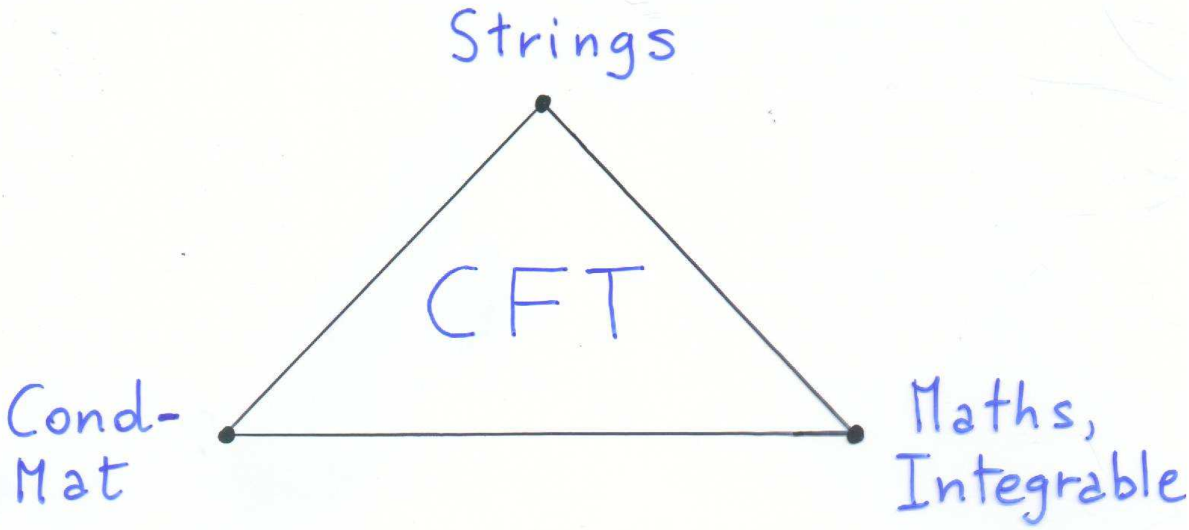
Boundary Conformal Field Theory

Outline

- Introduction
- Boundary states of Rational CFTs
  → conditions of modular covariance
- Orbifold technique for boundaries
  → Fractional branes, $c=1, 3/2$
- Boundary Renormalization-group flows
  → "g-theorem" conjecture

(B)CFT Scenario



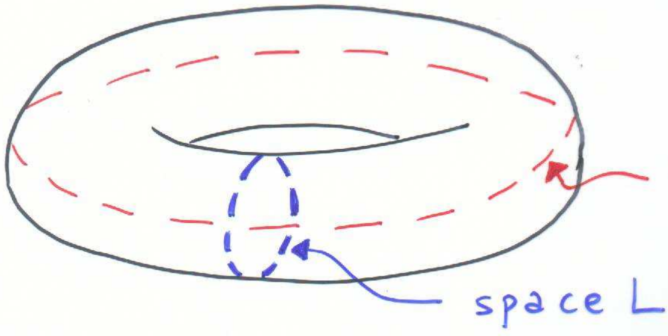
Issues:

- CFT: find & classify boundary states by algebraic methods.
- Cond-Mat: boundary interactions & RG Flows; e.g., Kondo effect, tunnelling in QHE.
- Strings: D-branes in general backgrounds;

  geometrical interpretation of CFT algebraic data;

  unstable branes, tachyon condensation, RG Flows.

Partition function on the torus



$\tau = \frac{iT}{L}$

$$Z(\tau) = \mathrm{tr}\left(e^{-T\mathcal{H}}\right) \qquad \mathcal{H} = \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right)$$

$$= \sum_{\text{reps } i,j} n_{ij}\, \chi_i(\tau)\, \overline{\chi_j(\tau)}$$

multiplicities

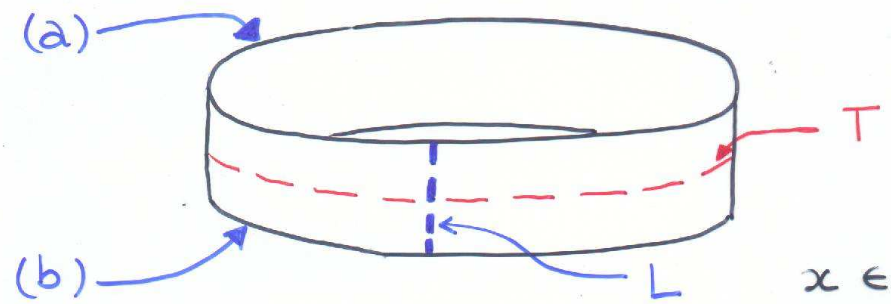
characters of Virasoro reps.

Modular invariance: $\tau \to -\frac{1}{\tau}$

- $Z(\tau) = Z(-\frac{1}{\tau})$
- $\chi_i(-\frac{1}{\tau}) = \sum_{j=1}^{N} S_{ij}\, \chi_j(\tau)$
- n_{ij} determined by a set of linear equations: $S n S^{+} = n$

Partition function on the annulus

boundary conditions (a) & (b):



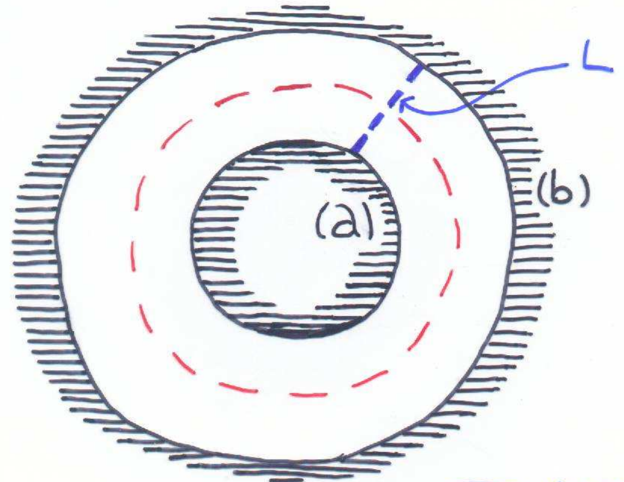
$x \in (0, L)$

- b.c. for stress tensor: $T_{01}(x,y) = 0\,,\quad x = 0, L$

  $\approx$ reality condition $\rightarrow$ one Virasoro

$$Z_{ab}(\tau) = \sum_{\text{reps } i}^{N} A^{i}_{ab}\, \chi_i(\tau)$$

- modular transformation:



same as torus

$$Z_{ab}(\tau) = \langle b^* |\, e^{-L\mathcal{H}}\, | a \rangle$$

boundary state

Properties of boundary states $|a\rangle$

- boundary condition: $T_{01} = 0 \approx T_{zz} - T_{\bar{z}\bar{z}} = 0$

$$(L_n - \bar{L}_{-n})\,|a\rangle = 0 \qquad \forall n$$

- standard solution for any pair of Virasoro reps. (i, i^*) in the bulk

$$|i\rangle\rangle = \sum_N \underbrace{|i;N\rangle}_{\chi_i} \otimes U\, \underbrace{\overline{|i^*,N\rangle}}_{\overline{\chi_{i^*}}}$$

(Ishibashi)

- boundary coefficients

$$|a\rangle = \sum_{i=1}^{M} B_{ai}\, |i\rangle\rangle$$

- modular covariance conditions

$$\sum_i^{N} A^i_{ab}\, S_{ij} = \langle b^*|j\rangle\rangle \langle\langle j|a\rangle = B_{bj}\, B_{aj}$$

- general solution for $Z_{torus} = \sum_i \chi_i \overline{\chi_{i^*}}$

$$B_{ai} = \frac{S_{ai}}{\sqrt{S_{0i}}}, \qquad A^i_{ab} = N^i_{ab}$$

(Cardy)

S-matrix for $\tau \to -\frac{1}{\tau}$ — Verlinde Fusion coeffs

"Cardy boundary states"

boundaries = # bulk sectors

Remarks:

- boundary states of Rational CFT represent D-branes on compact manifolds, e.g. group manifolds

  → geometrical interpretation of CFT data

  B_{ai} are $\propto$ mass & charges of D-brane

- boundary states for other, non-diagonal bulk theories are not known $\left(Z_{torus} \neq \sum_i \chi_i \bar{\chi}_{i^*}\right)$

  → classification of boundary states of RCFT

  $\approx$ D-branes on general, curved backgrounds

Done so far:

- $c<1$ Virasoro minimal models
  $\widehat{SU(2)}_k$ models } (Sagnotti et al; Zuber et al)

- $c=1$ & $c=3/2$ $N=1$ susy models (Schellekens et al.; Fuchs, Schweigert; A.C., G. D'Appollonio)

Being done:

- $N=2$ susy models & Gepner construction of Calabi-Yau models; (Schomerus et al; -----; Maldacena, Moore, Seiberg)

- $\widehat{SU(N)}_k$ models (Schellekens et al; Gannon et al; ...)

Ex: $c=1$ compactified scalar $\widehat{U(1)}_k$

Dirichlet: $\alpha_n - \bar{\alpha}_{-n} |a\rangle_D = 0$ — it breaks: momentum cons.

Neumann: $\alpha_n + \bar{\alpha}_{-n} |a\rangle_N = 0$ — winding no. cons.

- Ishibashi states:

"Bogoliubov states"

$$|n,0\rangle\rangle_D = e^{\sum_{j=1}^{\infty} \frac{\alpha_{-j}\bar{\alpha}_{-j}}{j}} |n,0\rangle$$

$$|0,m\rangle\rangle_N = e^{-\sum_{j=1}^{\infty} \frac{\alpha_{-j}\bar{\alpha}_{-j}}{j}} |0,m\rangle$$

- Rational CFT : compactification radius $R = \sqrt{2k}$

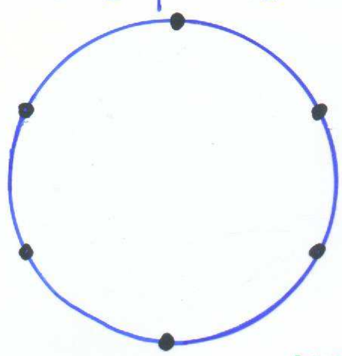
$$Z_{Torus} = \sum_{i=1}^{2k} \chi_i \bar{\chi}_{-i}$$ $2k$ sectors, $\widehat{U(1)}_k$ symm

- Cardy boundaries

$$|x\rangle = \frac{1}{(2k)^{1/4}} \sum_{n\in\mathbb{Z}} e^{-inx/R} |n,0\rangle\rangle_D \,, \quad x_j = 2\pi R \frac{j}{2k}$$

$2k$ D0 branes at discrete points on the circle

Ex $k=3$
$j = 1,2,\dots,6$

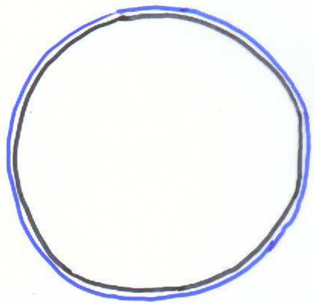


(S-matrix $S_{j\ell} = \frac{1}{\sqrt{2k}} e^{-i\frac{2\pi j\ell}{2k}}$)

2 non-Cardy boundaries from Neumann:

$$|\pm\rangle = \left(\frac{k}{2}\right)^{1/4} \sum_{\ell\in\mathbb{Z}} \left(|0, 2k\ell\rangle\rangle_N \pm |0, (2\ell+1)k\rangle\rangle_N \right)$$

two D1 branes with specific values of Wilson line



- left-right gluing Ω on the boundary, $\bar{J} - \Omega(J) = 0$, $\Omega = \pm 1$, is different from that of the bulk, $Z_{Torus} = \sum_i \chi_i \overline{\chi_{-i}}$:

  $\rightarrow$ symmetry-breaking boundaries

- boundary coefficients B_{ai} $\left(|a\rangle = \sum_i B_{ai} |i\rangle\rangle\right)$ do not change if boundary and bulk gluings are changed simultaneously;

$\rightarrow$ $|\pm\rangle_N \rightarrow |\pm\rangle_D$, symmetry-preserving boundaries of T-dual theory $Z'_{Torus} = \sum_{i=1}^{2k} \chi_i \bar{\chi}_i \quad (= Z_{Torus}(1/R))$

$i^* \equiv -i$, $i^* = i \mod 2k$ for $i = 0, k$ only

$\rightarrow$ 2 Ishibashi

Boundaries of orbifold CFTs

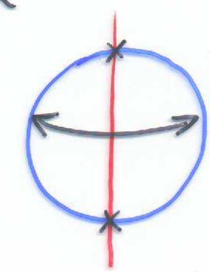
$$\text{Orbifold} = \frac{\text{Manifold}}{\text{Discrete group}}$$

In CFT : quotient by discrete symmetry of operator algebra

- can be extended to a map between the respective boundary states (A.C., D'Appollonio; -----)
- can yield non-Cardy, symmetry breaking boundaries
- convey some geometrical interpretation

Ex: $c=1$ $\quad \frac{S^1}{\mathbb{Z}_2}$ $\quad X(\sigma,t) \approx X + 2\pi R$

$\mathcal{P} : X \to -X$



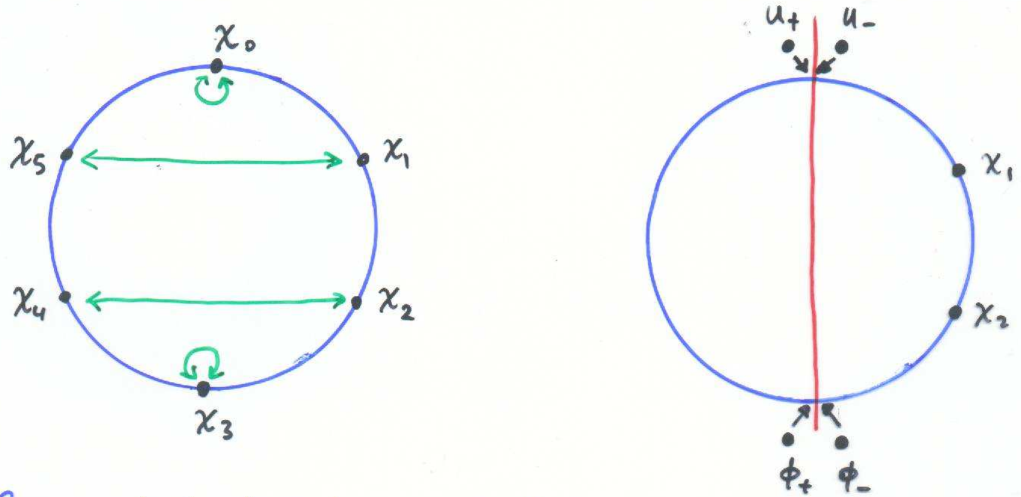
- two fixed points $X = 0, \pi R$
- chiral $\widehat{U(1)}$ symmetry is broken
- at $R^2 = 2k$, there are $k+7$ sectors

→ $k+7$ Cardy boundaries

Ex $k=3$

S^1: $2k=6$ D0
$\quad\quad 2$ D1

$S^1/\mathbb{Z}_2$ $\quad k+7=10$ Cardy boundaries orbifold



- Geometrical interpretation:
  - $|\chi_1\rangle, |\chi_2\rangle$ P-symmetric D0 branes;
  - $|u_\pm\rangle, |\phi_\pm\rangle$ fractional D0 branes at fixed points:

    properties
    - $|\chi_0\rangle, |\chi_3\rangle$ have splitted in pairs;
    - Fraction of circle charges Bam;
    - acquired charge in twisted sectors;
    - Fixed at $x=0, \pi R$ (no moduli).
  - $|\sigma_i\rangle, |\tau_i\rangle$, $i=0,1$, branes of $\mathbb{Z}_2$ twisted sectors come from splitting of 2 D1 branes of circle $\rightarrow$ fractional "twisted" D1 branes
- Used inverse map $S^1/\mathbb{Z}_2 \longrightarrow S^1$ given by another $\mathbb{Z}_2$ orbifold generated by a "simple current" (Schellekens et al.) extended to boundary states.

Extensions (A.C., G. D'Appollonio)

- remaining $c=1$ RCFTs: orbifolds $\frac{\widehat{SU(2)}_1}{G}$ with $G = T, O, I$ (Ginsparg): we found the boundaries and maps between them (non-trivial yet doable RCFT)

- $N=1$ Susy models at $c = 3/2$: one compactified superfield, four orbifold lines and six Ginsparg points.

- possible improvements
  - complete geometrical interpretation;
  - analyze $N=2$ Susy models $c=3$;
  - study boundaries with maximal symmetry breaking: $\frac{\widehat{SU(K)}_1}{SU(K)} = W_K$ minimal models at $c = K-1$

Ex: branes of $\widehat{SU(2)}_k$ models

Bulk theories are classified by A-D-E

- A_{k+1} theories

- Diagonal partition function $Z_{\text{Torus}} = \sum_{i=1}^{k+1} |\chi_i|^2$
- $k+1$ Cardy boundaries, $B_{ai} = \frac{S_{ai}}{\sqrt{S_{1i}}}$
- geometrical interpretation (Recknagel, Schomerus)

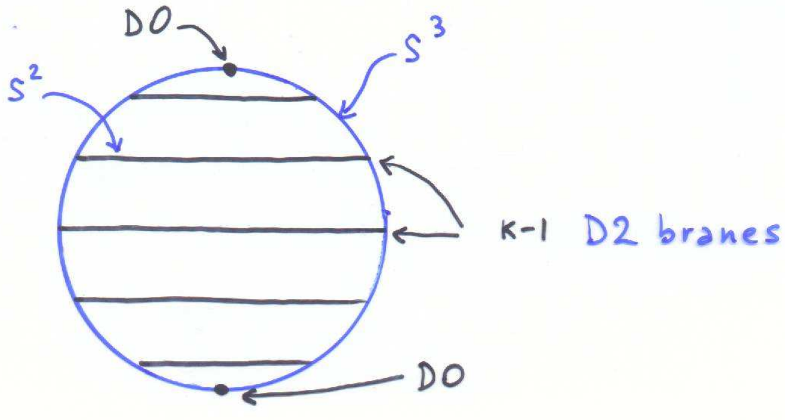
– take classical limit $k \to \infty$: $SU(2) \approx S^3$ sphere

– classical b.c. $J + \bar{J} = 0$, $J = g^{-1}\partial g$

it reads: $(g^{-1}\partial_x g)^{\perp} = 0$ Dirichlet b.c. in direction $\perp$ to conjugacy class of g

→ D2 brane extending over the conj. class $C(g) \approx S^2$

- flux of 2-form B is quantized by k : stability



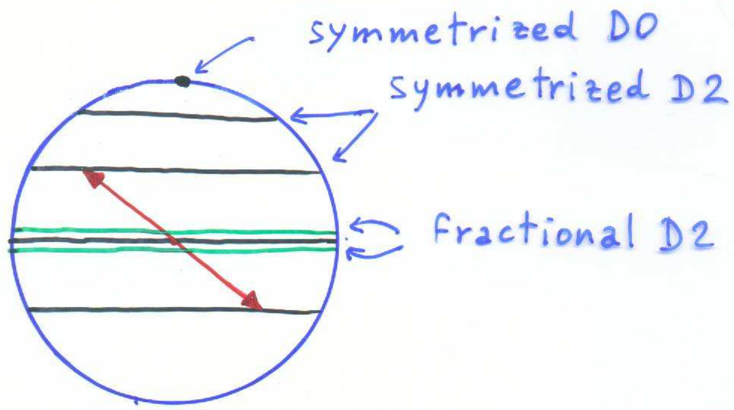
- finite k : D2 brane is fuzzy ; NC geometry

(Felder, Fröhlich, ...)

- D_n theories

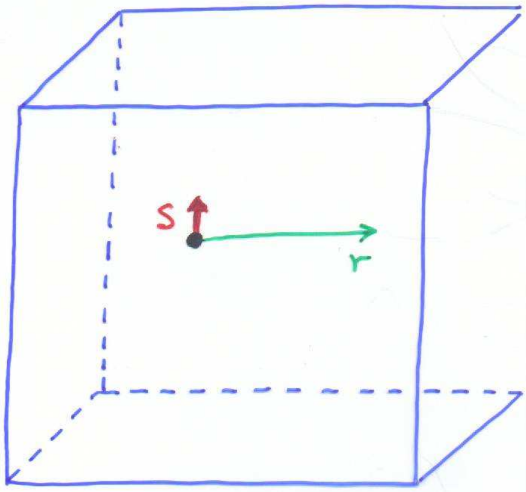
- Non-diagonal Z_{Torus} → non-Cardy boundaries
- Can be obtained as $\mathbb{Z}_2$ orbifold $\frac{SU(2)}{\mathbb{Z}_2} = SO(3)$ from the A_{k+1} theory (k even)
- Use orbifold map for boundaries:

  $\mathbb{Z}_2$ $\quad \mathcal{P}$ = antipodal reflection in S^3



Boundary interactions & RG flows

Ex: Kondo effect



- Diluted magnetic impurities
- take s-wave, only radial dependence $0 < r < \infty$
- free massless fermions + boundary interaction

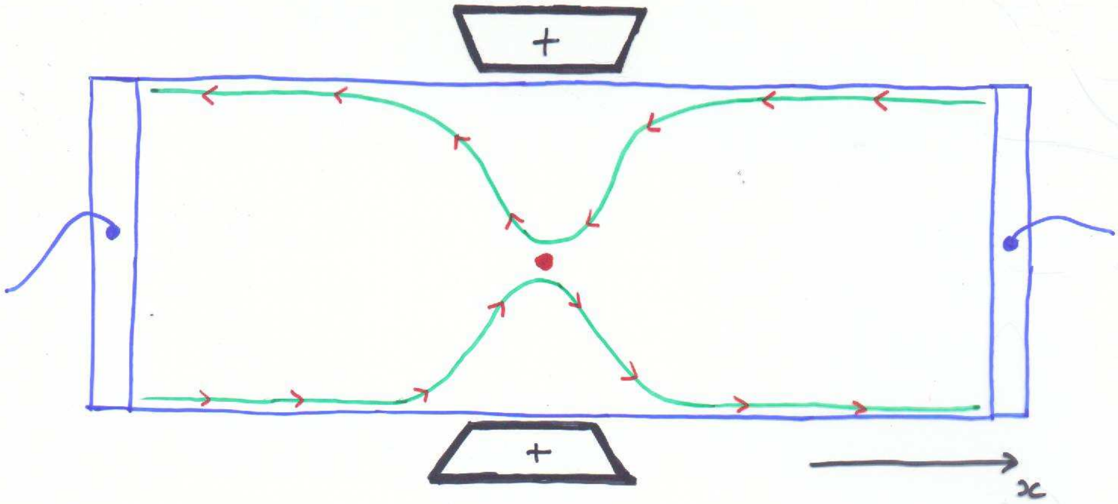
$$\mathcal{H} = \int_0^\infty dx \sum_{i=\pm} \psi_i^+ \partial_x \psi_i + \lambda \vec{S} \cdot \vec{J}(x=0), \qquad \vec{J} = \psi_i^+ \vec{\sigma}_{ij} \psi_j$$

- boundary RG flow in bulk critical theory
- non-trivial IR boundary state found by mapping free fermions (multicomponent) to Wess-Zumino. Non-perturbative RG flow
  $\lambda = 0 \longrightarrow \lambda = \lambda^* = O(1)$

(Affleck, Ludwig, 1991 –)

$\longrightarrow$ many original ideas & results in BCFT

Ex: resonant tunnelling in quantum Hall effect



- chiral excitations along the edge of the 2d electron density (no bulk excitations)
- chiral edge excitations described by $c=1$ chiral compactified bosonic theory
- electron density can be squeezed at one point: two edges can have interaction

$$S = \int dt\,dx\,(\partial_t+\partial_x)\varphi_L\,\partial_t\varphi_L + (\partial_t-\partial_x)\varphi_R\,\partial_t\varphi_R$$
$$+ \lambda \int dt \left(e^{i\frac{\varphi_R}{3}} e^{-i\frac{\varphi_L}{3}} + h.c.\right)\Big|_{x=0}$$

- relevant boundary interaction in boundary Sine-Gordon model (Fendley, Ludwig, Saleur, 1994–)
- integrable boundary RG flow (Goshal, Zamolodchikov)
- exact results for resonant transition amplitude and temperature effects (Thermodynamic Bethe Ansatz)
- experiment proving fractional charge of excit.

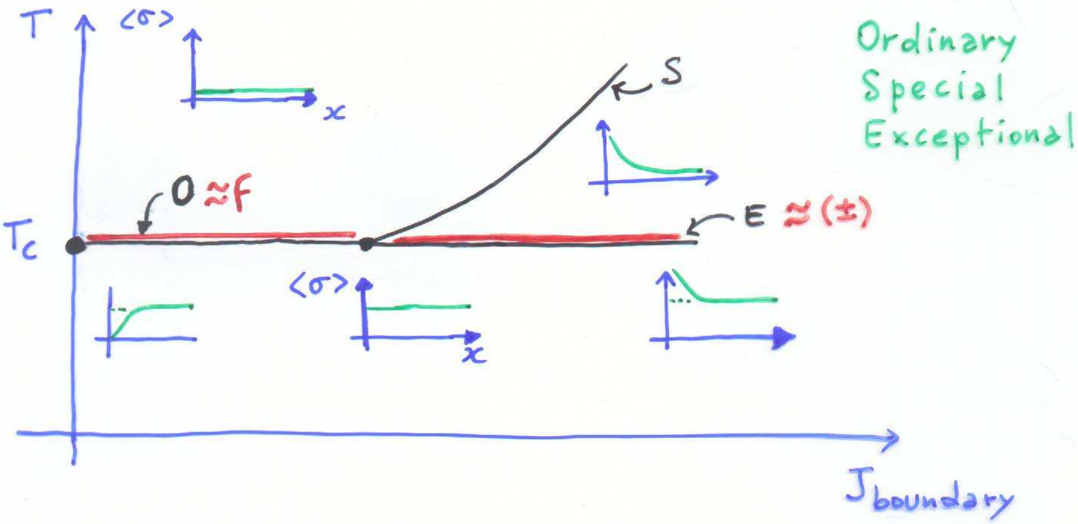
Ex: boundary RG flow in Ising model

$$Z_{\text{Torus}} = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{\frac{1}{16}}|^2$$

- 3 sectors : $\mathbb{1}$, ψ, σ
- 3 Cardy boundaries : $|+\rangle, |-\rangle$ fixed b.c.

  $|f\rangle$ free b.c.

- phase diagram



- other points on critical lines are non-conformal boundary conditions

- RG flow $(-) \longleftarrow\!\!\longleftarrow (f) \longrightarrow\!\!\longrightarrow (+)$

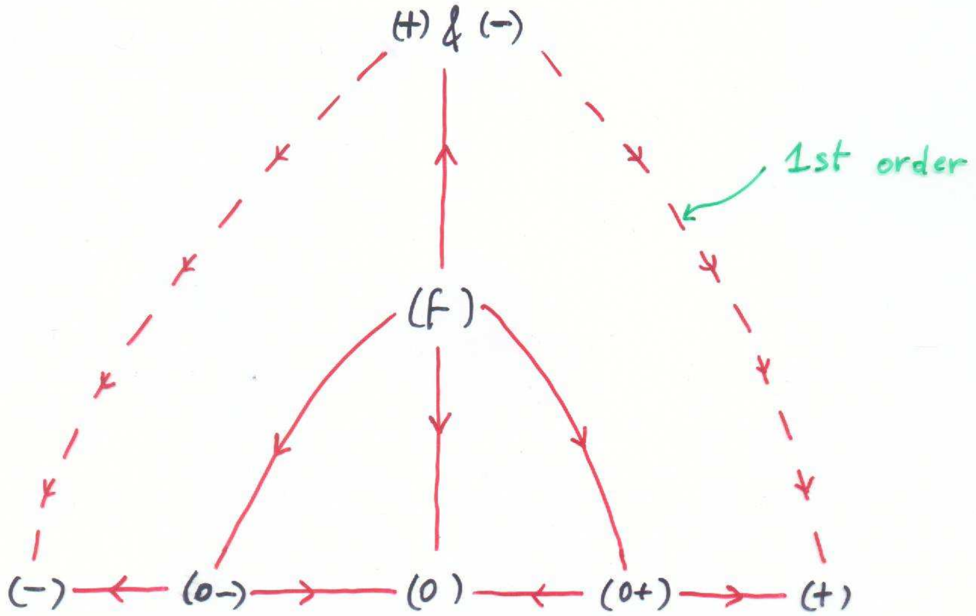
  driven by bondary magnetic field $H_b \gtrless 0$

Ex: boundary RG flow of tri-critical Ising

- Next Virasoro minimal model, first $N=1$ model
- 6 sectors, 6 Cardy boundaries
- Ising model with vacancies $\sigma_i = \pm 1, 0$
- boundaries : $|+\rangle, |-\rangle, |0\rangle$ fixed

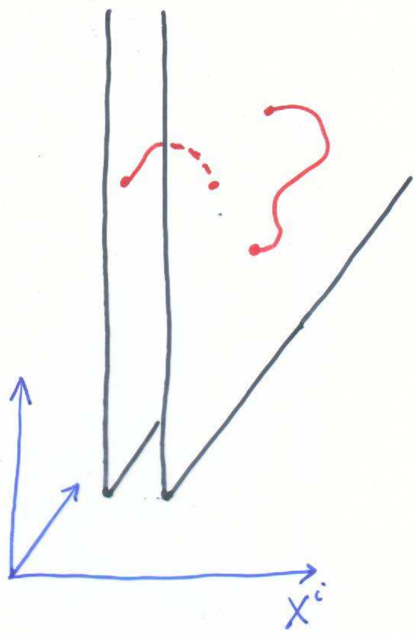
  $|0+\rangle, |0-\rangle$ partially fixed

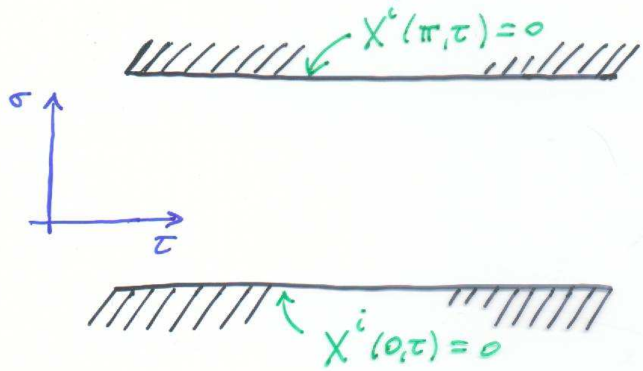
  $|F\rangle$ free



- compilation of results of integrable boundary interactions (Affleck)
- can flow from a Cardy state to a superposition of them (Recknagel et al.)

Unstable D-branes





- Certain (collections of) branes are unstable and could decay into other branes

- Change of boundary conditions on the strip described by boundary interaction and RG flow

$$S = S_{CFT}\Big|_D + \underset{\text{relevant}}{\lambda} \int d\sigma\, d\tau \; \underset{\text{tachyon field}}{T(\sigma,\tau)} \big(\delta(\sigma) + \delta(\sigma-\pi)\big)$$

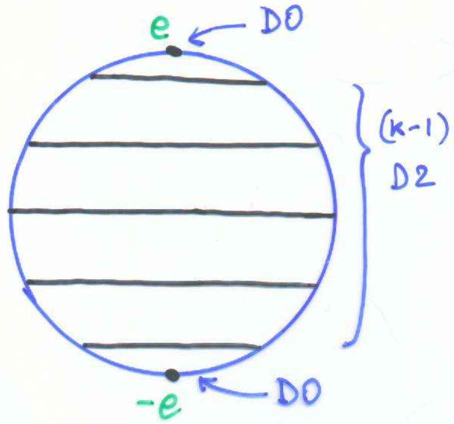
- "tachyon condensation" ≈ effective potential with non-trivial $\langle T \rangle$

  (Sen; Harvey, Kutasov, Moore, ...)

Ex: RG flow of $\widehat{SU(2)}_k$ branes (Fredenhagen, Schomerus)

- Some boundary flows in these CFTs have been found in the study of the Kondo effect

- one perturbative flow known in the semiclassical limit $k \to \infty$



- n D0 branes at e

$\downarrow$

one D2 at $\kappa = n-1$ position $(\kappa = 0, 1, 2, \ldots, k)$

N.B.: $n = k+1$ D0 produce the other D0 at $(-e)$, as being an anti-brane

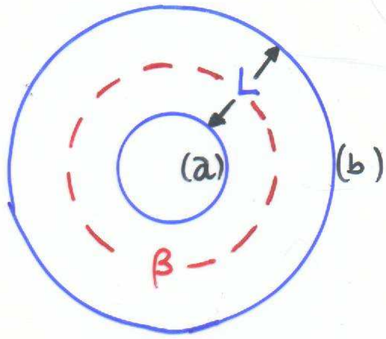
Conclusion: "charge" n of D0's in $\widehat{SU(2)}_k$ is defined modulo $k+2$, if interactions are taken into account

$\rightarrow$ $\mathbb{Z}_{k+2}$ topological charge in this background (S^3 with 3-form field strength) is accounted by K-theory (Bouwknegt, Mathai; ....)

Boundary entropy, "g-theorem" (Affleck, Ludwig)

$$Z_{ab} = \langle b^* | e^{-L\mathcal{H}} | a \rangle$$

$$\underset{L\to\infty}{\sim} e^{\frac{\pi}{6}\frac{L}{\beta}c} \underbrace{\langle b^* | 0 \rangle \langle 0 | a \rangle}_{g_b g_a}$$



$$\mathcal{S}_{ab} = \log Z_{ab} + \beta U_{ab} \qquad \text{entropy}$$

$$\underset{L\to\infty}{\simeq} \frac{\pi}{3}\frac{L}{\beta} c + \log(g_a g_b)$$

$$\boxed{\mathcal{S}_{ab}(T=0) = \log(g_a g_b)}$$

- Cardy boundaries: $\quad g_a = B_{a0} = \dfrac{S_{a0}}{\sqrt{S_{00}}}$ S-matrix

- regularized sum of degrees of freedom

$$Z_{ab} = \mathrm{tr}\left(e^{-\beta \mathcal{H}_{ab}}\right) = \sum_i A^i_{ab}\, \chi_i\!\left(\tfrac{\beta}{L}\right)$$

$$\underset{L\to\infty}{\simeq} \sum_i A^i_{ab}\, \text{"}\chi_i(0)\text{"} = \sum_i A^i_{ab}\, \mathrm{tr}_{(i)}(1)$$

- g-theorem conjecture:

  g decreases along boundary RG flows

- true in all known examples and calculations to date

- independent of c-theorem (# of "field components")