

PERTURBATION THEORY OF HIGHER-SPIN CONSERVED CURRENTS OFF CRITICALITY

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Minimal conformal theories perturbed by the least relevant field are an interesting and computable framework for studying the Zamolodchikov currents of higher spin, which suggest the integrability of the off-critical theory. After clarifying the perturbative technique, a generalization of the c -theorem for currents of any spin is given and applied in this context. Mixing of descendant fields in the infrared limit of the renormalization-group flow is shown. The Ward identity for conservation of the spin-four current is given, and shown to encode a dynamical symmetry of the theory away from criticality.

1. Introduction

The study of massive two-dimensional field theories has been pursued in the recent literature [1–10]. This is a natural step in the program of investigating the space of two-dimensional field theories, once conformal field theories (CFTs) are fairly understood [11]. One motivation is the search of exact results in field theory corresponding to integrable lattice models of statistical mechanics [12]. Moreover, understanding the renormalization group (RG) flow between two CFTs is a sine-qua-non condition to carry on the string program.

Zamolodchikov has opened three new avenues in the study of off-critical phenomena:

(i) He proves the so-called c -theorem [1], an exact result which follows from conservation of the stress tensor. Whenever two unitary CFTs are connected by a

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RG trajectory, this theorem implies the qualitative condition on the ultraviolet and infrared central charges $c_{UV} \geq c_{IR}$. This can be interpreted as a measure of the irreversibility of the RG in two dimensions. The theorem also implies a quantitative constraint, a sum rule for the correlation of the trace of the stress tensor, derived by Cardy [13].

(ii) He adapted the RG and the ϵ -expansion [14] to studying the scaling region around a CFT* [3]. This perturbation expansion around non-gaussian (yet, computable) theories will be called conformal perturbation theory.

(iii) Finally, he showed examples of minimal CFTs perturbed by relevant operators which possess additional conserved currents of higher spin [15], due to the existence of null vectors [5]. This observation enabled Zamolodchikov to conjecture the exact S -matrix [16] for the Ising model in a magnetic field [17]. This conjecture passed the numerical tests [18], and has been generalized [19–21].

Our work mostly elaborates on Zamolodchikov's ideas. In sect. 2, we present a self-contained description of CFT perturbed by a slightly relevant field. The canonical case is the minimal model with central charge $c(m) = 1 - 6/[m(m+1)]$ for $m \gg 1$, perturbed by the least relevant field $\Phi_{1,3}$. The infrared CFT has $c(m-1)$, i.e. it is at small distance in coupling space, therefore the whole region can be described by the perturbative RG technique. This subject has been discussed in the literature [1, 3, 4, 8], but for later purposes we need to be more specific and self-consistent on the renormalization conditions and clarify some general properties of the perturbative series. Actually this off-critical theory possesses higher-spin conserved currents, and sect. 2 gives us the basis for their study in sect. 4.

In sect. 3 we show how additional c -theorems follow from the conservation of higher-spin currents, and derive the corresponding sum rules. An easy example is given by the Ising model above the critical temperature, because it is equivalent to the free Majorana fermion, a trivial integrable theory.

In sect. 4 we describe the application to the interacting theory of sect. 2. We give renormalized expressions for the composite operators building the conserved current of spin four. We prove that the current does not renormalize, i.e. it does not develop an anomalous dimension, and therefore the extended c -theorem can be applied. This requires the understanding of the RG flow of renormalized descendant fields, like $L_{-2}\Phi_{1,3}$, which build the current, and how they fit in the Verma moduli of the IR CFT. Actually, at the end of the RG flow, descendant fields are finite mixtures of the corresponding fields in the IR CFT and derivatives of fields of lower dimensions, i.e. descendants reshuffle in the IR Verma module. We have a glance to how this conformal structure breaks and recover along the RG flow. Although mixing of primary fields of close dimension is well known, this

*These methods were also developed in ref. [4].

finite mixing of descendants has some peculiarities, which we analyse in detail. We show that it ensures the correct flow to the IR CFT of the null-vector equation of $\Phi_{1,3}$, and of the Ward identities associated to conservation of the currents off criticality.

In sect. 5 these Ward identities are discussed and some commutators of the spin-three conserved charge are derived. Conservation of the stress tensor implies off criticality the translation and rotation invariances. In the same way, conservation of the spin-four current leads to a differential equation for correlators which encodes a “dynamical” symmetry of the off-critical theory, yet unclear in physical terms.

Further comments are reported in the conclusions and details of the perturbative calculations are given in appendix A.

2. Conformal perturbation theory

In this section we shall discuss the renormalization of a two-dimensional conformal field theory, with action S_0 , perturbed with a slightly relevant scalar field Φ_0 ,

$$S = S_0 - \lambda_0 \int d^2x \Phi_0(x). \quad (2.1)$$

Bare expressions, i.e. in the CFT, are characterized by a zero subindex. The scaling dimension of Φ_0 is $\Delta = 2h = 2 - y$, $0 < y \ll 1$, and $\dim(\lambda_0) = y$.

For the sake of definiteness, we shall consider the example of minimal conformal models $c(m) = 1 - 6/m(m+1)$, perturbed with $\Phi_{1,3}$, $h_{1,3} = 1 - 2/(m+1)$. The theory has a mass scale $\sim \lambda_0^{1/y}$, i.e. it is no more scale invariant, and has both massive and massless excitations. It was shown in refs. [3,4], that it interpolates between the two CFTs $c(m)$ and $c(m-1)$, corresponding to its ultraviolet (UV) and infrared (IR) asymptotic limits respectively. Moreover, for large m the IR CFT is reachable for small coupling $\lambda_0 \sim O(y)$, within the domain of validity of the perturbative expansion. This setting is analogous to Wilson’s ϵ -expansion [14,22,23], but the perturbation is now done around a non-gaussian fixed point.

2.1. RENORMALIZED $\langle \Phi \Phi \rangle$

Renormalization is carried out in three steps, namely (i) regularization, (ii) renormalization and (iii) renormalization group improvement, which are easily illustrated by the example of the $\langle \Phi \Phi \rangle$ correlator. Generalization of eq. (2.1) to more than one coupling will be discussed later. The first term in the perturbative

expansion is

$$\begin{aligned}
 \langle \Phi_0(x) \Phi_0(0) \rangle &= \frac{\left\langle \Phi_0(x) \Phi_0(0) \exp\left(\lambda_0 \int \Phi_0\right) \right\rangle_0}{\left\langle \exp\left(\lambda_0 \int \Phi_0\right) \right\rangle_0} \\
 &= \langle \Phi_0(x) \Phi_0(0) \rangle_0 + \lambda_0 \int d^2x_1 \langle \Phi_0(x) \Phi_0(0) \Phi_0(x_1) \rangle_0 + O(\lambda_0^2) \\
 &= \frac{1}{(|x|^2)^{2h}} \left(1 + \lambda_0 \frac{4\pi b}{y} |x|^y A + O(\lambda_0^2) \right), \tag{2.2}
 \end{aligned}$$

where $b = C_{\phi\phi\phi}$ is the structure constant and connected correlators will be always implied. The integral to be computed is UV and IR convergent for $0 < \text{Re } y < 1$ yielding

$$A = \frac{\Gamma(1-y)\Gamma(1+y/2)^2}{\Gamma(1-y/2)^2\Gamma(1+y)} = 1 + O(y^3). \tag{2.3}$$

Therefore no regularization is needed as long as $0 < y \ll 1$.

Renormalization in this context amounts to removing the would-be singularities as $y \rightarrow 0$, i.e. when the perturbation becomes marginal [22]. We introduce a dimensionless renormalized coupling $g = g(\lambda_0)$ and a renormalized field $\Phi(x, y) = \Phi_0(x) / \sqrt{Z(g)}$, satisfying the requirement that correlators of $\Phi(x, g)$ have a finite limit as $y \rightarrow 0$ and g fixed. One can think of this procedure as a change of coordinates in the Wilson space of actions. There are many coordinates fulfilling the previous requirement, and the renormalization conditions choose arbitrarily one of these. This freedom introduces a principle of general covariance in the space of actions [1, 3], as we shall see later.

We adopt here the wave-function renormalization

$$\langle \Phi(x, g) \Phi(0, g) \rangle \Big|_{|x|=\mu^{-1}} \equiv \mu^4. \tag{2.4}$$

The renormalized coupling depends on the renormalization scale μ , $g(\mu) = \mu^{-y} \lambda(\mu)$, and coincides with the bare coupling at the scale of a lattice cutoff a^{-1} , $g(a^{-1}) = a^y \lambda_0$, where $\mu \ll a^{-1}$. It is defined together with the β -function as follows

$$\Theta(x, g) \equiv 2\pi\beta(g)\Phi(x, g), \tag{2.5}$$

$$\mu \frac{dg(\mu)}{d\mu} \equiv \beta(g) \rightarrow \int_{g_0=a^y\lambda_0}^g dg = \int_{a^{-1}}^\mu \beta \frac{d\mu}{\mu}, \tag{2.6}$$

where $\Theta = T_\mu^\mu$ is the trace of the stress tensor. From eqs. (2.2)–(2.4) it follows

$$\sqrt{Z} = \mu^{-y} \left[1 + \frac{2\pi bA}{y} \lambda_0 \mu^{-y} + O(\lambda_0^2) \right]. \tag{2.7}$$

The bare expression of the trace of the stress tensor can be obtained as in eq. (2.2) [8]

$$\Theta_0 = -2\pi y \lambda_0 \Phi_0. \tag{2.8}$$

Higher-order terms $O(\lambda_0^n)$ in eq. (2.8) are not important here, actually there are none in our example of minimal models, due to a dimensional argument [5]. The Ward identity for translation invariance implies that the stress tensor has no wave-function renormalization, $\Theta_0 = \Theta$ [24]*. Therefore, eqs. (2.5), (2.7), and (2.8) give

$$\beta(g(\lambda_0)) = -y \lambda_0 \mu^{-y} - 2\pi bA (\lambda_0 \mu^{-y})^2 + O(\lambda_0^3), \tag{2.9}$$

and from eq. (2.6) follows

$$g = \lambda_0 \mu^{-y} \left[1 + \frac{\pi b}{y} A \lambda_0 \mu^{-y} + O(\lambda_0^2) \right]. \tag{2.10}$$

The relation between bare and renormalized couplings must be inverted to the accuracy of the perturbation expansion and substituted back in all expressions. For eqs. (2.9) and (2.2) we find

$$\beta(g) = -yg - \pi b g^2 A + O(g^3), \tag{2.11}$$

$$\langle \Phi(x, g) \Phi(0, g) \rangle = \frac{\mu^4}{(|\mu x|^2)^{2h}} \left(1 + 4\pi b g \left(\frac{|\mu x|^y - 1}{y} \right) + O(g^2) \right). \tag{2.12}$$

Since these expressions have a finite limit $y \rightarrow 0$, we have succeeded in renormalizing the theory to the first-order perturbation. Existence of this limit, i.e. renormalizability, to higher order depends on more detailed properties of the theory. The dimensions of fields, structure constants, etc. must conspire to have exact cancellations of higher poles y^{-n} in eqs. (2.11) and (2.12). In our example, the $y \rightarrow 0$ theory is a $c = 1$ CFT perturbed by a marginal operator, which is reasonably renormalizable but not obviously a known theory of an interacting boson.

For $y \ll 1$ the first two terms of β are actually invariant under change of coupling coordinates, i.e. of renormalization conditions, of the form $g' = g(1 + \alpha g)$, where α is not $O(1/y)$. Therefore the renormalized $\beta(g)$ is also equal to the bare

* These two statements will be better explained later.

one $\beta_0(g_0) = -a dg_0/da$, which can be obtained by scaling of the free energy [8, 25].

2.2. RG-IMPROVED $\langle \Phi \Phi \rangle$

The zeroes of the beta-function correspond to scale invariant theories, $\Theta = 0$. In our case, eq. (2.11), these are the UV fixed point $g = 0$ we started with and $g^* = -y/\pi b$ the IR attractive one. If $y \ll 1$ the IR fixed point should be correctly described by perturbation theory. However, $\langle \Phi(x, g^*)\Phi(0, g^*) \rangle$ in eq. (2.12) is not yet sufficiently accurate for attaining the IR fixed point, because it does not show a power-law behaviour. The solution to this puzzle is well known – scaling behaviour is recovered by summing an infinite number of terms in the perturbative expansion. This is nothing other than the RG improvement and is implemented by solving the Callan–Symanzik equation. Its derivation and solution is standard, so we shall be schematic [24].

Let us consider the connected N -point correlator

$$G_N(x_i, g(\mu), \mu) = \sqrt{Z}^{-N/2} \langle \Phi_0(x_1) \dots \Phi_0(x_N) \rangle. \tag{2.13}$$

The Callan–Symanzik equation reads

$$\left(\sum_{i=1}^N \left(x_i^\mu \frac{\partial}{\partial x_i^\mu} + 2h(g) \right) + \beta \frac{\partial}{\partial g} \right) G_N(x_i, g(\mu), \mu) = 0, \tag{2.14}$$

where the “anomalous” dimension $\gamma(g)$ is

$$2h(g) = 2 + \gamma(g), \tag{2.15}$$

$$\gamma(g) = \frac{1}{2} \frac{d \log Z}{d \log \mu} = -y - 2\pi b g + O(g^2) + O(yg^3). \tag{2.16}$$

The technical derivation follows from

$$\mu \frac{d}{d\mu} \langle \Phi_0 \dots \Phi_0 \rangle = 0, \quad \text{where} \quad \mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g},$$

then the explicit variation with respect to μ is traded for a scale transformation. This recipe extends to other correlators, in particular by considering $\langle \Phi \Theta \rangle$ it follows $\gamma = \partial \beta / \partial g$, for the anomalous dimension of the perturbing field itself. When applied to the action together with the renormalization conditions, this gives the renormalized form* of eq. (2.1), $S = S_0 - \int d^2x f_0^g \Phi dg$.

* This formula is taken as a definition of Φ in ref. [3].

Physically, eq. (2.14) dictates covariance of the theory under scale transformations. At a scale invariant point g^* , $\beta(g^*) = 0$, and the Ward identities for dilatations in the CFT are recovered [11]. Scale invariance is lost off criticality but there is still a one-parameter invariance involving change of both scale and coupling. The solution of the Callan–Symanzik equation makes this fact apparent. Let us consider the particular case of the two-point function. Eq. (2.14) reads

$$\left(|x| \frac{\partial}{\partial |x|} + \beta \frac{\partial}{\partial g} \right) \bar{G}_2(x, g) = 0, \tag{2.17}$$

where

$$\bar{G}_2(x, g) = \langle \Phi(x, g), \Phi(0, g) \rangle \left(\exp \int^g \frac{2h(g)}{\beta(g)} dg \right)^2. \tag{2.18}$$

The solution is

$$\bar{G}_2(\bar{x}(t), \bar{g}(t)) = \bar{G}_2(\bar{x}(0), \bar{g}(0)), \quad \forall t, \tag{2.19}$$

where $\bar{x}(t)$ and $\bar{g}(t)$ are solutions of the flow

$$\frac{d\bar{x}}{dt} = \bar{x}, \quad \frac{d\bar{g}}{dt} = \beta(\bar{g}). \tag{2.20}$$

Eq. (2.12) gives boundary conditions for eq. (2.20), $\bar{x}(0) = x$ and $\bar{g}(0) = g(\mu)$. Moreover, let us consider eq. (2.19) for the value t_1 such that $|x|e^{t_1} = \mu^{-1}$, i.e. we bring $|x|$ to the renormalization point scale. Then the corresponding coupling $\bar{g}(t_1)$ is given by

$$\int_{g(\mu)}^{\bar{g}(t_1)} \frac{d\bar{g}}{\beta(\bar{g})} = t_1 = \log \frac{1}{\mu|x|}, \tag{2.21}$$

i.e. $\bar{g}(t_1) = g(|x|^{-1})$. Eq. (2.19) reads in this case

$$\bar{G}_2(|x|, g(\mu)) = \bar{G}_2(\mu^{-1}, \bar{g}(|x|^{-1})). \tag{2.22}$$

The r.h.s. of this equation makes explicit the one-parameter freedom of the Callan–Symanzik equation by moving all the $|x|$ -dependence into the running coupling constant. The l.h.s. provides us with a definition of $\langle \Phi(x)\Phi(0) \rangle$ for arbitrary values of $|x|$, such that it respects the scaling behaviour observed at $|x| \sim O(\mu^{-1})$. By plugging the explicit values of γ and β it follows

$$\langle \Phi(x, g)\Phi(0, g) \rangle = \frac{\mu^4}{(\mu|x|)^{4h(0)}} \frac{1}{[1 - \pi b g ((|\mu x|^y - 1)/y)]^4}, \tag{2.23}$$

and the running coupling constant is

$$\bar{g}(|x|^{-1}) = |\mu x|^y \frac{g}{1 - \pi b g((|\mu x|^y - 1)/y)}, \tag{2.24}$$

where $g = g(\mu)$ is the coupling constant in our renormalization scheme.

The above expression (2.23) is the RG improvement of eq. (2.12), containing the resummation of certain higher-order terms of the perturbative expansion. Now the correlator shows a power-law behaviour in both fixed-point theories, the last one with dimension $2h(g^*) = 2 + y$. In the minimal series, this means that $\Phi(x, g)$ interpolates between the conformal fields $\Phi_{1,3}$ at $c = c(m)$ and $\Phi_{3,1}$ at $c = c(m - 1)$ [3, 4]*. Moreover, the same power-law behaviours of $\langle \Phi \Phi \rangle$ are obtained for g fixed, in the asymptotic limits $|x| \rightarrow 0$ (UV) and $|x| \rightarrow \infty$ (IR), respectively. This is an explicit example of the statement that CFTs $c(m)$ and $c(m - 1)$ describe the asymptotic limits of the massive theory (2.1).

2.3. MORE COUPLINGS

Let us briefly show how our renormalization conditions extend to the multi-dimensional case and recover the results of refs. [1, 3, 4]. The UV theory S_0 actually contains other relevant fields Φ_{0i} , $i = 2, \dots, N$, besides the field Φ_0 we are perturbing with (call it Φ_{01}). Therefore we must prove that it is consistent to retain only one coupling, i.e. the other couplings g^i , $i \neq 1$ remain zero as we switch on g^1 . In other words, in the space of couplings there is a RG trajectory connecting the UV and IR CFTs lying straight along the g^1 axis. Within perturbation theory, we shall only consider slightly relevant fields Φ_{0i} , which can yield a flow to a close fixed point.

The action generalizes to $S = S_0 - \sum_{i=1}^N \lambda_i^0 \int \Phi_{0i} d^2x$, where $0 < y_i = \dim(\lambda_{0i}) \ll 1$ and of the same order. The renormalization conditions become

$$\Phi_{0i} = (Z(g)^{1/2})_i^j \Phi_j(g), \quad \langle \Phi_i(x, g) \Phi_j(0, g) \rangle \Big|_{|x|=\mu^{-1}} = \mu^4 G_{ij}(g), \tag{2.25}$$

$$\Theta(x, g) = 2\pi \beta^i(g) \Phi_i(x, g), \quad \mu dg^i(\mu)/d\mu = \beta^i(g), \tag{2.26}$$

where $g = \{g^i\}$ and $G_{ij}(g)$ is the Zamolodchikov metric [1]. Conditions (2.25) still respect reparametrization invariance in the space of actions, $g^i \rightarrow g^i + \epsilon^i(g)$, under which β^i transforms as a vector, Φ_i as a one-form and G_{ij} as a metric tensor. Therefore they must be supplemented with a choice of coordinate system to allow actual computations. Physical quantities must be invariant quantities. For the one-dimensional case we chose $G_{11} = 1$ without loss of generality. In multi-dimensional space, G_{ij} can have in general curvature and therefore be nontrivial in any

* Note that at the conformal points the dependence on μ can be cancelled by a scale transformation.

coordinate system. Whether this is actually the case is an interesting open problem[★]. We can choose normal coordinates at the point $g = 0$, $G_{ij} = \delta_{ij} + O(g^2)$, such that the space is flat to first order in perturbation theory. This leaves a rotational invariance to be fixed as more convenient. In normal coordinates we can repeat our previous treatment of the beta-function and obtain

$$\beta^i(g) = -y_i g^i - \pi \sum_{j,k} C^i_{jk} g^j g^k + \dots, \tag{2.27}$$

where $C^i_{jk} \sim C_{ijk} + O(y^3)$ are the structure constants, and no summation is implied on the index i . Moreover the anomalous dimensions, eq. (2.16), form the matrix $\gamma_i^j = \partial_i \beta^j$. Its eigenvalues are actually invariant quantities and give the anomalous dimensions of the fields at the IR CFT [27]. Let us consider eq. (2.27) for only one coupling $g^1 \neq 0$; if $C_{11j} = 0$ for all $j \neq 1$, then $\beta^i(g^1) \equiv 0$, $i \neq 1$ and the one-coupling RG flow is consistent. This is the case for minimal theories, $C_{(1,3),(1,3),j} = 0$ for $h_j \neq h_{(1,3)}$ and < 1 .

2.4. DIVERGENCES IN THE PERTURBATION EXPANSION

The first perturbative correction was shown to be an analytic function of y for $0 < \text{Re } y < 1$, the lower and upper bounds being dictated by UV and IR convergence of the integral respectively. Let us estimate the corresponding bounds for finiteness of the higher-order term,

$$I_n = \frac{\lambda_0^n}{n!} \int d^2x_1 \dots d^2x_n \langle \Phi_0(0) \Phi_0(x_1) \dots \Phi_0(x_n) \Phi_0(x) \rangle_0. \tag{2.28}$$

UV convergence. We can study the integral for $0 \leq |x_i| \leq |x|$ and by relabelling integration variables, the region $0 \leq |x_1| \leq |x_2| \leq \dots \leq |x|$ is sufficient. The operator product expansion (OPE) can be used to estimate the behaviour as x_i approaches x_{i-1} or x_{i+1} , $i = 1, \dots, n$, with $x_0 = 0$ and $x_{n+1} = x$. The OPE is of the form $\Phi_0(x) \Phi_0(0) = |x|^{-4h} + |x|^{-2h} \Phi_0(0) + \text{irrelevant fields}$, where the contribution of the identity operator must be discarded because the correlator is connected. The leading UV singularity is given by the field of lower dimension in the OPE, Φ_0 itself in our case. For any pair of approaching points, one integration factorizes as $\int_0^1 d^2(x_i - x_{i-1}) |x_i - x_{i-1}|^{-2h} \sim O(1/y)$. The leading singularity as $y \rightarrow 0$ is therefore $O(y^{-n})$, because n is the maximum number of approaches. This leading contribution $(\lambda_0/y)^n$ was effectively resummed by the RG in the previous sections. In summary, we found

$$y > 0, \quad \forall n \quad (\text{UV convergence}). \tag{2.29}$$

[★] Metrics of symmetric spaces have been obtained in the case of marginal perturbations [26].

IR convergence. The problem can be presented in a toy example. Consider the hypergeometric function in the Euler integral representation,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \quad (2.30)$$

where $a, b > 0$, $b - a$ a rational number $n_c - 1 < b - a < n_c$, n_c an integer. Suppose the full integral is not given, but rather its series expansion in z^{-n} , those coefficients are the beta-functions $\int_0^1 t^{b-1-a-n}(1-t)^{c-b-1} dt$. The correct $z \rightarrow \infty$ behaviour, schematically

$$z^a F(a, b; c; z) \sim 1 + z^{-1} + \dots + z^{-n_c+1} + (-z)^{-(b-a)}(1 + z^{-1} + \dots), \quad (\arg(-z) < \pi) \quad (2.31)$$

cannot be recovered from this series expansion. For all orders z^{-n_c-k} which are dominated by the rational power $(-z)^{-(b-a)}$, the coefficient of the series is “IR” singular at $t \rightarrow 0$ (the space-time variable would be $x = t^{-1} - 1$). The lesson is that IR singularities at and above the order z^{-n_c} signal the presence of non-analytic rational powers in the series. The power $(b - a)$ can be recovered by putting a cutoff (see later), but the finite coefficient in front can only be recovered by analytic continuation of the whole serie.

The same mechanism could happen for the series (2.28). The leading IR singularity is given by the region of integration $|x_i| \rightarrow \infty$, all at the same speed. Since $|x| \ll |x_i|$, we can use the OPE $\Phi_0(x)\Phi_0(0) \sim |x|^{-2h}\Phi_0(0)$ and estimate the multiple integral by power counting [7]. To first order it follows $I_1 \sim \lambda_0 |x|^{-2h} \int_1^\infty d^2x_1 |x_1|^{-2h}$, convergent for $y < 1$. To higher-order one finds

$$y < \frac{2}{n+1}, \quad (\text{IR convergence}). \quad (2.32)$$

Therefore the IR bound shrinks the convergence region to zero, and eventually hits any given value of y at order n_c . In our case of slightly relevant perturbation $y = 4/(m + 1)$, the IR singularity first appears at high order $n_c = [m/2]$, and the lower orders are well defined without any regularization. For strongly relevant perturbations $y > 1$, IR divergences are already present at the lowest order, and a regularization procedure is crucially needed. Dotsenko introduced a method of analytic continuation, which amounts to compute (2.28) in the CFT with shifted $c \rightarrow c - \epsilon$, and obtained correct results in case of logarithmic singularities occurring for $y = 1$ [7]. The extension of this method to rational powers is done in ref. [28].

The value of the rational power possibly occurring in the series can be inferred by cutting the integral to the correlation length $R_c(\lambda_0) \sim \lambda_0^{-1/y}$, as in refs. [7, 29].

This argument extends to our case, eq. (2.28), and yields

$$I_{n_c} \sim \frac{\lambda_0^{n_c}}{|x|^{2h}} R_c(\lambda_0)^{-2+y(n_c+1)} \sim \begin{cases} \lambda_0^{(m-1)/2} R_c^0 \sim \lambda_0^{(m-1)/2} \log \lambda_0, & m \text{ odd,} \\ \lambda_0^{m/2} R_c^{y/2} \sim \lambda_0^{(m-1)/2} & m \text{ even.} \end{cases} \quad (2.33)$$

In summary, this argument suggests that IR singularities yield non-analytic terms in the perturbation series. Their actual presence is unclear at the moment. Therefore in this paper we shall limit ourselves to the properties of the IR–UV safe terms of the perturbation expansion $0 \leq n < n_c$. We shall exhibit the basic properties of the off-critical theory, mainly of the higher-spin conserved currents occurring within the perturbative approach. This approximation is self-consistent for large- m minimal models, and the handling of IR divergences hopefully will not modify our results too much.

2.5. RENORMALIZATION IS NECESSARY FOR THE IR LIMIT

Since we are taking into account the IR–UV finite terms of the perturbation series, we could say that there is no actual need to perform the renormalization procedure of the previous sections. In a recent paper [27] the bare couplings were considered an admissible choice of coordinates in the space of actions. The principle of general covariance in the space of actions was emphasised and a covariant form was given for some RG-invariant quantities, like $\gamma_i^{-1}(g^*)$, which allows their computation in any coordinate system, including the bare theory.

However, the bare theory is not convenient because the bare fields have a singular IR limit. This is particularly important for us because we shall study in the next sections the IR behaviour of the Zamolodchikov higher-spin conserved currents. On the contrary, renormalized fields have both UV and IR smooth limits, eq. (2.23), i.e. the Zamolodchikov metric is well defined everywhere. Bare coordinates correspond to a metric with an unphysical singularity at the IR CFT point.

In order to show this, let us look for the IR point in the bare coupling $g_0^* = a^y \lambda_0^*$. The expression $\Theta = -2\pi y \lambda_0 \Phi_0$ may vanish at $\lambda_0^* = \pm \infty$ if $\Phi_0 \sim O(\lambda_0^{-1})$. Indeed, the IR fixed point corresponds to $\lambda_0^* = -\infty$ (the sign given by the beta-function). In order to attain this limit we need a resummed expression for $\langle \Phi_0 \Phi_0 \rangle$. Eq. (2.23) suggests

$$\langle \Phi_0(x) \Phi_0(0) \rangle = \frac{1}{|x|^{4h} (1 - g_0(\pi b/y)(x/a)^y)^4}. \quad (2.34)$$

Eq. (2.34) can be checked against the two-loop calculation of ref. [27], and indeed, it shows a power-law behaviour at $g_0 = -\infty$ with correct scaling dimensions $h + y$. The field is $\Phi_0 \sim O(g_0^{-2})$ confirming the vanishing of Θ .

In these coordinates, we have a trivial β -function $\beta_0(g_0) = -yg_0$ and a non-trivial Zamolodchikov metric

$$G_0(g_0) = a^{4h} \langle \Phi_0(a) \Phi_0(0) \rangle = \frac{1}{(1 - g_0 \pi b / y)^4}. \quad (2.35)$$

Since $G_0(g_0^*) = 0$, this is a singular coordinate system around the IR fixed point*. One can check that the distance between the two fixed points is invariant, $\int ds = \int_{-\infty}^0 \sqrt{G_0} dg_0 = y/\pi b$, and actually small in value. The renormalized coordinate is obtained by the change of variable $g = g_0 / (1 - (\pi b / y) g_0)$, which brings the metric to one, eq. (2.4). The β -function transforms as a covariant vector, and becomes non-trivial, eq. (2.11).

In summary, for $y = 0$ the renormalized coordinates are necessary for a meaningful description of the scaling region around UV CFT, while for $y > 0$ they are necessary for a correct mapping around the IR CFT.

3. Generalized c -theorems and sum rules

3.1. DERIVATION

Massive field theories interpolating between two conformal field theories are subject to certain qualitative as well as quantitative constraints which follow from the conservation of the stress tensor,

$$\partial_z T_2 + \partial_z \Theta_0 = 0, \quad (3.1)$$

where $T_{zz} = T_2$ and $T_{z\bar{z}} = \Theta_0 = \frac{1}{4} T_\mu^\mu$. Zamolodchikov's c -theorem [1] implies the qualitative property that the central charges c_{UV} and c_{IR} of the corresponding CFTs must obey

$$c_{UV} \geq c_{IR}. \quad (3.2)$$

From this theorem, Cardy derived the quantitative sum rule [13],

$$\Delta c = c_{UV} - c_{IR} = (12/\pi) \int d^2x x^2 \langle \Theta_0(x) \Theta_0(0) \rangle, \quad (3.3)$$

namely, the net change of c is equal to the dimensionless moment of a two-point correlator computed in the off-critical theory. Eq. (3.3) is particularly appealing since the l.h.s. contains data from the CFT and the r.h.s. is computed in the massive theory, for any value of $g \neq 0$, g^* . The sum rule is actually an identity in

* What we call here g_0 is not the running coupling constant evaluated at scale a^{-1} in subsect. 2.1., since that coupling corresponds to a flat metric too.

the dependence on the coupling constant, thus derivatives with respect to it generate more sum rules for multi-point correlators, as investigated in ref. [30].

Actually, the previous conditions have natural generalizations when the non-critical theory possesses additional conserved currents,

$$\partial_{\bar{z}} T_n + \partial_z \Theta_{n-2} = 0, \tag{3.4}$$

leading to a conserved charge P_{n-1} of spin $n - 1$,

$$P_{n-1} = \frac{1}{2\pi i} \oint (dz T_n + d\bar{z} \Theta_{n-2}) \tag{3.5}$$

and its conjugate \bar{P}_{n-1} . The proof parallels Zamolodchikov's derivation of the c -theorem [1]. Consider the off-critical correlators

$$\begin{aligned} \langle T_n(z, \bar{z}) T_n(0, 0) \rangle &= \frac{F(|z\bar{z}|)}{z^{2n}}, & \langle T_n(z, \bar{z}) \Theta_{n-2}(0, 0) \rangle &= \frac{G(|z\bar{z}|)}{z^{2n-1}\bar{z}} \\ \langle \Theta_{n-2}(z, \bar{z}) \Theta_{n-2}(0, 0) \rangle &= \frac{H(|z\bar{z}|)}{z^{2n-2}\bar{z}^2}. \end{aligned} \tag{3.6}$$

The conservation law (3.4) implies that the dimensionless amplitude

$$C_n(g, |x|\mu) = 2(F - (2n - 2)G - (2n - 1)H), \tag{3.7}$$

satisfies

$$|x| dC_n/d|x| = -4(2n - 2)(2n - 1)H \leq 0. \tag{3.8}$$

In euclidean space, reflection positivity implies that the r.h.s. of eq. (3.8) is negative. The next step uses the Callan–Symanzik equation for C_n ,

$$(|x| d/d|x| + \beta \partial/\partial g) C_n = 0.$$

Notice that no anomalous dimensions show up for C_n , because the conserved current eq. (3.4) is not renormalized, and neither is $T_{\mu\nu}$. This will be clarified in the next section. Finally, the quantity

$$c_n(g) = C_n(g, |x|\mu)|_{|x|=\mu^{-1}} \tag{3.9}$$

is decreasing along the RG flow, i.e. when the couplings are varied along a renormalized trajectory

$$-\beta \partial c_n/\partial g = -4(2n - 2)(2n - 1)H(g, |x|\mu)|_{|x|=\mu^{-1}} \leq 0. \tag{3.10}$$

Moreover, c_n is stationary at the CFT ($\Theta_n \equiv 0$), and its numerical value is given by the correlator $\langle T_n T_n \rangle$. This completes the generalization of the c -theorem. Clearly c_2 is the Virasoro central charge and eq. (3.2) is recovered.

The associated sum rule follows by integrating both sides of eq. (3.10),

$$\Delta c_n = 2 \frac{(2n-2)(2n-1)}{\pi} \int d^2z z^{2n-3} \bar{z} \langle \Theta_{n-2}(z, \bar{z}) \Theta_{n-2}(0, 0) \rangle. \quad (3.11)$$

This equation is a quantitative condition for the massive theory issued from CFT data. There are several interesting RG flows which could be tested. When one continuous symmetry is preserved along the flow, eq. (3.11) applies to the corresponding conserved current. For example, the fermionic partner of the stress tensor is conserved in the flow within supersymmetric minimal models [31]. In this case, one can check that eq. (3.11) is equal to eq. (3.3) by a rigid supersymmetric transformation off criticality.

More interesting are the cases of “hidden” or “dynamical” symmetries, like in the minimal models of W -algebras, and the integrable perturbations of $c < 1$ minimal models recently discovered by Zamolodchikov. We are mainly interested in the latter case, where eq. (3.11) can check proposals for the integrable field theory associated to a given RG flow.

We refer to integrable quantum field theories as those possessing an infinite set of conserved currents with spins $\{s_i\}$ [32]. If they have only massive particles (i.e. $c_{\text{IR}} = 0$), their one-shell dynamics is described by a factorized S -matrix [16]. Zamolodchikov considered certain perturbations of minimal models, which flow into purely massive phases, like the Ising model in a magnetic field. He argued that higher-spin currents are conserved due to null-vector equations of the CFT [5]. The lowest values of the spins $\{s_i\}$ were inferred by counting dimensions of Verma modules, and came out in a beautiful pattern: they are equal to the exponents of simple Lie algebras modulo the Coxeter number. Existence of an infinite set was conjectured and, consistently, an exact S -matrix was obtained [17]. This is a solution to the bootstrap equations with a minimal content of particles and consistent with an infinite set $\{s_i\}$ of higher-spin conserved currents [16].

Toda field theories built on affine extensions of the Lie algebras have exactly this set of classically conserved currents and are the most promising candidates for describing these perturbed CFTs [20, 21, 32]. Present investigations have considered the simply-laced algebras A_n, D_n, E_6, E_7, E_8 possessing a unique S -matrix compatible with the set of conserved currents. In these cases, the qualitative information of eq. (3.4) was sufficient. Quantitative conditions will be useful for future studies. In the case of non-simply laced algebras, the S -matrix bootstrap has multiple solutions and the relation with Toda field theories is problematic [21]. Moreover, two field theories may have the same S -matrix, which encodes the on-shell properties only.

The application of the generalized sum rules eq. (3.11) to these integrable theories is limited by the poor knowledge of correlators off criticality. However there are recent developments in their computation [33], which let us hope to present results in this direction in a future work.

In the following we illustrate the use of sum rules with the example of free massive fermions*. A free theory is a rather trivial case of integrable theory. It has the advantage that both sides of eq. (3.11) can be checked explicitly. Actually, the result takes a suggestive form of complete reconstruction of the off-critical $\langle \Theta \Theta \rangle$ correlator from CFT data. In the next section, we shall study the conserved currents by perturbation theory in the large- m minimal models, which provide a non-trivial computable theory. In order to apply the sum rule we have to understand the RG flow of the currents (T_n, Θ_{n-2}) .

3.2. EXAMPLE. MASSIVE MAJORANA FERMION

The lagrangian of the theory is

$$S = \int (\psi \partial_z \psi + \bar{\psi} \partial_z \bar{\psi} + im \bar{\psi} \psi) d^2z, \tag{3.12}$$

where $\psi, \bar{\psi}$ are one-component spinors. The linear equations of motion for this theory generate infinite series of higher-spin conserved currents. One set of even spin currents is [5]

$$\begin{aligned} T_{2n} &= -\pi : \partial_z^{n-1} \psi \partial_z^n \bar{\psi} :, \quad (n = 1, 2, \dots), \\ \Theta_0 &= i \frac{1}{2} \pi m : \bar{\psi} \psi :, \quad \Theta_{2n} = -\frac{1}{4} m^2 T_{2n}, \quad (n \geq 1). \end{aligned} \tag{3.13}$$

For (T_2, Θ_0) one recovers the stress tensor in the standard CFT notation.

From $\langle T_{2n} T_{2n} \rangle$ at $m = 0$ one computes

$$c_{2n} = \frac{1}{2} \Gamma(2n) \Gamma(2n - 1). \tag{3.14}$$

Decreasing of the c_n -charge is not particularly interesting here because the IR fixed point is trivial.

Let us now discuss the sum rules associated to these c_{2n} numbers. Each sum rule, for any $n \geq 1$, produces one condition for each correlator $\langle \Theta_{n-2} \Theta_{n-2} \rangle$. Notice that the currents (3.13) form a chain of relations which can be used in momentum space to express all $\langle \Theta_{n-2} \Theta_{n-2} \rangle$ in terms of the lowest correlator

* Some attempts in this direction were also made in ref. [34].

$\langle \Theta_0 \Theta_0 \rangle$, namely

$$\Delta c_{2n} = -2 \frac{(4n-2)(4n-1)}{\pi} 4^{2n-1} (\partial_{\bar{p}})^{4n-3} \partial_p \left(\frac{m^2 \bar{p}}{4 p} \right)^{2n-2} \langle \Theta_0(p) \Theta_0(-p) \rangle \Big|_{p=0}, \tag{3.15}$$

where

$$\langle \Theta_0(p) \Theta_0(-p) \rangle = \int d^2z \exp\left[-\frac{1}{2}i(\bar{p}z + p\bar{z})\right] \langle \Theta_0(z, \bar{z}) \Theta_0(0, 0) \rangle. \tag{3.16}$$

A more physical form of the previous equation is obtained by expressing $\langle \Theta_0 \Theta_0 \rangle$ in terms of its spectral representation [35],

$$\langle \Theta_0(p) \Theta_0(-p) \rangle = \frac{\pi}{3} \int_0^\infty d\lambda \rho(\lambda^2, m^2) \frac{\lambda^4}{p^2 + \lambda^2}. \tag{3.17}$$

The generalized sum rules, eq. (3.11), give the moments of the spectral density $\rho(\lambda^2, m^2)$,

$$\Delta c_{2n} = \frac{\Gamma(4n)}{6} m^{4n-4} \int_0^\infty d\lambda \frac{\rho(\lambda^2, m^2)}{\lambda^{4n-4}}, \quad n = 1, 2, \dots \tag{3.18}$$

Therefore, we have shown that the constraints coming from the infinite conservation laws conspire to reconstruct an off-critical correlator from CFT data.

Clearly, in a massive free theory the spectral density can also be computed by the imaginary part of a one-loop Feynman diagram. From eqs. (3.12) and (3.13), one finds

$$\rho(\lambda, m) = 6m^2 \frac{\sqrt{\lambda^2 - 4m^2}}{\lambda^4} \theta(\lambda - 2m), \tag{3.19}$$

where $\theta(x)$ is the step function. This equation gives an explicit check of eqs. (3.14) and (3.18).

4. The current of spin-four in the renormalized theory off criticality

Zamolodchikov analysed the conservation law for the spin-four current [5],

$$\partial_{\bar{z}} T_4(z) = \dots, \quad T_4(z) = L_{-2}^2 I(z) = \frac{1}{2\pi i} \oint_z d\eta \frac{T(\eta)T(z)}{\eta - z}, \tag{4.1}$$

in the minimal model $c(m)$ perturbed by the least relevant field of sect. 2. He assumed that the r.h.s. of eq. (4.1) has an expansion in integer powers of λ_0 times UV conformal fields of appropriate dimension. Then he showed that the $O(\lambda_0)$ term can be written as a derivative field by using the null-vector equation [11] of lowest level satisfied by the field $\Phi_{1,3}$

$$\chi_0 = [(h + 2)(h + 1)L_{-3} - 2(h + 1)L_{-1}L_{-2} + L_{-1}^3]\Phi_0. \quad (4.2)$$

A higher-order term can appear for odd m but it is a derivative too. Therefore he proved that the current remains conserved off criticality in the form

$$\begin{aligned} \partial_{\bar{z}}T_4(z) + \partial_{\bar{z}}\Theta_2(z) &= 0, \\ \Theta_2(z) &= -\frac{\lambda\pi y}{2}(a_1L_{-2} + a_2L_{-1}^2)\Phi_0(z) + a_3\lambda_0^{(m+1)/2}T(z) \end{aligned} \quad (4.3)$$

where the constants are

$$a_1 = 2(2 - y)/(6 - y), \quad a_2 = y(2 + y)(8 - y)/12(4 - y)(6 - y),$$

and $a_3 = 0$ for even m , with $y = 2 - 2h_{1,3} = 4/(m + 1)$.

As discussed in subsect. 2.4, the simple perturbative expansion for Θ_2 could be violated by IR singularities at high order $n \geq n_c = [m/2]$. On the other hand, the exact S -matrices conjectured on the basis of conservation of higher-spin currents were verified numerically [18], say in the case of the Ising model in a magnetic field. This suggests that the IR problem can be cured, at least when the off-critical theory is purely massive ($c_{IR} = 0$). We cannot address this interesting problem here.

We shall rather discuss the properties of the current (4.3) to lower orders $0 < n < n_c$, where the perturbative hypothesis holds*. We shall derive its expression in terms of renormalized fields and study the RG flow of the related descendant fields. Then we shall apply the generalized sum rule of sect. 3.

4.1. NULL-VECTOR WARD IDENTITY

Let us consider the perturbative expansion of a correlator involving the null vector (4.2),

$$\langle \chi_0(z) \dots \rangle = \langle \chi_0(z) \dots \rangle_0 + \lambda_0 \int d^2x_1 \langle \chi_0(z) \dots \Phi_0(x_1) \rangle_0 + \dots \quad (4.4)$$

It vanishes to any order in perturbation theory. Eq. (4.4) can be interpreted as a dynamical Ward identity of the off-critical theory. Any perturbed minimal CFT

*Therefore we shall drop the term $O(\lambda_0^{(m+1)/2})$ in the current (4.3).

possesses an infinity of them. Let us put it in another form:

$$(h + 2)(h + 1)\langle L_{-3}\Phi_0(z)\Phi_0(x_1)\dots\Phi_0(x_N)\rangle - 2(h + 1)\partial_z\langle L_{-2}\Phi_0(z)\Phi_0(x_1)\dots\Phi_0(x_N)\rangle = -\partial_z^3\langle \Phi_0(z)\Phi_0(x_1)\dots\Phi_0(x_N)\rangle. \tag{4.5}$$

Notice that contact terms are impossible on dimensional grounds. This equation certainly makes sense in the bare off-critical theory up to $O(\lambda_0^{n_c})$, because both sides are finite meromorphic functions of y . It implies that: (i) Θ_2 does not develop an anomalous dimension; (ii) $L_{-3}\Phi$, $L_{-2}\Phi$ and Φ mix in the IR CFT.

In sect. 2 we saw that Φ is multiplicatively renormalized, i.e. the poles y^{-k} are cancelled by the Z -factor independently of the other fields in the correlators. Let us assume that $L_{-2}\Phi_0$ and $L_{-3}\Phi_0$ are also multiplicatively renormalized. Then their Z -factors are both equal to Z of Φ , because eq. (4.5) holds and there are two linear independent fields among Φ_0 , $L_{-2}\Phi_0$ and $L_{-3}\Phi_0$,

$$L_{-2}\Phi(x, g) = L_{-2}\Phi_0(x)/\sqrt{Z}, \quad L_{-3}\Phi(x, g) = L_{-3}\Phi_0(x)/\sqrt{Z}. \tag{4.6}$$

Therefore the renormalized form of the component Θ_2 of the current (4.3) is

$$\Theta_2(z) = \frac{1}{2}\pi\beta(g)(a_1L_{-2}\Phi(z, g) + a_2\partial_z^2\Phi(z, g)) + o(g^{[m/2]-1}), \tag{4.7}$$

that is Θ_2 does not develop an anomalous dimension, and neither does Θ . Actually, this property was assumed in the proof of the extended c -theorem in sect. 3.

Let us now discuss the IR limit of the renormalized null-vector equation $0 = \langle \chi(g)\dots \rangle = \langle \chi_0\dots \rangle/\sqrt{Z}$, eqs. (4.2) and (4.5). Let us simplify the notation by denoting fields of the IR CFT $c(m - 1)$ with prime indices. Since the renormalized field $\Phi(g)$ flows into $\Phi(g^*) = \Phi_{3,1}^{(m-1)} \equiv \Phi'$ and the null-vector equation holds all along the flow, it should end up in the corresponding equation of the IR CFT, $0 = \langle \chi_{3,1}^{(m-1)}\dots \rangle \equiv \langle \chi'\dots \rangle$. However, χ' has different coefficients than χ in eq. (4.2), because $h_{1,3}^{(m)} \rightarrow h_{3,1}^{(m-1)} = h'$. Therefore $L_{-2}\Phi$ and $L_{-3}\Phi$ cannot flow in the corresponding descendants of the IR CFT, rather to finite linear combinations of them which adjust the coefficients of the null-vector equation. We shall call this phenomenon “IR mixing”.

Let us show some sample computations. The simplest one is

$$\begin{aligned} \langle L_{-2}\Phi_0(z)\Phi_0(0)\rangle &= \langle L_{-2}\Phi_0(z)\Phi_0(0)\rangle_0 \\ &+ \lambda_0 \int d^2x_1 \langle L_{-2}\Phi_0(z)\Phi_0(0)\Phi_0(x_1)\rangle_0 + \dots \\ &= \frac{h}{z^2|z|^{4h}} \left[3 + \lambda_0 \frac{4\pi b}{y} (3 - y + O(y^2)) + O(\lambda_0^2) \right]. \end{aligned} \tag{4.8}$$

Let us compare it with $\langle \Phi_0 \Phi_0 \rangle$ in eq. (2.2). The $O(1/y)$ term is cancelled by the same Z -factor eq. (2.7), as expected, while the finite part is different. The renormalized correlator is defined according to eq. (4.6), and satisfies the same Callan–Symanzik equation (2.17) as $\langle \Phi \Phi \rangle$. Therefore this correlator is completely determined by its value at the scale $|x| = \mu^{-1}$, see eqs. (2.17)–(2.23):

$$z^2 \langle L_{-2} \Phi(z, g) \Phi(0, g) \rangle \Big|_{|x|=\mu^{-1}} = \mu^4 (3h(g) - \pi bg + O(yg) + O(g^2)). \quad (4.9)$$

Moreover the IR limit is simply obtained by letting $g \rightarrow g^*$. Eq. (4.9) shows the mixing phenomenon, because the correlator $\langle L_2^{(m-1)} \Phi_{3,1}^{(m-1)} \Phi_{3,1}^{(m-1)} \rangle \equiv \langle L_{-2} \Phi' \Phi' \rangle \sim 3h'$. There are two fields with the same dimension in the IR Verma module, therefore we expect

$$L_{-2} \Phi(x, g^*) = (1 + \nu(g^*)) L_{-2} \Phi'(x) + \rho(g^*) \partial_z^2 \Phi'(x). \quad (4.10)$$

In order to compute the mixing coefficients we need another correlator. By proceeding as before we get

$$z^4 \langle L_{-2} \Phi(z) L_{-2} \Phi(0) \rangle \Big|_{|x|=\mu^{-1}} = \mu^4 (9h^2(g) + 22h(g) + \frac{1}{2}c(g) - 22\pi bg + \dots). \quad (4.11)$$

Eqs. (4.9)–(4.11) yield

$$L_{-2} \Phi(x, g^*) = (1 - \frac{2}{3}\pi bg^*) L_{-2} \Phi'(x) + \frac{1}{6}\pi bg^* \partial_z^2 \Phi'(x) + O(yg, g^2). \quad (4.12)$$

Let us also check the flow of the null-vector equation (4.5), in the case of $\langle \chi(z) \Phi(0) \rangle$. We need another correlator, computed as in eq. (4.8),

$$z^3 \langle L_{-3} \Phi(z) \Phi(0) \rangle \Big|_{|x|=\mu^{-1}} = \mu^4 (-4h(g) + 2\pi bg + \dots). \quad (4.13)$$

Putting all correlators together, we verify the following equation to the accuracy $O(g, y)$

$$\begin{aligned} 0 &= \langle \chi(\mu^{-1}, g) \Phi(0, g) \rangle \\ &= (h+2)(h+1) \langle L_{-3} \Phi(\mu^{-1}, g) \Phi(0, g) \rangle - 2(h+1) \partial_z \langle L_{-2} \Phi \Phi \rangle + \partial_z^3 \langle \Phi \Phi \rangle \\ &= \mu^7 [(h(g)+2)(h(g)+1)(-4h(g)) - 2(h(g)+1)(-3h(g)(2h(g)+2)) \\ &\quad + (-2h(g))(2h(g)+1)(2h(g)+2)]. \quad (4.14) \end{aligned}$$

At the UV CFT, $\langle \chi(\mu^{-1})\Phi(0) \rangle$ is a function of h which vanishes identically. Away from criticality, mixing ensures that the dependence on g combines with those of h to have the same vanishing function of $h(g)$.

One could ask whether there is a more natural definition of $L_{-2}\Phi(g), L_{-3}\Phi(g)$ off criticality such that they do not mix. We shall argue for the negative answer, and try to show the difference between our problem and the usual mixing of primary fields of close dimensions. In the RG literature [36], (primary) fields mix when the anomalous dimensions matrix γ_i^j is non-diagonal, see eq. (2.25). Its eigenvalues and eigenvectors are the new scaling dimensions and fields respectively. The field corresponding to a zero eigenvalue is called RG invariant combination. In our case, this would be Θ_2 in eq. (4.7). However, any choice of constant coefficients a_1, a_2 would enjoy this property, because both $L_{-2}\Phi$ and Φ are RG eigenstates and have the same anomalous dimension. This makes the difference with the standard literature. We miss a characterization of descendant fields in the RG.

At CFTs it is clear that scale invariance alone cannot characterise descendant fields, Virasoro representations are needed to range them. Here we have an analogous situation away from criticality, and we must go beyond scale covariance for finding a criterion. We can think of a dynamical symmetry, which is encoded in the Ward identity for conservation of the spin-four current. Some steps in this direction are presented in sect. 5. Within the two-dimensional vector space $\{L_{-2}\Phi, L_{-1}^2\Phi\}$ at level two, there is a linear combination of fields with coefficients depending on h which flow into its IR analogous. It is somehow “covariant”, and should have a particular meaning. In sect. 5, we show that it is the commutator $[P_3, \Phi]$, such that the mixing ensures the correct flow of this expression in the IR CFT*.

Let us finally note that we could define a field flowing to itself by a g -dependent linear combination. By eqs. (4.10) and (4.12) this field is for $g < g^*$

$$(L_{-2}\Phi)_{(2)}(x, g) = (1 - \nu(g)) [L_{-2}\Phi(x, g) - \rho(g) \partial_z^2 \Phi(x, g)]. \quad (4.15)$$

However this satisfies a matricial Callan–Symanzik equation, because $\beta \partial/\partial g$ acts on the coefficients of the linear combination, i.e. it is not a RG eigenstate.

4.2. NORMAL-ORDERING AWAY FROM CRITICALITY

The calculation of the correlator $\langle T(z)\Phi(w)\Phi(u) \rangle$ illustrates that the previous results on mixing follow by normal ordering the renormalized fields away from criticality. The (non-commutative) normal ordering is defined by

$$:\mathcal{A}\mathcal{B}:(z, g) = \lim_{w \rightarrow z} (\mathcal{A}(w, g)\mathcal{B}(z, g) - \text{singular terms}). \quad (4.16)$$

* Note that this is not the quasi-primary field at level two.

This limit exists only if the composite operator $:\mathcal{A}\mathcal{B}:$ does not have extra anomalous dimension. As a counterexample consider the renormalized fields $\Phi_{2,2}$ and $\Phi_{3,3}$, which interpolates between the corresponding UV and IR primary fields. In the Landau–Ginzburg description of the UV CFT [37], $\Phi_{2,2}$ is φ , the elementary field, and $\Phi_{3,3}$ is the composite field

$$:\varphi^2: = \lim\left[\varphi(w)\varphi(z)|w-z|^{2(h_{2,2}-h_{3,3})} - \text{singular terms}\right].$$

The composite field develops an extra anomalous dimension off criticality, because one finds that $2h_{2,2}(g) - h_{3,3}(g) \neq 2h_{2,2}(0) - h_{3,3}(0)$. Therefore the normal ordering is made finite by putting the additional factor

$$(\mu|w-z|)^{2\gamma_{2,2}-\gamma_{3,3}} \sim 1 + \text{const.} \times g \log \mu|w-z| + \dots$$

After this remark, let us present the perturbative calculation. The UV correlators are

$$\langle T(0)\Phi_0(z)\Phi_0(w)\rangle_0 = \left(\frac{1}{z^2} + \frac{1}{w^2} - \frac{2}{zw}\right) \frac{h}{(|z-w|^2)^{2h}}, \tag{4.17}$$

$$\langle L_{-2}\Phi_0(z)\Phi_0(0)\rangle_0 = \langle :T\Phi_0:(z)\Phi_0(0)\rangle_0 = 3h/z^2(|z|^2)^{2h}. \tag{4.18}$$

The first-order correction involves an integral over x_1 of $\langle T(0)\Phi(z)\Phi(w)\Phi(x_1)\rangle$. This correlator possesses holomorphic poles and double poles which are harmless. A regularization is introduced which respects rotational invariance, and can be removed after angular integration*. The result is

$$\begin{aligned} \langle T(0)\Phi(z)\Phi(w)\rangle &= Z\langle T(0)\Phi_0(z)\Phi_0(w)\rangle \\ &= \frac{\mu^4 h}{(\mu|z-w|)^{4-2y}} \left\{ \left(\frac{1}{z^2} + \frac{1}{w^2} - \frac{2}{zw}\right) \left[1 + \frac{4\pi b g}{y} ((\mu|z-w|)^y - 1)\right] \right. \\ &\quad \left. + (-1/zw)(-2\pi b g)(\mu|z-w|)^y + I(z,w) + O(g^2) \right\}, \end{aligned} \tag{4.19}$$

* See appendix A for details.

where $I(z, w)$ is a non-polynomial function[★]

$$I(z, w) = \begin{cases} |z| \sim |w| \sim O(\mu^{-1}) \left(\frac{1}{z^2} + \frac{1}{w^2} \right) (-\pi b g) (\mu |z - w|)^y (1 + O(y)), \\ w = 0 \quad (1/z^2) (-2\pi b g) (\mu |z|)^y (1 + O(y)). \end{cases} \quad (4.20)$$

Observe that eq. (4.19) has a finite $y \rightarrow 0$ limit, confirming that T does not require renormalization. Using the first limit in eq. (4.20) we can compute

$$\langle T(0)\Phi(z)\Phi(w) \rangle \stackrel{|z| \sim |w| \sim \mu^{-1}}{\sim} \left(\frac{1}{z^2} + \frac{1}{w^2} - \frac{2}{zw} \right) \frac{h(g)}{(|z - w|^2)^{2h(g)}} (1 + O(\beta(g))). \quad (4.21)$$

For fields at finite distance, this off-critical correlator has the naive form. The Callan–Symanzik equation of sect. 2 tells us that this expression has IR limit by letting $g \rightarrow g^*$. Therefore the fields at finite distance do not experience any mixing.

Let us consider now the limit $w \rightarrow 0$ in eq. (4.19). The second limit in eq. (4.20) is finite, therefore the normal order of $T\Phi$ according to eq. (4.16) is finite, i.e. $:T\Phi:$ does not develop an anomalous dimension besides the one of Φ . We obtain

$$\langle :T\Phi:(z)\Phi(0) \rangle = \mu^4 \frac{3h(g) - \pi b g}{z^2 (|\mu z|^2)^{2h(g)}} (1 + O(\beta(g))). \quad (4.22)$$

Eq. (4.22) is identical to eq. (4.9), and suggests that the field $:T\Phi:$ normal-ordered off criticality equals $L_{-2}\Phi$ defined previously in eqs. (4.6) and (4.8).

In summary, this example suggests the following interpretation of IR mixing of descendants. Away from criticality, correlators of T and Φ are highly non-polynomial, therefore the normal ordering procedure yields a result depending on the coupling constant which does not commute with the RG flow,

$$\lim_{g \rightarrow g^*} :T\Phi:(z, g) \neq \left(\lim_{g \rightarrow g^*} T(w, g)\Phi(z, g) \right) :. \quad (4.23)$$

The l.h.s. of this equation is a composite renormalized operator having $L_{-2}\Phi_0$ as UV limit. The r.h.s. is $L_{-2}\Phi'$ of the IR CFT.

[★]The full expression of $I(z, w)$, given in the appendix, fulfills the Ward identity coming from conservation of the stress tensor.

4.3. THE FLOW OF CHARGES c_n

Let us discuss the generalized c -theorems of sect. 3 in perturbation theory. Their result, eq. (3.10), establishes that the variation of c_n along the flow is given by the $\langle \Theta_{n-2} \Theta_{n-2} \rangle$ correlator evaluated at scale μ^{-1} .

In the case of Zamolodchikov's c -theorem ($c_2 = c$), the use of eqs. (2.4), (2.5) and (2.11) into eq. (3.10) gives [1,3]

$$-\beta \partial c(g) / \partial g = -6\pi^2 \beta^2. \tag{4.24}$$

For one-coupling flows, this equation can always be integrated as

$$c(g) - c(0) = 6\pi^2 \int_0^g \beta(g) dg = -3\pi^2 yg^2 - 2\pi^3 bg^3 + O(g^4, y^3 g^3) \tag{4.25}$$

and the net change of the central charge is

$$\Delta c \equiv c(0) - c(g^*) = \frac{y^3}{b^2} + O(y^4) = \frac{12}{m^3} + O(m^{-4}) = c(m) - c(m-1) + O(m^{-4}) \tag{4.26}$$

in agreement with the values of $\langle TT \rangle$ at the UV and IR CFTs.

Let us now discuss the current (T_4, Θ_2) in eq. (4.3), defining the charge c_4 . Its value at the CFT is

$$c_4 = 2z^8 \langle T_4(z) T_4(0) \rangle = c^2 + 80c. \tag{4.27}$$

The first condition of the extended c -theorem, $(c_4)_{UV} > (c_4)_{IR}$, is therefore a trivial consequence* of $c_{UV} > c_{IR}$. Inspection of the conserved current at level 6, given in refs. [5, 10], also suggests that c_{2k} is a polynomial of order k in c with large positive coefficients, such that this condition is always satisfied.

More interesting is the computation of the net flow of c_4 . The substitution of eqs. (4.7), (4.9) and (4.11) into eq. (3.10) yields

$$-\beta \partial c_4 / \partial g = -4 \times 6 \times 7 \langle \Theta_2(\mu^{-1}) \Theta_2(0) \rangle \mu^{-8} = -98 \times 6\pi^2 \beta^2 (1 + O(g, y)). \tag{4.28}$$

* For non-unitary theories none of the two bounds holds. Nevertheless, the quantitative prediction of the sum rule eq. (3.11) is still valid.

The integration gives*

$$c_4(0) - c_4(g^*) = 98 \Delta c + O(y^4). \quad (4.29)$$

Therefore we find that it does not match the expected value of eq. (4.27):

$$c_4(m) - c_4(m-1) \sim (\partial c_4 / \partial c) \Delta c \sim 81 \Delta c. \quad (4.30)$$

In order to explain this discrepancy, we suppose that the RG flow carries the current $:T^2:(z, g)$ into a mixture of IR conformal fields, as in the case of descendants of Φ . There are two fields at level 4 in the IR Verma module of the identity, therefore we expect

$$:T^2:(z, g^*) = (1 + \rho \Delta c) :T^2:(z) + \lambda \Delta c \partial_z^2 T' + O(y^4). \quad (4.31)$$

The coefficients ρ and λ are computed by applying the c -theorem to another charge,

$$\tilde{c}_4 = 2z^8 \langle T_4(z) \partial_z^2 T(0) \rangle = 7 \times 36c \quad \text{at CFT}. \quad (4.32)$$

Eq. (3.10) reads in this case

$$-\beta \partial \tilde{c}_4 / \partial g = -6 \times 7 \langle \Theta_2(\mu^{-1}) \partial_z^2 \Theta(0) \rangle \mu^{-8} = -280 \times 6\pi^2 \beta^2 (1 + O(g, y)), \quad (4.33)$$

and it yields

$$\tilde{c}_4(0) - \tilde{c}_4(g^*) = 280 \Delta c + O(y^4), \quad (4.34)$$

again different from the expected value (4.32). By computing the correlators of $:T^2:(z, g^*)$ in the IR CFT according to eq. (4.31) and matching the flows of c_4 and \tilde{c}_4 , it follows that

$$:T^2:(z, g^*) = \left(1 + \frac{2}{27} \Delta c\right) :T^2:(z) - \frac{1}{18} \Delta c \partial_z^2 T' + O(y^4). \quad (4.35)$$

This is the main result of this subsection. Notice that this c -theorem argument is quite effective in computing a higher-order effect $O(g^2, g^3)$, which would be rather cumbersome by straightforward perturbative expansion of $\langle T_4 T_4 \rangle$.

Let us remark the difference between the cases $c_{\text{IR}} = 0$ and $c_{\text{IR}} \neq 0$. In the former case of a purely massive theory, the extended c -theorems hold in their

* Due to non-renormalization of the current Θ_2 , the same result can be obtained in the bare theory, by integrating the sum rule, eq. (3.11), and substituting at the end of the calculation the renormalized coupling by eq. (2.10).

naive form of subsect. 3.2, and give constraints on the RG flow along integrable directions. In the latter case, there is additional freedom in the parameters specifying the flow of descendants. These arguments can still give constraints on the RG flow if supplemented with additional informations. These can be obtained from three-point functions or from the Ward identities for conservation of higher-spin currents, to be discussed in the next section. Conversely, the extended c -theorems can give the flow of currents once the nature of the IR CFT is known.

5. Ward identities for currents and their flow

Let us now discuss the Ward identities (WI) associated to conservation of the stress tensor and of the spin-four current away from criticality. This subject is interesting in itself, in particular the occurrence of invariance equations due to conservation of the latter current (to be defined later in more clear terms). However, a complete analysis is beyond the scope of this work. Therefore we shall limit ourselves to write the equations, study their RG flow and derive consistency checks for the mixings of operators previously found; in short how all fit in the IR CFT.

5.1. THE WARD IDENTITY OF THE STRESS TENSOR

By performing a conformal transformation within the path integral, the following fundamental relation can be derived [25]

$$\begin{aligned} \partial_z \langle T(z) \Phi_0(x_1) \dots \Phi_0(x_n) \rangle + \frac{1}{4} \partial_z \langle \Theta(z) \Phi_0(x_1) \dots \Phi_0(x_n) \rangle \\ = \pi \sum_{i=1}^n \left(\delta(z-x_i) \partial_{x_i} - \partial_z \delta(z-x_i) h_i \right) \langle \Phi_0(x_1) \dots \Phi_0(x_n) \rangle. \end{aligned} \quad (5.1)$$

Renormalization replaces $\Phi_0 \rightarrow \Phi$. Let us remind that this equation states covariance of the theory under conformal transformations, therefore it exists whether there is conformal invariance or not.

At the UV CFT, the Θ -term drops out and this equation gives the analyticity of the correlator $\langle T(z) \Phi \dots \rangle$ in the z complex plane. Together with the behaviour at infinity, this is enough to reconstruct the full function [11]

$$\langle T(z) \Phi_0(x_1) \dots \Phi_0(x_n) \rangle = \sum_{i=1}^n \left(\frac{h_i}{(z-x_i)^2} + \frac{1}{(z-x_i)} \partial_{x_i} \right) \langle \Phi_0(x_1) \dots \Phi_0(x_n) \rangle. \quad (5.2)$$

Away from criticality, analyticity of T is lost and there is no analogue of eq. (5.2).

Nevertheless, both at and off criticality, we can integrate eq. (5.1), eliminate the correlators involving T and Θ , and establish differential relations involving $\langle \Phi(x_1) \dots \Phi(x_n) \rangle$ only. These encode the true symmetries of the theory, those leaving the vacuum invariant, i.e. $\delta S = 0$.

At the conformal point, invariance transformations belong to the group $SL(2, \mathbb{C})$. In the non-critical theory, only translations and rotations survive. For example, consider the case of translations. Upon integration of eq. (5.1) by $\int_{\Sigma_1} d^2z/2i$, where Σ_1 is the strip containing the point x_1 and bounded by the lines $t = t_1 - \epsilon$ and $t = t_1 + \epsilon$, it follows that

$$[P, \Phi(x_1, g)] = \partial\Phi(x_1, g)/\partial x_1. \tag{5.3}$$

Clearly $P \equiv P_1$ is the momentum, the charge associated to conservation of the stress tensor, eqs. (3.4) and (3.5).

Let us now integrate eq. (5.1) over the whole space. On the l.h.s., the contour goes to infinity and vanishes because the correlator goes sufficiently fast to zero. In operator notation this means that $P|0\rangle = 0$, the vacuum is translational invariant [11]. The r.h.s. gives the differential equation of translation invariance

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \langle \Phi(x_1, g) \dots \Phi(x_n, g) \rangle = 0. \tag{5.4}$$

How is the WI (5.1) modified when we flow to the IR CFT? Notice that the contact term depends explicitly on the UV conformal dimension. In order to understand this point, let us study covariance under scale transformations and recover the Callan–Symanzik equation (2.14). Integration of eq. (5.1) by $\int_{\mathbb{R}^2} d^2z$ gives

$$-\frac{1}{4} \int_{\mathbb{R}^2} d^2z \langle \Theta(z) \Phi(x_1) \dots \Phi(x_n) \rangle = \pi \sum_{i=1}^n \left(x_i \frac{\partial}{\partial x_i} + h_i \right) \langle \Phi(x_1) \dots \Phi(x_n) \rangle. \tag{5.5}$$

Let us now add the conjugate equation, and replace $f\Theta = 2\pi\beta f\Phi = -2\pi\beta \partial S/\partial g$. It is not difficult to prove by the definitions of sect. 2 that

$$\beta \partial\Phi(x, g)/\partial g = 2(h(0) - h(g))\Phi(x, g). \tag{5.6}$$

The Callan–Symanzik equation (2.14) is indeed recovered by combining eqs. (5.5) and (5.6). While this equation has a smooth IR limit into the WI for scale invariance of the IR CFT, the original WI (5.1) does not. By inspection of all the terms, one convinces himself that the trace of the stress tensor vanishes as an

operator of the IR CFT, but it leaves contact terms

$$\begin{aligned} & \langle \Theta_2(z)\Phi(x_1) \dots \Phi(x_n) \rangle |_{\text{IR CFT}} \\ &= 4\pi \sum_{i=1}^n (h_i(g^*) - h_i(0))\delta(z - x_i)\langle \Phi(x_1) \dots \Phi(x_n) \rangle, \end{aligned} \tag{5.7}$$

which replace $h \rightarrow h'$ in the r.h.s. of eq. (5.1).

5.2. THE WARD IDENTITY OF THE SPIN-FOUR CURRENT

We proceed by analogy with the case of the stress tensor. The following OPE at CFTs can be derived by the methods of ref. [11] for the generic field ϕ_0 of dimension h :

$$\begin{aligned} :T^2:(\eta)\phi_0(z) &= \frac{2}{(\eta - z)}(L_{-1}L_{-2} + (h - 1)L_{-3})\phi_0(z) \\ &+ \frac{1}{(\eta - z)^2}(2hL_{-2} + L_{-1}^2)\phi_0(z) + \frac{1}{(\eta - z)^3}(2h + 1)L_{-1}\phi_0(z) \\ &+ \frac{h(h + 2)}{(\eta - z)^4}\phi_0(z) + \text{regular terms.} \end{aligned} \tag{5.8}$$

The WI for the spin-four current off criticality will contain a correlation function with insertion of the conservation law, and the contact terms reproducing eq. (5.8). In fact, contact terms are not modified off criticality until we hit the IR point, as in the previous discussion of the stress tensor. Therefore we have

$$\begin{aligned} & \partial_{\bar{z}} \langle :T^2:(z)\phi(x_1) \dots \phi(x_n) \rangle + \partial_z \langle \Theta_2(z)\phi(x_1) \dots \phi(x_n) \rangle \\ &= \pi \sum_{i=1}^n \left\{ \delta(z - x_i) 2[\partial_{x_i} L_{-2}(x_i) + (h_i - 1)L_{-3}(x_i)] \right. \\ &\quad \left. - \partial_z \delta(z - x_i) [2h_i L_{-2}(x_i) + \partial_{x_i}^2] + \partial_z^2 \delta(z - x_i) \frac{2h_i + 1}{2} \partial_{x_i} \right. \\ &\quad \left. - \partial_z^3 \delta(z - x_i) \frac{h_i(h_i + 2)}{6} \right\} \langle \phi(x_1) \dots \phi(x_n) \rangle. \end{aligned} \tag{5.9}$$

This equation establishes covariance of the theory under transformations generated by this current, which are, roughly speaking, squares of conformal transforma-

tions. A subset of them leave the off-critical vacuum invariant, i.e. they are symmetry transformations. We can derive the equations establishing the invariance of the theory, as we did in the case of translations, even if this symmetry is not clear to us in physical terms.

Upon integration of eq. (5.9) on a strip containing x_1 , we get the commutation relation

$$[P_3(g), \phi(x_1, g)] = 2(\partial_{x_1} L_{-2}\phi(x_1, g) + (h-1)L_{-3}\phi(x_1, g)). \quad (5.10)$$

For the perturbing field Φ , we can use the null-vector equation (4.2) and find

$$[P_3(g), \Phi(z, g)] = \frac{2}{h+2} \partial_z \left(3hL_{-2}\Phi(z, g) - \frac{h-1}{h+1} \partial_z^2 \Phi(z, g) \right). \quad (5.11)$$

When eq. (5.9) is integrated over the whole space, its l.h.s. vanishes, as in the case of T . In fact the contour correlator goes to zero sufficiently fast, and gives $P_3|0\rangle = 0$. The r.h.s. yields the invariance equation

$$\sum_{i=1}^n [\partial_{x_i} L_{-2}(x_i) + (h_i - 1)L_{-3}(x_i)] \langle \phi(x_1, g) \dots \phi(x_n, g) \rangle = 0. \quad (5.12)$$

This is the analogue of the WI for translation invariance (5.4) and constraints off-critical correlators involving ϕ , $L_{-2}\phi$ and $L_{-3}\phi$. Since the spin-four current is not a Noether current, i.e. it cannot be derived by a variation of the action, it should be considered as an equation of motion of the theory [5]. In the same way, the equation $P_3|0\rangle = 0$ must be thought as a dynamical symmetry. Both of them may not have a geometrical interpretation. Technically, these are remnants of the conformal structure surviving off criticality. The main question is to translate eq. (5.12) in the language of integrable quantum field theory [32].

The argument leading from eq. (5.1) to the WI for rotations can be generalized similarly. The analogue of the Callan–Symanzik equation can also be obtained, see eqs. (5.5) and (5.6), but the insertion of $f\Theta_2$ in the correlator remains explicit.

5.3. IR LIMIT AND CHECKS OF MIXING COEFFICIENTS

Let us study the infrared limit of the previous commutators and their interplay with IR mixing of descendant fields derived in sect. 4. Let us first notice that the canonical commutator $[P, \Phi] = \partial_z \Phi$ not only ensures that T keeps the canonical dimension, but also says that it has a smooth IR limit as Φ does. The substitution

of the IR mixing of $L_{-2}\Phi$, eq. (4.12), into $[P_3, \Phi]$ gives*

$$\begin{aligned}
 [P_3(g^*), \Phi] &= 2\partial_z \left[\left(1 - \frac{1}{3}y\right) L_{-2}\Phi(g^*) + \frac{1}{12}y \partial^2 \Phi(g^*) \right] \\
 &= 2\partial_z \left[\left(1 + \frac{1}{3}y\right) L_{-2}\Phi' - \frac{1}{12}y \partial^2 \Phi' \right] (1 + O(y^2)) = [P'_3, \Phi'] (1 + O(y^2)).
 \end{aligned}
 \tag{5.13}$$

Therefore, to the accuracy $O(y)$ of our calculation, we have checked that the IR mixing conspires to give to this commutator the correct expression in the IR CFT, up to a possible harmless overall factor.

We can repeat the analysis for $[P_3, T]$. We need the OPE at CFT

$$:T^2:(\eta)T(z) = \frac{1}{\eta - z} \partial_z \left[3:T^2:(z) + \frac{c-1}{6} \partial_z^2 T(z) \right] + \dots$$

This yields the commutator

$$[P_3(g), T(z, g)] = \partial_z \left(3:T^2:(z, g) + \frac{1}{6}(c(m) - 1) \partial_z^2 T(z, g) \right). \tag{5.14}$$

Again, at the end of the RG flow, this commutator does not have the correct form of the IR CFT. This is recovered by substitution of the IR mixing of $:T^2:(g^*)$, eq. (4.35), which was computed by the extended c -theorem,

$$\begin{aligned}
 [P_3(g^*), T(g^*)] &= \partial_z \left[3\left(1 + \frac{2}{27} \Delta c\right) :T_2:' + \frac{1}{6}(c(m) - \Delta c - 1) \partial^2 T' \right] \\
 &= \left(1 + \frac{2}{27} \Delta c\right) \partial_z \left[3:T_2:' + \frac{1}{6}(c(m) - 1) \partial^2 T' \right] \\
 &= \left(1 + \frac{2}{27} \Delta c\right) [P'_3, T']
 \end{aligned}
 \tag{5.15}$$

where $\Delta c = 12/m^3 + O(m^{-4})$.

In summary, the WI for the current of spin-four contains explicit data of the UV CFT, but its flow to the IR CFT is correctly recovered when mixing of descendant fields is taken into account. Notice that an overall constant is present, which cancels in IR CFT equations.

Conversely, we can assume the correct IR limit of the charge in the general form $P_3(g^*) = \kappa P'_3$, and obtain exact mixing coefficients of various fields, up to the overall undetermined constant κ . Cross-checks with the flow of other field equations may fix it. We hope to exploit the full structure of the field theory off criticality in a future work.

* Recall $h = 1 - y/2$ and $h' \sim 1 + y/2$.

6. Conclusions

Perturbation of CFT by a slightly relevant field was shown a computable non-trivial playground for understanding not scale-invariant theories. The presence of higher-spin conserved currents manifests the fact that the CFT structure is not completely destroyed. More work is needed to connect this to known structures of integrable field theories [10, 32, 38]. Nevertheless, we can address technical questions on the RG flow of composite operators and on off-critical Ward identities, and hopefully understand physical issues, like the dynamical symmetries associated to higher-spin conservation laws.

In this respect, the case of the strongly relevant perturbations $y > 1$ seems more difficult. Besides the exciting work on the construction of exact S -matrices [16, 17, 19–21], analytical results are rather difficult to obtain [7, 33]. The understanding of IR singularities is crucial and hopefully it will not imply the complete breakdown of the CFT structure. In short, at the critical point minimal theories are simpler for small m , off criticality they seem simpler for large m .

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Appendix A

INTEGRALS

Correlators of descendant operators at CFT can be computed by eq. (5.2) and the definition

$$L_{-n}\Phi_0(z) = \oint_z d\eta \frac{T(\eta)\Phi_0(z)}{(\eta - z)^{n-1}}. \tag{A.1}$$

The integrals in the lower order of conformal perturbation theory can be done by

$$\int d^2x \frac{(\bar{a} - \bar{x})^\gamma (\bar{x} - \bar{b})^\delta}{(|a - x|^2)^\alpha (|x - b|^2)^\beta} = \frac{\pi (\bar{a} - \bar{b})^{\gamma+\delta}}{(|a - b|^2)^{\alpha+\beta-1}} \frac{\Gamma(\alpha + \beta - 1)}{\Gamma(\alpha)\Gamma(\beta)} \times \frac{\Gamma(1 - \beta + \delta)\Gamma(1 - \alpha + \gamma)}{\Gamma(2 + \delta + \gamma - \beta - \alpha)}. \tag{A.2}$$

More difficult integrals show up in $\int d^2x \langle T(0)\Phi_0(z)\Phi_0(w)\Phi_0(x) \rangle_0$ of subsect. 4.2, for the term

$$\begin{aligned}
 J(w, z) &= \frac{\mu^{2y}(1-y/2)}{|z-w|^{4-2y}} I(w, z) \\
 &= \lambda_0 h \int d^2x \left(\frac{1}{x^2} - \frac{1}{x(z-x)} - \frac{1}{x(w-x)} \right) \frac{1}{(|z-x||w-x|)^{2-y}}, \quad (\text{A.3})
 \end{aligned}$$

where $I(z, w)$ appears in eqs. (4.19) and (4.20). We regularize the holomorphic poles and the double pole by the replacement $1/x \rightarrow \bar{x}/(|x|^2)^{1+\epsilon}$, which respects rotation invariance, and compute the integral by using the Feynman parameters. Then we can take the limit $\epsilon \rightarrow 0$, i.e. the holomorphic poles are harmless, and obtain

$$\begin{aligned}
 J(w, z) &= - \frac{(\pi/2)\lambda_0 b h y (1 + O(y))}{|w-z|^{4-3y}} \\
 &\quad \times \int_0^1 ds \frac{(s(1-s))^{y/2-1} (\bar{w}s + \bar{z}(1-s))^2}{(w^2s + z^2(1-s))^2} F(y, 2; 3; \xi), \quad (\text{A.4})
 \end{aligned}$$

where $\xi = 1 - [s(1-s)(w-z)^2/(w^2s + z^2(1-s))]$. Since $0 < \xi < 1$ and $F(y, 2; 3, 1) = 1 + O(y)$, we can approximate $F \sim 1 + O(y)$ inside the integral (A.4). Then the limits in eq. (4.20) can be obtained in terms of hypergeometric functions and expanded for $y \rightarrow 0$ as well. The conservation of the stress tensor provides a test of eq. (A.4),

$$\partial_{\bar{\eta}} \langle T(\eta)\Phi_0(z)\Phi_0(w) \rangle = \partial_{\bar{\eta}} J(z-\eta, w-\eta) = \frac{1}{2}\pi y \lambda_0 \partial_{\bar{\eta}} \langle \Phi_0(\eta)\Phi_0(z)\Phi_0(w) \rangle. \quad (\text{A.5})$$

Eq. (A.5) can be checked explicitly at the symmetric point $|w|^2 = |z|^2 = \frac{1}{2}|w-z|^2 = \kappa^2, \eta = 0$.

References

- [1] A.B. Zamolodchikov, JETP Lett. 43 (1986) 730
- [2] H. Saleur and C. Itzykson, J. Stat. Phys. 48 (1987) 449
- [3] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 46 (1987) 1090

- [4] A.A. Ludwig and J.L. Cardy, Nucl. Phys. B285 (1987) 687
- [5] A.B. Zamolodchikov, JETP Lett. 46 (1987) 161; Int. J. Mod. Phys. A3 (1988) 743;
A.B. Zamolodchikov, Integrable field theory from conformal field theory, Proc. of the Taganuchi Symposium, Kyoto 1988, to appear in Adv. Studies Pure Math.
- [6] J.L. Cardy, J. Phys. A20 (1987) L891
- [7] V.I. Dotsenko, Nucl. Phys. B314 (1989) 687
- [8] J.L. Cardy, *in* Champs, cordes et phénomènes critiques, Proc. of the 1988 Les Houches summer school, ed. E. Brézin and J. Zinn-Justin, (North-Holland, Amsterdam, 1989)
- [9] A. Bondi, G. Curci, G. Paffuti and P. Rossi, Metric and central charge in the perturbative approach to two dimensional fermionic models, preprint Pisa Univ. IFUP-TH 23/89;
D. Kutasov, Phys. Lett. B227 (1989) 68
- [10] T. Eguchi and S.-K. Yang, Phys. Lett. B224 (1989) 373
- [11] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B241 (1984) 333;
see also Conformal invariance and applications to statistical mechanics, ed. C. Itzykson, H. Saleur and J.B. Zuber (World Scientific, Singapore, 1988);
A. Cappelli, *in* Proc. of the XXIVth Int. Conf. on High energy phys., ed. R. Kotthaus and J. Kühn (Springer, Berlin, 1989) and ref. therein
- [12] R.J. Baxter, Exactly solved models in statistical mechanics (Academic Press, New York, 1982);
V. Pasquier and H. Saleur, Nucl. Phys. B330 (1990) 523
- [13] J.L. Cardy, Phys. Rev. Lett. 60 (1988) 2709
- [14] K.G. Wilson and J. Kogut, Phys. Rep. 12C (1974) 75
- [15] A.M. Polyakov, Phys. Lett. B72 (1977) 224
- [16] A.B. Zamolodchikov and A.I. Zamolodchikov, Ann. Phys. 120 (1979) 253
- [17] A.B. Zamolodchikov, Int. J. Mod. Phys. A4 (1989) 4235
- [18] M. Henkel and H. Saleur, J. Phys. A22 (1989) L513;
G. v. Gehlen, Nucl. Phys. B330 (1990)
- [19] J.L. Cardy and G. Mussardo, Phys. Lett. B225 (1989) 275;
H.W. Braden, E. Corrigan and P.E. Dorey, Phys. Lett. B227 (1989) 411; preprint UDCPT 89/53;
P. Christe and G. Mussardo, Nucl. Phys. B330 (1990) 465;
V.A. Fateev and A.B. Zamolodchikov, Int. J. Mod. Phys. A5 (1990) 1025;
A.B. Zamolodchikov, Landau preprint (1989)
- [20] T. Hollowood and P. Mansfield, Phys. Lett. B226 (1989) 73;
M. Henkel and H. Saleur, J. Phys. A22 (1989) L513
- [21] P. Christe and G. Mussardo, Nucl. Phys. B330 (1990) 465
- [22] E. Brézin, J.C. Le Guillou and J. Zinn-Justin, Phys. Rev. D8 (1973) 434
- [23] G. Parisi, Statistical field theory (Addison-Wesley, Reading, MA, 1988)
- [24] C. Itzykson and J.-B. Zuber, Quantum field theory (McGraw-Hill, New York, 1980)
- [25] A.M. Polyakov, Gauge fields and strings (Harwood, New York, 1987)
- [26] D. Kutasov, Phys. Lett. B220 (1989) 153
- [27] M. Lässig, Nucl. Phys. B334 (1990) 652
- [28] V.I. Dotsenko, Proc. Conf. on Yang–Baxter equation, conformal invariance and integrability, Camberra 1989, Int. J. Mod. Phys. A, to be published
- [29] A.M. Polyakov, JETP 30 (1970) 151
- [30] J.L. Cardy, J. Phys. A21 (1988) L797;
J.L. Cardy and H. Saleur, J. Phys. A22 (1989) L601;
A. Cappelli and J.I. Latorre, unpublished
- [31] R.G. Pogosyan, Sov. J. Nucl. Phys. 48 (1988) 763;
D.A. Kastor, E.J. Martinec and S.H. Shenker, Nucl. Phys. B316 (1989) 590
- [32] R. Sasaki and I. Yamanada, *in* Adv. Studies Pure Math. 16 (1988) 271
- [33] F.A. Smirnov, Reduction of sine-Gordon model as perturbation of minimal models of conformal field theory, Lomi preprint E-4-89;
J.L. Cardy and G. Mussardo, Form factors of descendent operators in perturbed CFT, preprint UCSBTH 90-2

- [34] J. Kodaira, Y. Sasai and H. Sato, Conservation laws away from criticality beyond the lowest order, preprint Hiroshima (1989)
- [35] D. Friedan, unpublished
- [36] P. Pascual and R. Tarrach, QCD: Renormalization for the practitioner, Lecture Notes in Phys. 194 (Springer, Berlin, 1983);
J.C. Collins, Renormalization (Cambridge Univ. Press, Cambridge, 1984);
D.J. Gross, *in* Methods in field theory, Proc. of 1975 Les Houches summer school, ed. R. Balian and J. Zinn-Justin (North-Holland, Amsterdam, 1976)
- [37] A.B. Zamolodchikov, Sov. J. Nucl. Phys. 44 (1986) 529
- [38] M.T. Grisaru, A. Lerda, S. Penati and D. Zanon, Phys. Lett. B234 (1990) 88; preprint MIT CPT-1820 (1990) Nucl. Phys. B, to be published; preprint MIT CPT-1850 (1990)