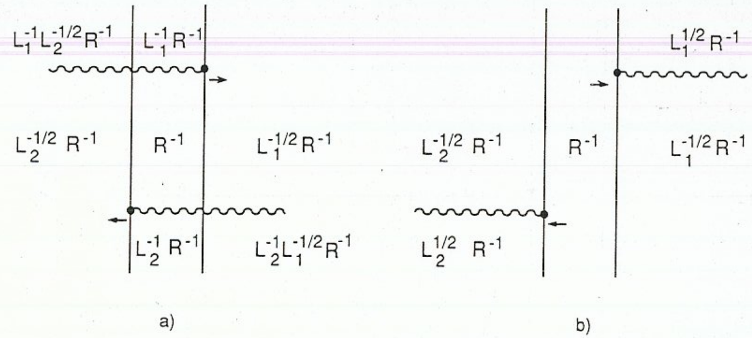


Fig. 9. (a) $t > 0$ solution for the Λ-mapping and (b) final configuration after switching of tails.

Christoffel connection. The main difference with the $t < 0$ situation is that Λ is non-trivial in the internal region where the wavefronts have crossed each other. In particular, we have

$$\Lambda = \begin{cases} R^{-1}, & |X| < T, \quad |Y| < b/2 \\ L_1^{-1/2}R^{-1}, & X > T, \quad Y > b/2 \\ L_2^{-1/2}R^{-1}, & X < -T, \quad Y < b/2 \end{cases} \quad (T > 0) \tag{6.28}$$

where the relevant regions are parametrized in X-coordinates.

On eqs. (6.24-6.24) and (6.28), we still have to impose the $t = 0$ matching conditions of absence of rotation at infinity in the external x-coordinates. There are two such conditions, one for the wavefront of particle (1) seen externally to particle (2) $(X > T)$ and vice-versa, corresponding to the matching of $L_2^{1/2}$ with $L_1^{-1/2}R^{-1}$ and of $L_1^{1/2}$ with $L_2^{-1/2}R^{-1}$. Since in either case the wavefront in X-coordinates points asymptotically in the Y-direction, we obtain the equations

$$L_2^{-1/2}e_2 = RL_1^{1/2}e_2 \qquad \left(e_2 \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{wavefront (1)}\right)$$

$$L_1^{-1/2}e_2 = RL_2^{1/2}e_2 \qquad (\text{wavefront(2)}). \tag{6.29}$$

By using the representations (5.16) for $L_1^{1/2}$ and $L_2^{1/2}$, eqs. (6.29) are solved by a pure rotation $R(\theta)$, where the rotation angle θ is given by

$$\tan\frac{\theta}{2} = \frac{E}{2} = \frac{\sqrt{s}}{4} \qquad (8\pi G_N = 1). \tag{6.30}$$

Notice also that $R(\theta)$ satisfies the relations

$$L_2^{1/2}RL_1^{1/2} = L_2^{-1/2}R^{-1}L_1^{-1/2} = B(\chi), \tag{6.31}$$

where the boost $B(\chi)$ leaves e_2 invariant, and the pseudo angle is $\chi = \log(1 + E^2/4)$.

The final configuration after scattering is better seen by switching the strings outwards (fig. 9b), i.e. by performing the local Lorentz transformation

$$\mathcal{L}(x) = \begin{cases} L_1 & Y > b/2 \\ 1 & |Y| < b/2 \\ L_2 & Y < -b/2 \end{cases}. \tag{6.32}$$

Such a transformation does not affect the metric, because it is constant across the wavefronts, and yields the final form of the Λ-mapping*:

$$\begin{aligned} X &= R^{-1}x, & |X| &< T \\ X - b/2 &= L_1^{\pm 1/2}R^{-1}(x - b_R/2), & X &> T \\ X + b/2 &= L_2^{\mp 1/2}R^{-1}(x + b_R/2), & X &< -T \end{aligned} \quad (T > 0) \tag{6.33}$$

where $b_R = R \cdot b$.

We see from eq. (6.33) that the final state differs from the initial state just by the R^{-1} transformation in between the particles, which defines the "S-matrix" of the problem. Thus the final momenta are

$$p^{\mu}_{(r)}(\infty) = R^{\mu}{}_{a}P^{a}_{(r)} = R^{\mu}{}_{\nu}(\theta)p^{\nu}_{(r)}(-\infty) \quad (r = 1, 2), \tag{6.34}$$

and eq. (6.30) yields the c.m. scattering angle to all orders in $G_N\sqrt{s}$.

We notice that the final expression for θ is the same as in eq. (5.23) for the geodesic of particle (1) [(2)] in the field of particle (2) [(1)] given by the Aichelburg–Sexl metric. The complete result is, however, different. In fact the geodesic S-matrices are given, according to eq. (5.24), by $L_2^{-1/2}$ or $L_1^{-1/2}$. Such transformations do not conserve the energy [23,24], while the complete solution shows just a rotation, which implies energy conservation.

7. Discussion

We have shown that our many-particle solutions to the Chern–Simons field on the one hand give an explicit realization of Deser–Jackiw–'t Hooft space-times [1,3] in singular X-coordinates (sects.3 and 4). Moreover, they also give rise to a well-defined scattering problem in smooth x-coordinates (sects.5,6).

*We concentrate here on the Lorentz part of Λ.Th ecomplete solution [32] shows also a $|b|$-dependent coordinate shift which does not affect the scattering angle.

The explicit form of the mapping from X- to x-coordinates given in particular cases (sect. 6) sheds some light on the problem of scattering in our framework and points towards its natural resolution.

First, we have made it clear that the particle trajectories are arbitrary in the Chern–Simons theory, where one should talk of particle exchange, which is topologically meaningful, rather than scattering.

Secondly, we have pointed out that one should set proper smoothness and asymptotic requirements on the metric, in order to define the scattering problem for the Einstein theory. In particular we have required the metric to be single valued, and to reduce asymptotically to a superposition of one-particle metrics, each one isotropic in the respective rest frame.

Based on such requirements, we have found that in the two-body massless case the c.m. scattering angle is given by $\tan(\theta/2) = 2\pi G_N\sqrt{s}$. This value recalls the geodesic scattering angle in an Aichelburg–Sexl metric, but the detailed description of scattering (subsect. 6.4) differs from the geodesic one.

The comparison of our scattering results with previous work on the two-body problem [3,7] is not easy, because it seems to us that they were mostly focused on the quantum problem, and did not clarify their results on classical scattering and reference frames.

We only comment on a possible scattering interpretation of the O-loop holonomy L_1L_2, which defines globally the invariant mass $\mathcal{M}$ of eq. (4.13), and involves a rotation, up to a similarity transformation. One can interpret such a rotation in singular X-coordinates with tails on the same side as the explicit deficit angle of a cut Minkowski space, for distances much larger than the particle separation. Is this "deficit angle" $\mathcal{M}$ directly related to a two-body scattering angle of the Einstein theory?

Our approach to this question takes a different route. In fact, we have looked for a smooth metric, where no tail appears, first for one-particle, and then for two, with a proper asymptotic condition. This led us to a "c.m. reference frame" different from the one above. In our frame, the one particle metrics are isotropic in their rest-frame but the two-body metric is anisotropic because of the relative speed, leading to shock-waves in the massless limit. This explains why our scattering angle is not related to $\mathcal{M}$ in a simple way, except to first-order in $G_N\sqrt{s}$.

The above remarks lead to the more general question of whether, and how, asymptotic states and scattering matrix can be defined in three-dimensional quantum gravity. At the classical level explored so far it was clear that metric requirements, and in particular the smoothness of $g_{\mu\nu} = O(\eta_{\mu\nu})$ were essential to define the scattering problem in the Einstein theory. In contrast, only topological observables appeared in the Chern–Simons theory, which does not need a metric, so that $e^a{}_\mu$ was allowed to be singular and non-invertible.

A similar question arises at the quantum level. The cut Minkowski space-

time is useful, because it allows a simple computation of particle amplitudes [3,10,11,25]. Actually, the amplitude for N particles can probably be obtained, at least in principle, by the relation between Chern–Simons theory and two-dimensional conformal field theory [27–29]. However, in order to interpret them as measurable scattering amplitudes, one needs to transform them into physical smooth coordinates, fulfilling proper asymptotic conditions. Therefore, we think the main problem is to understand how this smooth Einstein metric emerges in the quantum theory.

Our classical analysis suggests that, if the Chern–Simons theory (coupled to matter) leads to a consistent quantization of gravity, then it should be broken at large distances so as to build a scale and a smooth Einstein metric, and to allow the definition of asymptotic states and the scattering matrix. In the opposite situation of unbroken gauge invariance it is probable that no scattering matrix exists, due to the topological nature of the Chern–Simons theory, and the classical scattering features explored so far would appear to be just an artificial gauge fixing.

The answer to this alternative is not known so far and remains to be investigated, perhaps along the lines of a recent four-dimensional model [30]. We only remark that also string theory calculations [15], if extended to low dimensions, could shed some light on the quantum version of gravity, and check its classical limit.

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References

[1] S. Deser, R. Jackiw and G. 't Hooft, Ann. Phys. (NY) 152 (1984) 220
[2] A. Staruszkiewicz, Acta Phys. Pol. 24 (1963) 735;
J.R. Gott and M. Alpert, Gen. Rel. Grav. 16 (1984) 243;
S. Giddings, J. Abbot and K. Kuchar, Gen. Rel. Grav. 16 (1984) 751
[3] G. 't Hooft, Commun. Math. Phys. 117 (1988) 685
[4] S. Deser and R. Jackiw, Commun. Math. Phys. 118 (1988) 495
[5] P. Sousa Gerbert and R. Jackiw, Commun. Math. Phys. 124 (1989) 229
[6] E. Witten, Nucl. Phys. B311 (1988-89) 46; B323 (1989) 113
[7] S. Carlip, Nucl. Phys. B324 (1989) 106
[8] V. Moncrief, J. Math. Phys. 30 (1989) 2907
[9] S.P. Martin, Nucl. Phys. B327 (1989) 178
[10] P. Sousa Gerbert, Nucl. Phys. B346 (1990) 440
[11] K. Koehler, F. Mansouri, C. Vaz and L. Witten, Nucl. Phys. B348 (1991) 373
[12] J.E. Nelson and T. Regge, Nucl. Phys. B328 (1989) 190, B339 (1990) 516
[13] M. B. Green, J.H. Schwartz and E. Witten, Superstring theory (Cambridge Univ. Press, Cambridge, 1987)
[14] D. Amati, M. Ciafaloni and G. Veneziano, Int. J. Mod. Phys. A3 (1988) 1615;
D.J. Gross and P.F. Mende, Nucl. Phys. B303 (1988) 407

[15] D. Amati, M. Ciafaloni and G. Veneziano, Nucl. Phys. B347 (1990) 550
[16] A. Einstein, L. Infeld and B. Hoffmann, Ann. Math. 39 (1938) 65
[17] J. Goldberg, *in* Gravitation, Introduction to current research, ed. L. Witten (Wiley, New York, 1962) p.102
[18] P. Menotti and D. Seminara, Ann. Phys. (N.Y.) 208 (1991) 449
[19] G. Modanese and M. Toller, J. Math. Phys. 31 (198) 452
[20] A.P. Balachandran,M. Bourdeau and S. Jo, Int. J. Mod. Phys. A5 (1990) 2423
[21] K. Gupta and A. Stern, Spin and statistics in 2 + 1 gravity, preprint Syracuse 1991, SU-4228-463
[22] G. Grignani and G. Nardelli, Phys. Lett. B264 (1991) 45; Gravity in 2+1 dimensions coupled to point particles: a flat Chern–Simons theory equivalent to Einstein's, preprint MIT CPT-1953 (1991)
[23] P.C. Aichelburg and R.U. Sexl, Gen. Rel. Grav. 2 (1971) 303;
T. Dray and G. 't Hooft, Nucl. Phys. B253 (1985) 173
[24] V. Ferrari, P. Pendenza and G. Veneziano, Gen. Rel. Grav. 20 (1988) 1185
[25] F. Wilczek and Y.S. Wu, Phys. Rev. Lett. 65 (1990) 13;
E. Guadagnini, M. Martellini and M. Mintchev, Nucl. Phys. B336 (1990) 581
[26] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 235
[27] E. Witten, Commun. Math. Phys. 121 (1989) 351;
S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B326 (1989) 108;
A. Schwimmer, *in* Recent developments of CFT, ICTP Trieste October 1989, ed. S. Randjbar-Daemi et al. (World Scientific, Singapore, 1990)
[28] E. Verlinde, A note on braid statistics and the non-abelian Aharonov–Bohm effect, preprint IASSNS-HEP-90/60, Proc. Int. Colloq. on modern QFT, Bombay, January 1990
[29] G.V. Dunne, A. Lerda and C.A. Trugenberger, Int. J. Mod. Phys. B5 (1991) 1675
[30] D. Amati and G. Veneziano, Nucl. Phys. B204 (1982) 451;
D. Amati and J. Russo, Symmetry restoration in spontaneously broken induced gravity, SISSA preprint 86 EP (1990)
[31] A. Cappelli, M. Ciafaloni and P. Valtancoli, Classical scattering in 2+1 gravity with N spinning sources, preprint CERN TH-6248/91, Phys. Lett.B, to be published
[32] A. Cappelli, M. Ciafaloni and P. Valtancoli, to be published