MODULAR INVARIANT PARTITION FUNCTIONS OF SUPERCONFORMAL THEORIES

A. CAPPELLI [1]

Service de Physique Théorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 29 October 1986

The modular invariant partition functions of two-dimensional minimal superconformal theories are obtained by extending a systematic method developed for conformal theories. They are classified in three infinite series and a few exceptional cases and labelled by simply laced Lie algebras.

In a recent paper [1] (hereafter referred as I), a systematic method has been developed which yields modular invariant partition functions of two-dimensional minimal conformal invariant theories with central charge $c<1$ [2,3]. There are strong indications that this is a complete classification, where each solution is labelled by a pair of simply laced Lie algebras. Superconformal minimal theories [4] are invariant under a larger class of local transformations satisfying a pair of superconformal Neveu–Schwarz–Ramond algebras and have $c<\frac{3}{2}$.

The tricritical Ising model in two dimensions is an example of the simplest superconformal theory [4]; since it has $c=\frac{7}{10}$ it is also a minimal conformal theory. Amazingly, it provides a realization of $N=1$ supersymmetry in nature.

In this letter we obtain modular invariant partition functions for superconformal theories by extending the methods in I. The two mathematical problems are very similar and we shall see that superconformal solutions are made by the same building blocks as the conformal one.

The modular invariance problem has been settled in ref. [5] and two simpler solutions have been obtained. The unitarity condition for representations of the superconformal algebra constrains the values of $c<\frac{3}{2}$ to the discrete series [4]

$$c=\tfrac{3}{2}[1-8/m(m+2)], \quad m=3,4,\dots . \tag{1}$$

The values of the highest weights can be consistently constrained to a finite set (the Kac table):

$$h_{rs}=h_{m-r,m+2-s}=\{[(m+2)r-ms]^2-4\}/8m(m+2)+\tfrac{1}{32}[1-(-)^{r-s}],$$
$$1\leqslant r\leqslant m-1, \quad 1\leqslant s\leqslant m+1. \tag{2}$$

The superconformal algebra and its representations split into two sectors, the Neveu–Schwarz (NS) and Ramond (R) sectors, which are selected by antiperiodic or periodic boundary conditions on fermionic fields, respectively. In eq. (2), the NS states have $r-s$ even and the R states $r-s$ odd. The NS vacuum state has $h=0$, i.e. $r=s=1$, while the R "vacuum" [‡1] has $h=\frac{1}{24}c$ and appears for even m only, at the self-symmetric point of the Kac table $(r,s)=(\frac{1}{2}m,\frac{1}{2}m+1)$.

A fundamental domain Δ is a set of independent h values in each sector: $\Delta_{NS}=\{h_{rs}\,|\,1\leqslant s\leqslant r\leqslant m-1\}$, $r-s$ even and $\Delta_R=\{h_{rs}\,|\,1\leqslant s\leqslant r-1$ for $1\leqslant r\leqslant[\frac{1}{2}(m-1)]$ and $1\leqslant s\leqslant r+1$ for $[\frac{1}{2}(m+1)]\leqslant r\leqslant m-1\}$.

[1] On leave of absence from Dipartimento di Fisica and INFN, Sezione di Firenze, Largo E. Fermi 2, I-50125 Florence, Italy.
[‡1] This is the state of lowest energy in the R sector and it is globally supersymmetric invariant.

0370-2693/87/$03.50 © Elsevier Science Publishers B.V.
(North-Holland Physics Publishing Division)

As in conformal theories [3], the modular invariance conditions on the partition function of the theory defined on a torus yield the possible values of scaling dimensions $(h, \bar{h})$ of primary (super) fields, with h and $\bar{h}$ taken from the Kac table. The partition function is the sum of four terms for periodic (+) and antiperiodic (−) boundary conditions. Let us take a torus with periods ω_1, ω_2, $\tau=\omega_2/\omega_1$. Im $\tau>0$ and denote $Z(\alpha, \beta)$ the term pertaining to condition α along ω_2 and β along ω_1, $\alpha, \beta=\pm$. We have

$$Z_{(-,-)}(\tau)=Z^{\mathrm{NS}}=\mathrm{Tr}(\mathscr{T})_{\mathrm{NS}}=\sum_{h,\bar{h}}\mathscr{N}^{\mathrm{NS}}_{h\bar{h}}\chi^{\mathrm{NS}}_{h}(\tau)\left(\chi^{\mathrm{NS}}_{\bar{h}}(\tau)\right)^{*}, \tag{3a}$$

$$Z_{(+,-)}(\tau)=Z^{\widetilde{\mathrm{NS}}}=\mathrm{Tr}\left(\mathscr{T}(-)^{F}\right)_{\mathrm{NS}}=\sum_{h,\bar{h}}\mathscr{N}^{\widetilde{\mathrm{NS}}}_{h\bar{h}}\chi^{\widetilde{\mathrm{NS}}}_{h}(\tau)\left(\chi^{\widetilde{\mathrm{NS}}}_{\bar{h}}(\tau)\right)^{*}, \tag{3b}$$

$$Z_{(-,+)}(\tau)=Z^{\mathrm{R}}=\mathrm{Tr}(\mathscr{T})_{\mathrm{R}}=\sum_{h,\bar{h}}\mathscr{N}^{\mathrm{R}}_{h\bar{h}}\pm\chi^{\mathrm{R}}_{h}(\tau)\left(\chi^{\mathrm{R}}_{\bar{h}}(\tau)\right)^{*}, \tag{3c}$$

$$Z_{(+,+)}=Z^{\tilde{\mathrm{R}}}=\mathrm{Tr}\left(\mathscr{T}(-)^{F}\right)_{\mathrm{R}}=\mathrm{Tr}(-)^{F}, \tag{3d}$$

In eqs. (3), the ω_1 boundary conditions select the sector (NS or R) of states in the transfer matrix $\mathscr{T}=\exp\{2\mathrm{i}\pi[\tau(L_0-\frac{1}{24}c)-\tau^{*}(\bar{L}_0-\frac{1}{24}c)]\}$, while periodic ω_2 conditions yield the sign $(-)^F$ for the fermionic states. In the last term, x^J_h are the characters of the irreducible superconformal representations of weight h [6] for $J=\mathrm{R}$, NS and $\widetilde{\mathrm{NS}}$ sectors. They include the prefactor $\exp(-2\pi\mathrm{i}\tau c/24)$ for all J and for $J=\widetilde{\mathrm{NS}}$ they contain the sign $(-)^F$ in the trace over NS states, as it will be explained later. Summations extend to the h values in eq. (2) of the NS or R sectors.

The decomposition of the trace into irreducible representations yields the non-negative integer matrices $\mathscr{N}^{J}_{h,\bar{h}}=\mathscr{N}^{J}_{\bar{h},h}$. In particular, for $J=\widetilde{\mathrm{NS}}$ signs may arise in the decomposition, but they are included in the definition of $\chi^{\widetilde{\mathrm{NS}}}_{h}$ characters, in such a way that modular invariance will give $\mathscr{N}^{\mathrm{NS}}_{h\bar{h}}=\mathscr{N}^{\widetilde{\mathrm{NS}}}_{h\bar{h}}$. Therefore all matrices will be positive. In the following, the modular invariance conditions will determine them.

Since the R states, excluding the vacuum, are doubly degenerate and of opposite "chirality" $\Gamma=(-)^F$, a factor $\sqrt{2}$ will be included in the definition of χ^{R}_{h} in eq. (3c); in eq. (3d) their contributions cancel leaving the constant $Z^{\tilde{\mathrm{R}}}$ which is not determined by modular invariance [5].

The modular group $\Gamma=\mathrm{PSL}(2, \mathbb{Z})$ of τ-transformations [7]

$$\Gamma\ni A=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad A\colon \tau\to{}^{A}\tau=\frac{a\tau+b}{c\tau+d} \tag{4}$$

is generated by the transformations S: $\tau\to-1/\tau$ and T: $\tau\to\tau+1$, which may change boundary conditions. We have

$$Z^{\mathrm{NS}}(\tau+1)=Z^{\widetilde{\mathrm{NS}}}(\tau),\quad Z^{\mathrm{NS}}(-1/\tau)=Z^{\mathrm{NS}}(\tau),\quad Z^{\widetilde{\mathrm{NS}}}(\tau+1)=Z^{\mathrm{NS}}(\tau),\quad Z^{\widetilde{\mathrm{NS}}}(-1/\tau)=Z^{\mathrm{R}}(\tau),$$

$$Z^{\mathrm{R}}(\tau+1)=Z^{\mathrm{R}}(\tau),\quad Z^{\mathrm{R}}(-1/\tau)=Z^{\widetilde{\mathrm{NS}}}(\tau),\quad Z^{\tilde{\mathrm{R}}}(\tau+1)=Z^{\tilde{\mathrm{R}}}(\tau),\quad Z^{\tilde{\mathrm{R}}}(-1/\tau)=Z^{\tilde{\mathrm{R}}}(\tau). \tag{5}$$

It follows that modular invariance requires

$$Z=a\left(Z^{\mathrm{NS}}+Z^{\widetilde{\mathrm{NS}}}+Z^{\mathrm{R}}\right)+bZ^{\tilde{\mathrm{R}}}, \tag{6}$$

with a, b free constants up to the normalization. From eqs. (3), we see that this corresponds to the projection $\Gamma=(-)^F=1$ in the NS sector; by taking $b=\pm a$ we may have $\Gamma=\pm1$ in the R sector. These projections yield consistent local theories called "spin models" in ref. [4]. The subgroup Γ_2 of $\Gamma_1\Gamma_2=\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)\mathrm{mod}\ 2\}$, is generated by T^2 and ST^2S and transforms each term into itself; the local fermionic theory in the NS sector, corresponding to $Z=Z^{\mathrm{NS}}$, is Γ_2 invariant only.

Table 1
List of known partition functions in terms of affine $A_1^{(1)}$ characters (from I).

$k \geqslant 1$	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	A_{k+1}
$k+2=4\rho+2$, $\rho \geqslant 1$	$\sum_{\lambda \text{ odd}=1, \lambda \neq 2\rho+1}^{4\rho+1} \lvert\chi_\lambda\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2 + \sum_{\lambda \text{ odd}=1}^{2\rho-1} (\chi_\lambda \chi_{4\rho+2-\lambda}^* + \text{c.c.}) = \sum_{\lambda \text{ odd}=1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$D_{2\rho+2}$
$k+2=4\rho$, $\rho \geqslant 2$	$\sum_{\lambda \text{ odd}=1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda \text{ even}=2}^{2\rho-2} (\chi_\lambda \chi_{4\rho-\lambda}^* + \text{c.c.})$	$D_{2\rho+1}$
$k+2=12$	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	E_6
$k+2=18$	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_8+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3+\chi_{15})\chi_9^* + \text{c.c.}]$	E_7
$k+2=30$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	E_8

Here we discuss the invariance under the full modular group and we present matrices $\mathcal{N}_{h,h}^J$ solutions of eqs. (5). Our results are natural extensions of those in the conformal case, which were analyzed in details in I. Let us state them:

(i) the characters χ_h^J carry a unitary projective representation of the finite group $M_{4N} = \Gamma/\Gamma_{4N} = \text{PSL}(2, \mathbb{Z}/4N\mathbb{Z})$ for m odd and M_N for m even, where $N = 2m(m+2)$;

(ii) the solution factorizes in the tensor product of a pair of modular invariants matrices of the affine $A_1^{(1)}$ Kac–Moody algebra [8], for representations of level $k = m-2$ and $k' = m$, respectively;

(iii) the positive solutions are made by pairs of positive affine invariants (they are recalled in table 1). In I they were labelled by simply laced Lie algebras, because the values of the index r in the diagonal matrix terms were recognized as the Betti numbers of such algebras. Superconformal solutions are therefore labelled by a pair of simply laced Lie algebras and are listed in table 2.

For m odd only the diagonal solution $\mathcal{N}_{h,\bar{h}} = \delta_{h,\bar{h}}$ appears because it is the unique choice for both k and k' odd (algebras (A_{m-1}, A_{m+1})). For even m two further infinite series appear. They are for $m = 4\rho$: $(A_{m-1}, D_{2\rho+2})$ $\rho \geqslant 1$ and $(D_{2\rho+1}, A_{m+1})$ $\rho \geqslant 2$; for $m = 4\rho+2$: $(A_{m-1}, D_{2\rho+3})$ $\rho \geqslant 1$ and $(D_{2\rho+2}, A_{m+1})$ $\rho \geqslant 1$. The (D, D) pair is equivalent to one of the two. Solutions (A, A), $(A, D_{2\rho+2})$ and $(D_{2\rho+2}, A)$ were already found in ref. [5]. In addition, there are exceptional cases for $m = 10$, 12, 16, 18, 28, 30, by combining (A, E) or (D, E) affine invariants. There are two more solutions for $m = 10$ $((A_9, E_6), (D_6, E_6))$ and $m = 12$ $((E_6, A_{13}), (E_6, D_8))$, one more solution for $m = 16$ $((A_{15}, E_7) \approx (D_9, E_7))$, $m = 18$ $((E_7, A_{19}) \approx (E_7, D_{11}))$. $m = 28$ $((A_{27}, E_8) \approx (D_{15}, E_8))$ and $m = 30$ $((E_8, A_{29}) \approx (E_8, D_{17}))$.

As our analysis parallels that of I and sometimes reduces to it, it seems natural to extend the two conjectures made there: namely,

(i) our methods yield all invariants including those with negative signs [‡2];

(ii) the subset of positive invariants is labelled by simply laced Lie algebras. In the supersymmetric case however, this labelling is not unique, since $D_{2\rho+1}$ combinations are sometimes degenerate with A_m ones; we shall clarify this point later on.

These solutions yield the operator content of superconformal theories as follows. In the R-sector, there exist primary conformal fields $\Theta_{h,\bar{h}}^{\pm}$, called "spin-fields" in ref. [4], with multiplicity given by the matrix elements $\mathcal{N}_{h\bar{h}}^{R} \neq 0$. In the NS sector, primary superfields $\Phi_{h,\bar{h}}$ corresponding to $\mathcal{N}_{h,\bar{h}}^{NS} \neq 0$ decompose into bosonic and fermionic components (conformal) fields and their descendants; fermionic components are cancelled by the projection $1 + (-)^F$ leaving only bosonic combinations.

‡2 A proof exists for conformal solutions when m has no square factors (unpublished).

Table 2
List of known partition functions in terms of superconformal characters: $\chi^{\mathrm{NS}}=\chi$, $\widetilde{\chi^{\mathrm{NS}}}=\tilde{\chi}$, $\chi^{\mathrm{R}}=\hat{\chi}$; (p', p) correspond to $(m, m+2)$ or $(m+2, m)$ by exchanging $r \leftrightarrow s$.

$m \geqslant 3$	$\frac{1}{4}\sum_{\substack{r=1\\ r-s}}^{m-1}\sum_{\substack{s=1\\ \text{even}}}^{m+1}(\lvert\chi_{rs}\rvert^2+\lvert\tilde{\chi}_{rs}\rvert^2)+\frac{1}{4}\sum_{\substack{r=1\\ r-s}}^{m-1}\sum_{\substack{s=1\\ \text{odd}}}^{m+1}\lvert\hat{\chi}_{rs}\rvert^2$	$(\mathrm{A}_{m-1}, \mathrm{A}_{m+1})$
$p'=4\rho$, $\rho \geqslant 1$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\left(\sum_{\substack{s=1\\ \text{odd}}}^{2\rho-1}\lvert\chi_{r,s}+\chi_{r,p-s}\rvert^2+2\lvert\chi_{r,2\rho+1}\rvert^2+\{\chi\to\tilde{\chi}\}\right)$	
	$+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\left(\sum_{\substack{s=1\\ \text{odd}}}^{2\rho-1}\lvert\hat{\chi}_{r,s}+\hat{\chi}_{r,p-s}\rvert^2+2\lvert\hat{\chi}_{r,2\rho+1}\rvert^2\right)$	$(\mathrm{A}_{p'-1}, \mathrm{D}_{2\rho+2})$
$p'=4\rho$, $\rho \geqslant 2$	$\frac{1}{4}\sum_{\substack{s=1\\ \text{odd}}}^{p-1}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}(\lvert\chi_{r,s}\rvert^2+\lvert\tilde{\chi}_{rs}\rvert^2)+\frac{1}{4}\sum_{\substack{s=2\\ \text{even}}}^{p-2}\left(\lvert\chi_{2\rho,s}\rvert^2+\sum_{\substack{r=2\\ \text{even}}}^{2\rho-2}(\chi_{rs}\chi^{*}_{p'-rs}+c.c.)+\{\chi\to\tilde{\chi}\}\right)$	
	$+\frac{1}{4}\sum_{\substack{s=2\\ \text{even}}}^{p-2}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\lvert\hat{\chi}_{rs}\rvert^2+\frac{1}{4}\sum_{\substack{s=1\\ \text{odd}}}^{p-1}\left(\lvert\hat{\chi}_{2\rho,s}\rvert^2+\sum_{\substack{r=2\\ \text{even}}}^{2\rho-2}(\hat{\chi}_{rs}\hat{\chi}^{*}_{p'-r,s}+c.c.)\right)$	$(\mathrm{D}_{2\rho+1}, \mathrm{A}_{p-1})$
$p'=10$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}[\lvert\chi_{r1}+\chi_{r7}\rvert^2+\lvert\chi_{r5}\,\chi_{r11}\rvert^2+(\chi\to\tilde{\chi})]=\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\left(\lvert\chi_{r4}+\chi_{r8}\rvert^2+\{\chi\to\tilde{\chi}\}\right)$	
	$+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}(\lvert\hat{\chi}_{r1}+\hat{\chi}_{r7}\rvert^2+\lvert\hat{\chi}_{r5}+\hat{\chi}_{r11}\rvert^2)+\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\lvert\hat{\chi}_{r4}+\hat{\chi}_{r8}\rvert^2$	$(\mathrm{A}_{p'-1}, \mathrm{E}_6)$
$p'=10$ $=4\rho+2$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}(\lvert\chi_{r1}+\chi_{r5}+\chi_{r7}+\chi_{r11}\rvert^2+\lvert\tilde{\chi}_{r1}+\tilde{\chi}_{r5}+\tilde{\chi}_{r7}+\tilde{\chi}_{r11}\rvert^2+2\lvert\hat{\chi}_{r4}+\hat{\chi}_{r8}\rvert^2)$	$(\mathrm{D}_{2\rho+2}, \mathrm{E}_6)$
$p'=16$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{\lvert\chi_{r1}+\chi_{r17}\rvert^2+\lvert\chi_{r5}+\chi_{r13}\rvert^2+\lvert\chi_{r7}+\chi_{r11}\rvert^2+\lvert\chi_{r9}\rvert^2+[(\chi_{r3}+\chi_{r15})\chi^{*}_{r9}+\mathrm{c.c.}]\}$	
	$+\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{\chi\to\tilde{\chi}\}+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'=2}\{\chi\to\hat{\chi}\}$	$(\mathrm{A}_{p'-1}, \mathrm{E}_7)$
$p'=28$	$\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}(\lvert\chi_{r1}+\chi_{r11}+\chi_{r19}+\chi_{r29}\rvert^2+\lvert\chi_{r7}+\chi_{r13}+\chi_{r17}+\chi_{r23}\rvert^2)$	
	$+\frac{1}{4}\sum_{\substack{r=1\\ \text{odd}}}^{p'-1}\{\chi\to\tilde{\chi}\}+\frac{1}{4}\sum_{\substack{r=2\\ \text{even}}}^{p'-2}\{\chi\to\hat{\chi}\}$	$(\mathrm{A}_{p'-1}, \mathrm{E}_8)$

The main difference with respect to conformal solutions is the possibility of half-integer spins in the NS sector, i.e. $\mathcal{N}^{\mathrm{NS}}_{h,\bar{h}} \neq 0$ for $s=h-\bar{h}\in\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$. The exceptional solutions (A, E) and (D, E) indeed contain half-integer spin superfields: this only means that they have fermionic and bosonic components interchanged; fermionic components are still projected out.

Let us first write the characters and their modular transformations. The first step is to trade the vector index (r, s) which labels characters modulo $(m, m+2)$ and $m, -m-2)$, into the scalar $\lambda=(m+2)r-ms$ modulo $N=2m(m+2)$. We need to consider the cases m odd and even separately.

For m odd, m and $m+2$ are coprimes, then the discussion in I applies directly: setting $\bar{\lambda}=(m+2)r+ms$, we have

$$\bar{\lambda}=\omega_0\lambda \mod N, \quad \omega_0=m+1-\tfrac{1}{2}N, \quad \omega_0^2=1 \mod 4N. \tag{7}$$

The superconformal characters [6] are written in our notations as follows:

$$\chi_\lambda^{\mathrm{NS}}(\tau)=\left[N_\lambda(\tfrac{1}{2}\tau)-N_{\bar{\lambda}}(\tfrac{1}{2}\tau)\right]\exp(-\tfrac{2}{48}\pi\mathrm{i})\eta(\tfrac{1}{2}(\tau+1))\eta^{-2}(\tau) \quad \lambda \text{ even}, \tag{8a}$$

$$\widetilde{\chi_\lambda^{\mathrm{NS}}}(\tau)=\chi_\lambda^{\mathrm{NS}}(\tau+1)=\left[N_\lambda(\tfrac{1}{2}(\tau+1))-N_{\bar{\lambda}}(\tfrac{1}{2}(\tau+1))\right]\exp(-\tfrac{2}{16}\pi\mathrm{i})\eta(\tfrac{1}{2}\tau)\eta^{-2}(\tau) \quad \lambda \text{ even}, \tag{8b}$$

$$\chi_\lambda^{\mathrm{R}}(\tau)=\sqrt{2}\left[N_\lambda(\tfrac{1}{2}\tau)-N_{\bar{\lambda}}(\tfrac{1}{2}\tau)\right]\eta(2\tau)\eta^{-2}(\tau) \quad \lambda \text{ odd}, \tag{8c}$$

in terms of the functions

$$N_\lambda(\tau)=\sum_{n=-\infty}^{+\infty}\exp\left[2\mathrm{i}\pi\tau(Nn+\lambda)^2/2N\right], \quad \eta(\tau)=\exp(\tfrac{2}{24}\mathrm{i}\pi\tau)\prod_{n=1}^{\infty}\left[1-\exp(2\mathrm{i}\pi\tau n)\right]. \tag{9}$$

Due to the symmetry properties

$$\chi_\lambda^J=\chi_{-\lambda}^J=\chi_{\lambda+N}^J=-\chi_{\bar{\lambda}}^J, \quad \text{for all } J, \tag{10}$$

the matrix problem in eqs. (3)–(5) requires λ modulo N. The matrices $\mathcal{N}^J_{\lambda,\lambda'}$ are extended out of the fundamental domains Δ_J according to these symmetries; we shall remove this degeneracy at the end of the discussion. Intermediate steps of the calculations require sometimes λ modulo $2N$ but these extensions do not lead to ambiguities.

The characters are traces of $\exp(2\mathrm{i}\pi\tau L_0)$ over states of weight h, which have degeneracy $d(n)$ at level n. We have

$$\widetilde{\chi_h^{\mathrm{NS}}}(\tau)\left(\widetilde{\chi_{h'}^{\mathrm{NS}}}(\tau)\right)^*=|\exp(-2\pi\mathrm{i}\tau\tfrac{1}{24}c)|^2\exp[2\pi\mathrm{i}(h-h')]$$
$$\times\sum_{n,n'=0}^{\infty}d(n)d(n')\exp[2\mathrm{i}\pi(\tau n-\tau^* n'+n+n')], \tag{11}$$

where n, n' run over integer and half-integer values for NS states. Therefore the sign into eq. (3b) is given by $(-)^F=\exp[2\pi\mathrm{i}(h-h'+n+n')]$: since modular invariance constrains $h-h'$ to be integer or half integer, the sign is correct for the lowest level $n=n'=0$, the highest weight; additional negative signs appear for higher level states built by an odd number of fermionic operators.

The modular transformation of characters can be written as

$$T: \quad \widetilde{\chi_\lambda^{\mathrm{NS}}}(\tau+1)=\exp\left[2\pi\mathrm{i}(\lambda^2/2N-\tfrac{1}{8})\right]\chi_\lambda^{\mathrm{NS}}(\tau), \quad \chi_\lambda^{\mathrm{R}}(\tau+1)=\exp(2\pi\mathrm{i}\lambda^2/4N)\chi_\lambda^{\mathrm{R}}(\tau),$$

$$S: \quad \chi_\lambda^{\mathrm{NS}}(-1/\tau)=\sqrt{2/N}\sum_{\substack{\lambda'=2\\ \text{even}}}^{N}\exp(2\pi\mathrm{i}\lambda\lambda'/2N)\chi_{\lambda'}^{\mathrm{NS}}(\tau) \quad \lambda \text{ even},$$

$$\widetilde{\chi_\lambda^{\mathrm{NS}}}(-1/\tau)=\exp\left[2\pi\mathrm{i}(\lambda^2/4N-\tfrac{1}{16})\right]\sqrt{2/N}\sum_{\substack{\lambda'=1\\ \text{odd}}}^{N}\exp(2\pi\mathrm{i}\lambda\lambda'/2N)\chi_{\lambda'}^{\mathrm{R}}(\tau) \quad \lambda \text{ even},$$

$$\chi_\lambda^{\mathrm{R}}(-1/\tau)=\sqrt{2N}\sum_{\substack{\lambda'=2\\ \text{even}}}^{N}\exp\left[2\pi\mathrm{i}(\lambda\lambda'/2N-\lambda'^2/4N+\tfrac{1}{16})\right]\widetilde{\chi_{\lambda'}^{\mathrm{NS}}}(\tau) \quad \lambda \text{ odd}. \tag{12}$$

By using the symmetries in eq. (10), it can be checked that the S transformation reduces to the form in ref.

[5], and it can be written as a matrix acting on a vector of all characters whose square is the identity. Therefore the characters carry an unitary representation of the group Γ. Moreover the results of appendix B in I apply to the characters in eqs. (8), (9) and show that they transform by a λ- and τ-independent phase under transformations of the subgroup $\Gamma_{4N} \subset \Gamma$ [7]. They carry therefore a unitary projective representation of the quotient group $M_{4N} = \Gamma/\Gamma_{4N}$.

From eqs. (12), (5), the modular invariance conditions for the $\mathcal{N}^{J}_{\lambda,\lambda'}$ matrices are

$$T: \quad \mathcal{N}^{\rm NS}_{\lambda,\lambda'} = \mathcal{N}^{\widetilde{\rm NS}}_{\lambda,\lambda'} \neq 0, \quad \text{only if} \quad \lambda^2 = \lambda'^2 \mod 2N, \tag{13a}$$

$$\mathcal{N}^{\rm R}_{\lambda\lambda'} \neq 0, \quad \text{only if} \quad \lambda^2 = \lambda'^2 \mod 4N, \tag{13b}$$

and

$$S: \quad \frac{2}{N} \sum_{\eta,\eta'=2\ \text{even}}^{N} \exp[\mathrm{i}2\pi(\eta\lambda - \eta'\lambda')/2N]\, \mathcal{N}^{\rm NS}_{\eta,\eta'} = \mathcal{N}^{\rm NS}_{\lambda,\lambda'} \qquad \lambda,\ \lambda' \ \text{even} \tag{14a}$$

$$\frac{2}{N} \sum_{\eta,\eta'=2\ \text{even}}^{N} \exp\{\mathrm{i}2\pi[(\eta\lambda - \eta'\lambda')/2N - (\eta^2 - \eta'^2)/4N]\}\, \mathcal{N}^{\widetilde{\rm NS}}_{\eta\eta'} = \mathcal{N}^{\rm R}_{\lambda,\lambda'} \qquad \lambda,\ \lambda' \ \text{odd.} \tag{14b}$$

The modular equations in the conformal case (eq. (3.8) in I) are similar to eqs. (13), (14): indeed it can be checked that the conformal solution $\mathcal{N}^{\rm conf.}_{\lambda,\lambda'}$ for $N = 2m(m+2)$, m odd, also satisfies eqs. (13), (14), by identifying $\mathcal{N}^{\rm NS}_{\lambda,\lambda'} = \mathcal{N}^{\widetilde{\rm NS}}_{\lambda,\lambda'} = \mathcal{N}^{\rm conf.}_{\lambda,\lambda'}$ for λ, λ' even, and $\mathcal{N}^{\rm R}_{\lambda,\lambda'} = \mathcal{N}^{\rm conf.}_{\lambda,\lambda'}$ for λ, λ' odd. Let us recall its expression (eq. (3.14) in I):

$$\mathcal{N}^{\rm conf.}_{\lambda,\lambda'} = \sum_{\substack{\alpha:\,\alpha^2 | N/2 \\ \alpha|\lambda,\alpha|\lambda'}} \sum_{\mu} C(\alpha,\mu) \sum_{\xi=0}^{\alpha-1} \delta_{\lambda',\mu\lambda+\xi N/\alpha \bmod N}. \tag{15}$$

The $C(\alpha, \mu)$ are free coefficients of the linear combination of solutions, which are characterized by the pairs (α, μ) where α^2 is a factor of $\frac{1}{2}N$ and μ are numbers mod N/α^2 such that $\mu^2 = 1 \bmod 2N/\alpha^2$. Actually, for m odd α is also odd and it can be shown that $\mu^2 = 1 \bmod 4N/\alpha^2$. It follows that only integer spin solutions appear.

The subsequent analysis of I applies completely: for odd m the unique positive solution is the diagonal one for both the NS and the R sector. We shall not discuss here the possibility of half integer spin solutions for the NS sector only, which are modular invariants on the subgroup Γ_2.

Let us now discuss the even-m case, starting again from eq. (7): now $\lambda = (m+2)r - ms$ is always even and it is also invariant under $(r, s) \to (r + \frac{1}{2}m,\ s + \frac{1}{2}m + 1)$. However, this translation changes the $r - s$ parity, i.e. it connects the two sectors R and NS. Therefore we may use again the variable λ modulo N within each sector. We have

$$l = (\tfrac{1}{2}m + 1)r - \tfrac{1}{2}ms, \quad \tilde{l} = (\tfrac{1}{2}m + 1)r - \tfrac{1}{2}m_s,$$

$$\text{NS:} \quad \tilde{l} = \omega_0 l \mod \tfrac{1}{2}N, \quad \text{R:} \quad \tilde{l} = \omega_0 l + \tfrac{1}{4}N \mod \tfrac{1}{2}N, \tag{16}$$

with $\omega_0 = m + 1$ and $\omega_0^2 = 1 \bmod \frac{1}{2}N$.

This allows to define characters and to obtain modular transformations and equations. They are as in eqs. (12)–(14) with summations on even λ in all sectors. The subgroup of modular transformations acting by a global phase is now Γ_N.

For m even, there are both integer spin solutions to eqs. (13), (14), namely $l^2 = l'^2 \bmod N$ for R, NS, and half-integer spin solutions, $l^2 = l'^2 \bmod \frac{1}{2}N$ for NS, $l^2 = l'^s \bmod N$ for R. Both solutions may be obtained by the methods of I. For integer spin solutions, it can be checked that the conformal invariant eq. (15) for the variable $l \bmod \frac{1}{2}N$ satisfies eqs. (13), (14) for m even. Half-integer spin solutions can be obtained by some modifications of the methods. Amazingly, it can be shown that both kinds of solutions factorize into a pair of affine $A_1^{(1)}$ invariants provided that they are contracted with characters of given $r-s$ parity (i.e. in the partition function there are no cross terms like $\chi_\lambda^{NS}(\chi_{\lambda'}^{R})^*$). We obtain

$$\mathcal{N}^{NS}_{\lambda,\lambda'} = \mathcal{N}^{\widetilde{NS}}_{\lambda,\lambda'} = \mathcal{N}_{rr'}\mathcal{N}_{ss'}, \quad r-s=r'-s'=0 \mod 2,$$
$$\mathcal{N}^{R}_{\lambda,\lambda'} = \mathcal{N}_{rr'}\mathcal{N}_{ss'}, \quad r-s=r'-s'=1 \mod 2, \tag{17}$$

for pairs $(\mathcal{N}_{rr'}, \mathcal{N}_{ss'})$ of affine invariants of level $k=m-2$ and $k'=m$, respectively (see table 1); $\lambda=(m+2)r-ms$ and $\lambda'=(m+2)r'-ms'$ have values in the fundamental domains Δ_J.

Conversely, superconformal solutions can be obtained by taking pairs of affine invariants; they are split into NS, $\widetilde{NS}$ and R sectors according to $r-s$ parity (see table 2). This construction yields sometimes degenerate solutions for combinations containing the affine invariant $D_{2\rho+1}$. The symmetry of the Kac table in eq. (2) implies $\mathcal{N}_{r's'rs}=\mathcal{N}_{r's',m-r,m+2-s}$, therefore only symmetrized tensor products yield independent solutions in eq. (17): $\mathcal{N}_{r's'rs}=\mathcal{N}_{rr'}\mathcal{N}_{ss'}+\mathcal{N}_{r',m-r}\mathcal{N}_{n',m+2-s}$. Let us consider now the affine invariants which satisfy $\mathcal{N}_{s's}=\mathcal{N}_{s',m+2-s}$: their combinations with the $D_{2\rho+2}$ solution $(\mathcal{N}_{r'r})$ or the A one $(\delta_{rr'})$ are equivalent because $\mathcal{N}_{r'r}+\mathcal{N}_{r',m-r}=\delta_{r'r}$. This degeneracy does not appear in the conformal solutions because these $D_{2\rho+2}$ pairs are not allowed.

In table 2, solutions are normalized to have a non-degenerate vacuum $(h,\bar{h})=(0,0)$, i.e. $\mathcal{N}^{NS}_{0,0}=\mathcal{N}^{\widetilde{NS}}_{0,0}=1/2$. Since the $|\chi^R|^2$ contain a two factor for double degeneracy of states, the R representations have integer coefficients $2\mathcal{N}^{R}_{h,\bar{h}}$, as expected from the decomposition of the trace of the transfer matrix in eq. (3); if the R "vacuum" $(c/24, c/24)$ appears in the solution, the term in eq. (3d) is also different from zero, then $(\mathcal{N}^{R}_{c/24,c/24}\pm Z^{\tilde{R}})/2\in\mathbb{Z}$. (The constant $Z^{\tilde{R}}$ is omitted in table 2).

In conclusion, we hope this classification will be useful in searching for a microscopic 2D statistical model which realizes superconformal invariance. Up to now only the tricritical Ising model ($m=3$) fits the classification [4,5] and for $m=4$ there are attempts with the Ashkin–Teller model [9] and the gaussian model [10].

Claude Itzykson and Jean-Bernard Zuber are acknowledged for good suggestions and discussions. The Angelo Della Riccia foundation is also acknowledged for partial support.

References

[1] A. Cappelli, C. Itzykson and J.B. Zuber, Saclay preprint PhT 86/122, to be published in Nucl. Phys. B.
[2] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. B 241 (1984) 333.
[3] J.L. Cardy, Nucl. Phys. B 270 [FS16] (1986) 186;
C. Itzykson and J.B. Zuber, Nucl. Phys. B, to be published;
G. Gepner, Princeton preprint (1986).
[4] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. 52 (1984) 1575; Phys. Lett. B 151 (1985) 37;
M. Bershadsky, V. Knizhnik and M. Teitelman, Phys. Lett. B 151 (1985) 31;
Z. Qiu, Nucl. Phys. B 270 (1986) 205.
[5] D. Kastor, Chicago preprint EFI 86-20;
Y. Matsuo and S. Yahikozawa, preprint UT-485 (June 1986).
[6] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103 (1986) 105.
[7] R.C. Gunnings, Lectures on Modular forms (Princeton U.P., Princeton, NJ, 1962).
[8] G. Gepner and E. Witten, Princeton preprint (1986).
[9] G. von Gehlen and V. Rittenberg, Bonn preprint (1986).
[10] D. Friedan and S.H. Shenker, Chicago preprint EFI 86-17.