Topological Insulators in 3D and Bosonization

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Outline

- Topological states of matter: bulk and edge
- Fermions and bosons on the (1+1)-dimensional edge
- Effective actions and partition functions
- Fermions on the (2+1)-dimensional edge
- Effective BF theory and bosonization in (2+1) dimensions

Topological States of Matter

- System with bulk gap but non-trivial at energies below the gap
- global effects and global degrees of freedom:

→ massless edge states, exchange phases, ground-state degeneracies

- described by topological gauge theories
- quantum Hall effect is chiral (B field breaks Time-Reversal symmetry)
- Topological Insulators are non-chiral (Time-Reversal symmetric)
- other systems: QAnomalousHE, Chern Insulators, Topological Superconductors, in D=1,2,3
- Non-interacting fermion systems: ten-fold classification using band theory
- Interacting systems: effective field theories & anomalies

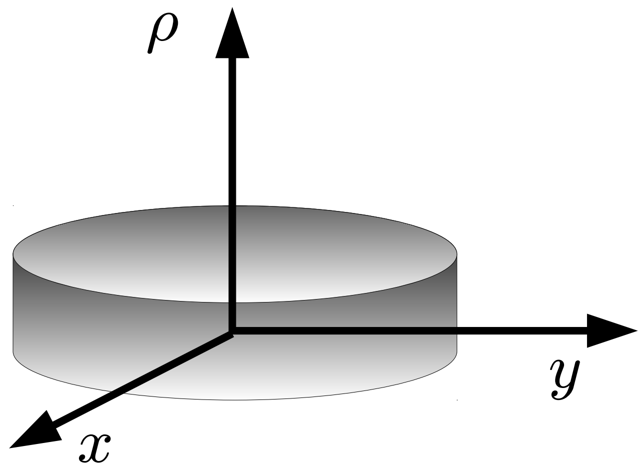
Topological band states have been observed in D=1,2,3

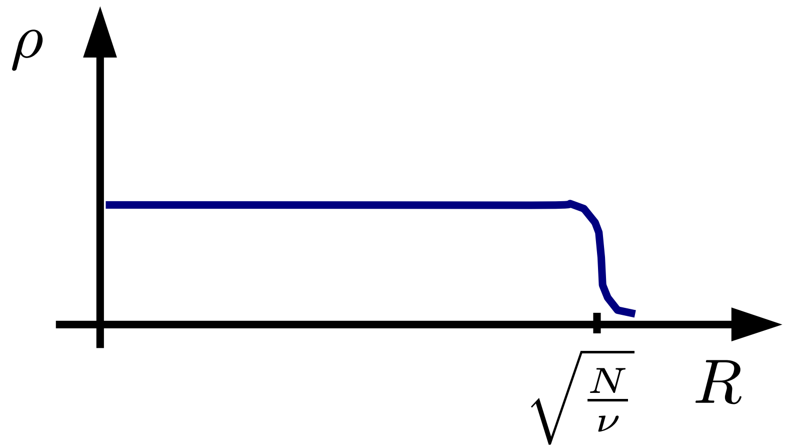
(Molenkamp et al. '07; Hasan et al. '08 – now)

Quantum Hall effect and incompressible fluids

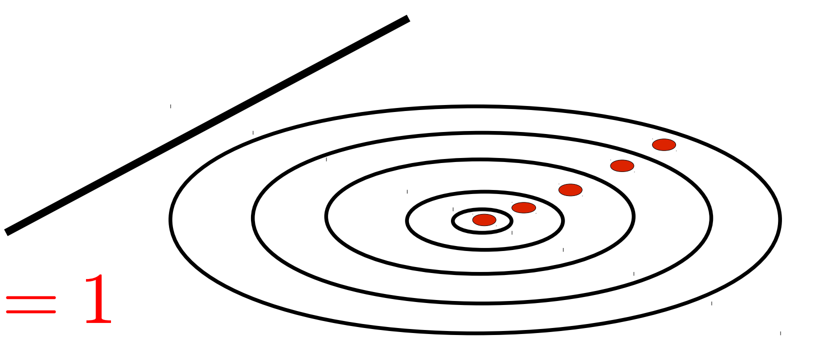
Electrons form a droplet of fluid:

→ incompressible: gap → fluid: $\rho(x,y) = \rho_o = \text{const.}$

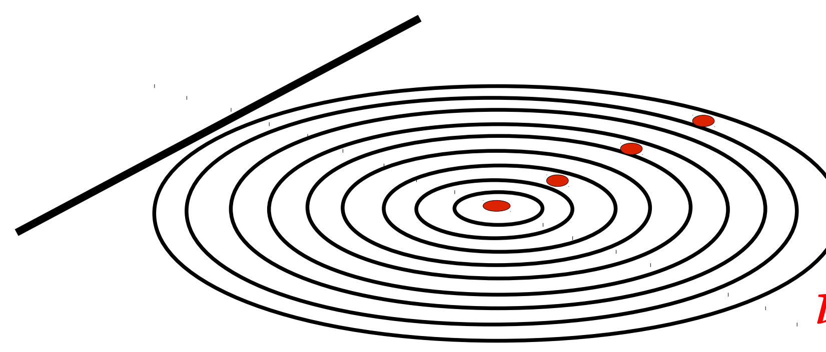




filling fraction $\nu = 1, \frac{1}{3}, \ldots$



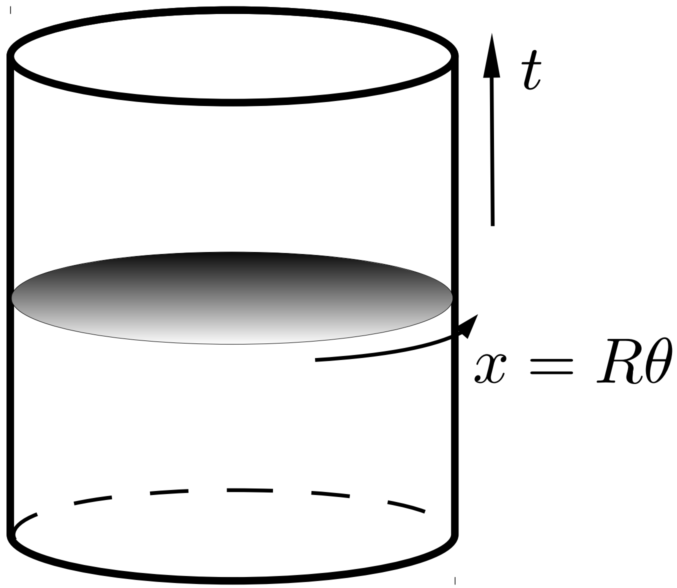
$\nu = 1$

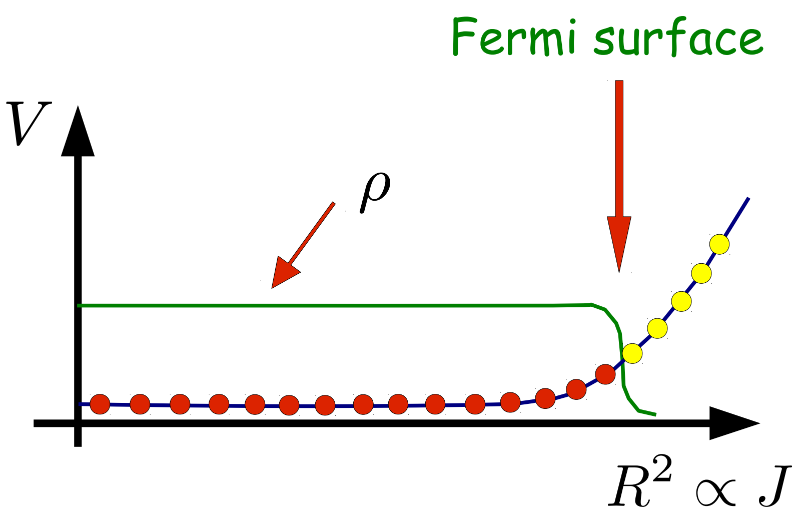


$\nu = \frac{1}{3}$

Edge excitations

The edge of the droplet can fluctuate: massless edge excitations





edge ~ Fermi surface: linearize energy $\varepsilon(k) - \varepsilon_F = vk = \frac{v}{R}n, \quad n \in \mathbb{Z}$

relativistic field theory in (1+1) dimensions with chiral excitations (X.G.Wen, '89)

→ Weyl fermion (non interacting) $\nu = 1$

→ Interacting fermion $\nu = \frac{1}{k}$ chiral boson (Luttinger liquid)

Bosonic effective action

- Express matter current in terms of Wen's hydrodynamic field a_μ

$$j^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho}\partial_\nu a_\rho$$

- Guess simplest action: topological, no dynamical degrees of freedom in (2+1) d

$$S_{eff}[a,A] = -\frac{k}{4\pi}\int \varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + \int A_\mu j^\mu = S_{\rm matt} + \text{e.m. coupling}$$

$$S_{ind}\,[A] = \frac{1}{4\pi k}\int dx^3\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho = \frac{1}{4\pi k}\int AdA$$

$$\rho = \frac{\delta S}{\delta A_0} = \frac{\nu}{2\pi}\mathcal{B}, \quad J^i = \frac{\delta S}{\delta A_i} = \frac{\nu}{2\pi}\varepsilon^{ij}\mathcal{E}^j$$ Density & Hall current $\nu = \frac{1}{k} = 1, \frac{1}{3}, \ldots$

- Sources of a_μ field are anyons (Aharonov-Bohm phases $\frac{\theta}{\pi} = \nu = \frac{1}{3}, \cdots$)
- Gauge invariance requires a boundary term in the action: add relativistic dynamics

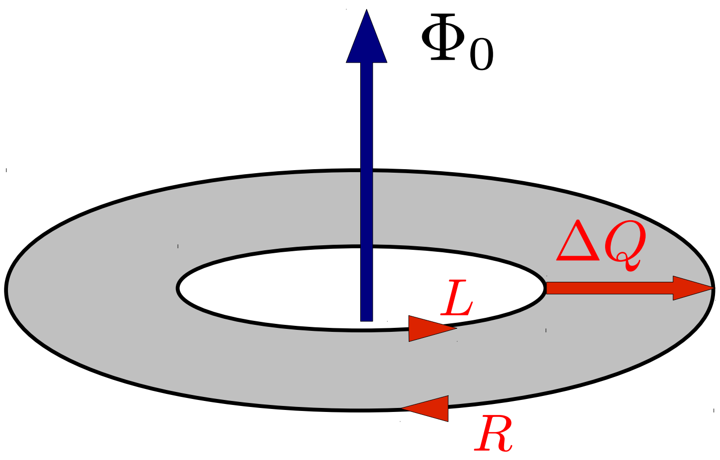
$$S_{eff} = -\frac{k}{4\pi}\int_D ada + \frac{k}{4\pi}\int_{\partial D}\partial_x\varphi\partial_0\varphi - v\,(\partial_x\varphi)^2, \qquad a_i|_{\partial D} = \partial_i\varphi, \quad a_0|_{\partial D} = 0$$

→ Chiral boson: $\nu = 1$ bosonization of Weyl fermion, $\nu = \frac{1}{k}$ interacting fermion

Anomaly at the boundary of QHE

- edge states are chiral
- chiral anomaly: boundary charge is not conserved
- bulk (B) and boundary (b) compensate:

$$\partial_i J^i + \partial_t \rho = 0, \;\rightarrow\; \oint dx J_B + \partial_t Q_b = 0$$



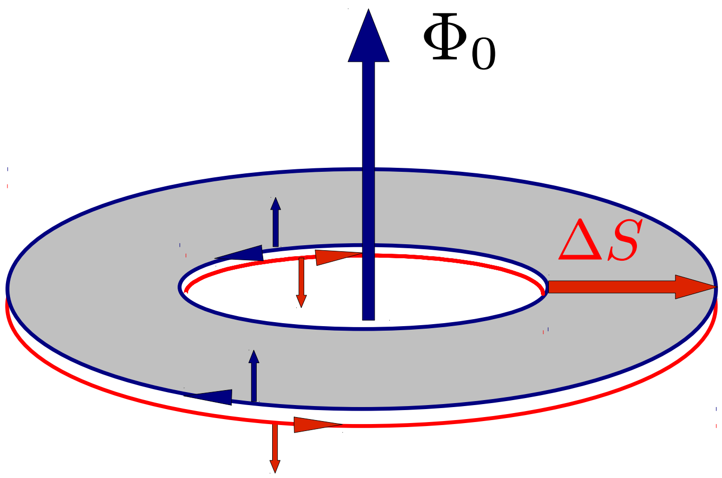
- adiabatic flux insertion (Laughlin, '82)

$$\Phi \rightarrow \Phi + \Phi_0, \qquad \Phi_0 = \frac{hc}{e}$$

$$Q_R \rightarrow Q_R + \Delta Q_b = \nu, \qquad \Delta Q_b = \int_{-\infty}^{+\infty} dt \oint dx\; \partial_t \rho_R = \nu \int F_R = \nu\, n$$

chiral anomaly

Topological insulators in 2D



Quantum Spin Hall Effect

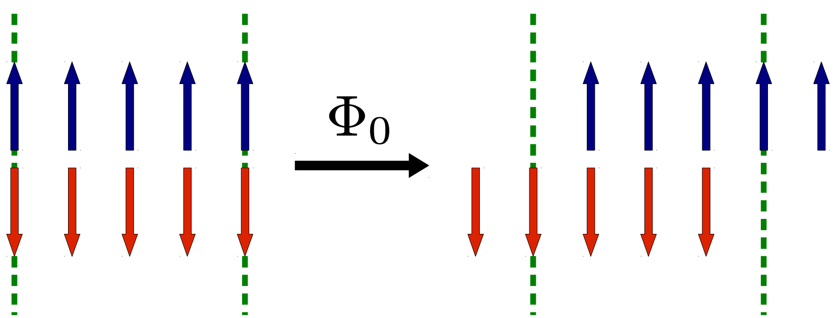
- take two $\nu = 1$ Hall states of spins ↑↓
- system is Time-Reversal invariant:

$$\mathcal{T}: \ \psi_{k\uparrow} \to \psi_{-k\downarrow}\ , \qquad \psi_{k\downarrow} \to -\psi_{-k\uparrow}$$

→ non-chiral theory

- flux insertion pumps spin

(Fu, Kane, Mele '06)

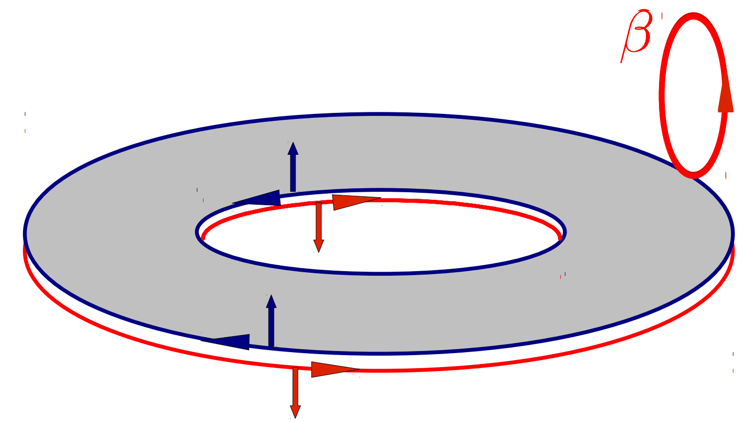


$$\Delta Q = \Delta Q^{\uparrow} + \Delta Q^{\downarrow} = 1 - 1 = 0$$

$$\Delta S = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1$$

- in Topological Insulators spin is not conserved (spin-orbit inter.)
- $\mathcal{T}$ remains a good symmetry: $\mathcal{T}^2 = (-1)^F = (-1)^{2\Delta S}$ (spin parity)
- $\frac{\Phi_0}{2}$ generates $\Delta S = \frac{1}{2}$ excitation: $\mathcal{T}^2 = -1$ degenerate Kramers pair

Partition Function of Topological Insulators



- Compute partition function of a single edge, combining the two chiralities, on $S^1 \times S^1$
- Four sectors of fermionic systems

$$NS,\ \widetilde{NS},\ R,\ \widetilde{R},\ \text{risp.}\ (AA),\ (AP),\ (PA),\ (PP)$$

- Neveu-Schwarz sector describes ground state and integer flux insertions:

$$Z_{NS}(\tau,\zeta) = Z_{NS}(\tau,\zeta+\tau), \qquad V : \zeta \to \zeta + \tau \quad \text{adds a flux} \quad \Phi \to \Phi + \Phi_0,$$

$$\tau = \frac{i\beta+\delta}{L}, \qquad \zeta = \beta(iV_o + \mu)$$

- Ramond sector describes half-flux insertions: $\frac{\Phi_0}{2}, \frac{3\Phi_0}{2}, \ldots$

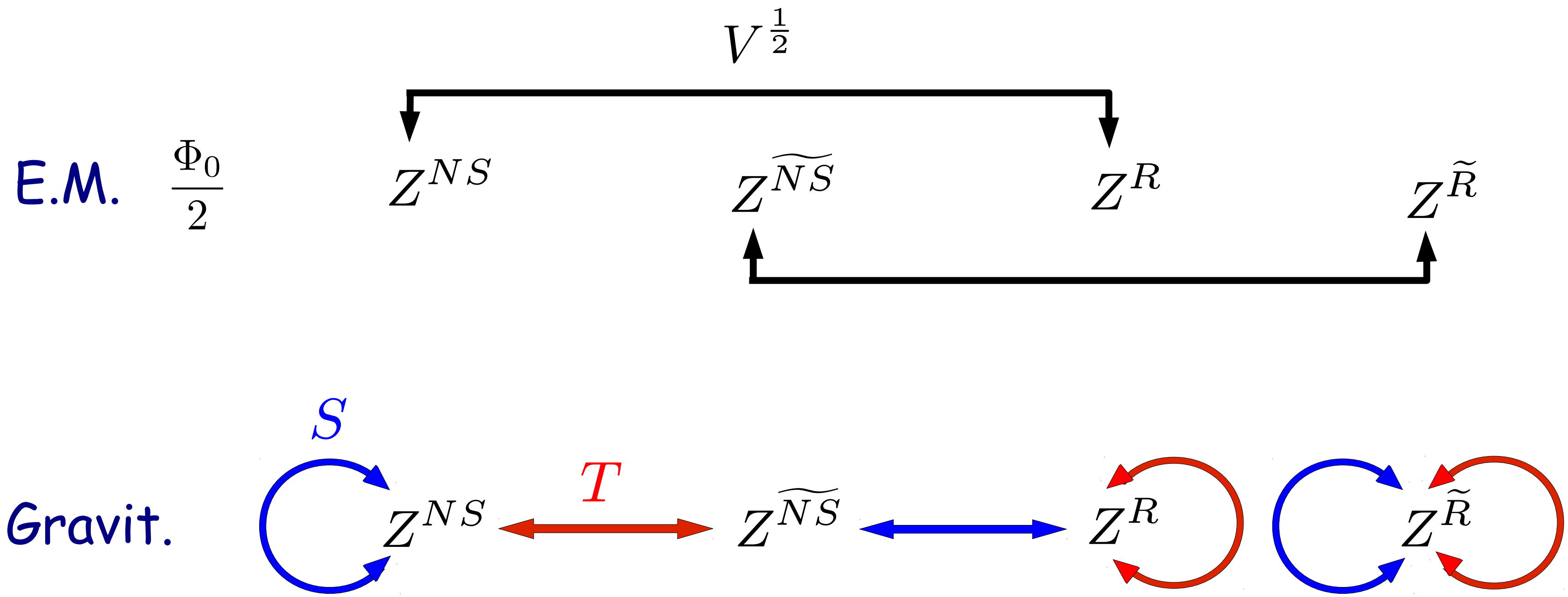
$$V^{\frac{1}{2}} : Z_{NS}(\tau,\zeta) \ \to \ Z_{NS}\left(\tau, \zeta + \frac{\tau}{2}\right) = Z_R(\zeta,\tau)$$

low lying Kramers pair

$$(-1)^{\Delta S} = -1$$

- Standard calculation of Z using CFT; boson and fermion representations
- bosonization is an exact map in (1+1) dimensions

Responses to background changes



$Z = Z(\tau, \zeta)$ $\quad$ $V : \zeta \to \zeta + \tau$ $\quad$ $T : \tau \to \tau + 1$ $\quad$ $S : \tau \to -\frac{1}{\tau}$

Flux addition $\quad$ Modular transformations

Ten-fold classification (non interacting)

class\δ	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+	+	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−	+	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−	−	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+	−	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

space dim. d

IQHE → A

Top. Ins. → AII

A, AIII: period 2

AI–CI: period 8 (Bott)

(A. Kitaev; Ludwig et al. 09)

- Study $\mathcal{T}, \mathcal{C}, \mathcal{P}$ symmetries of quadratic fermionic Hamiltonians
- Ex: $\mathcal{T}$ symmetry forbids a mass term $m\,\psi_\uparrow^\dagger \psi_\downarrow$ for a single fermion
- Matches classes of disordered systems/random matrices/Clifford algebras
- How to extend to interacting systems?

Topological insulators in 3D

- Fermion bands with level crossing

  - zoom at low energy near the crossing

  - approx. translation & Lorentz invariances

  - massive Dirac fermion with kink mass $m(x)$

- boundary (2+1)-dimensional massless fermion localized at $x = 0$ (Jackiw-Rebbi)

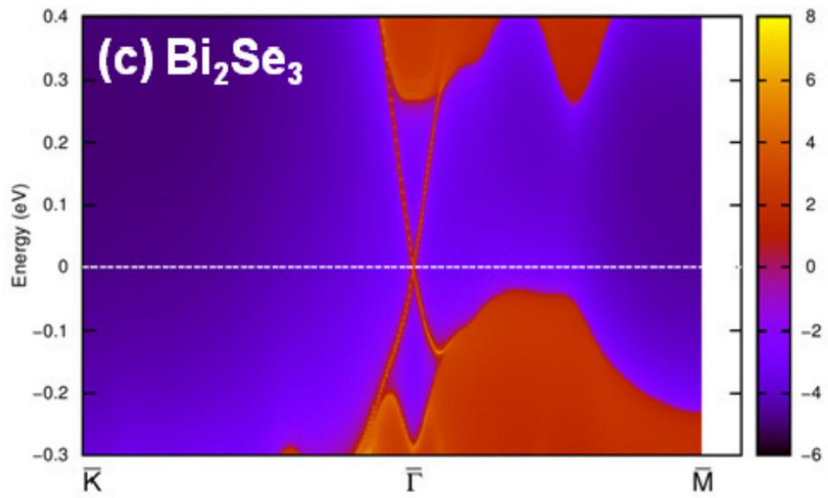
- Spin is planar and helical $\langle \vec{S} \cdot \vec{p} \rangle = 0$

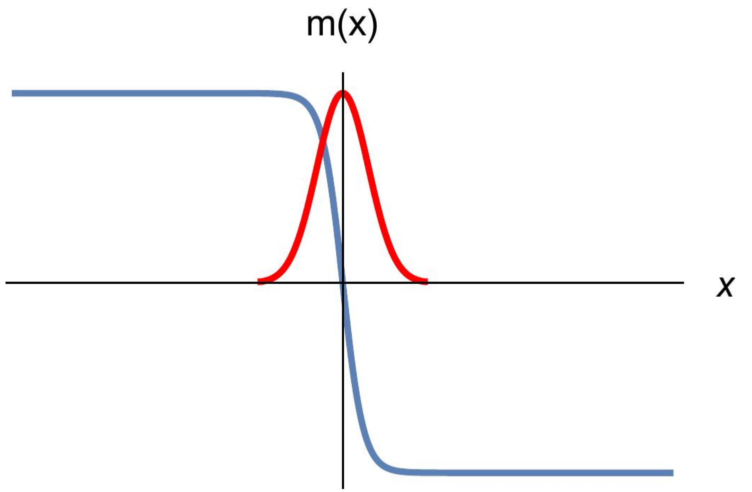
- Single species has $\mathcal{P}, \mathcal{T}$ anomaly

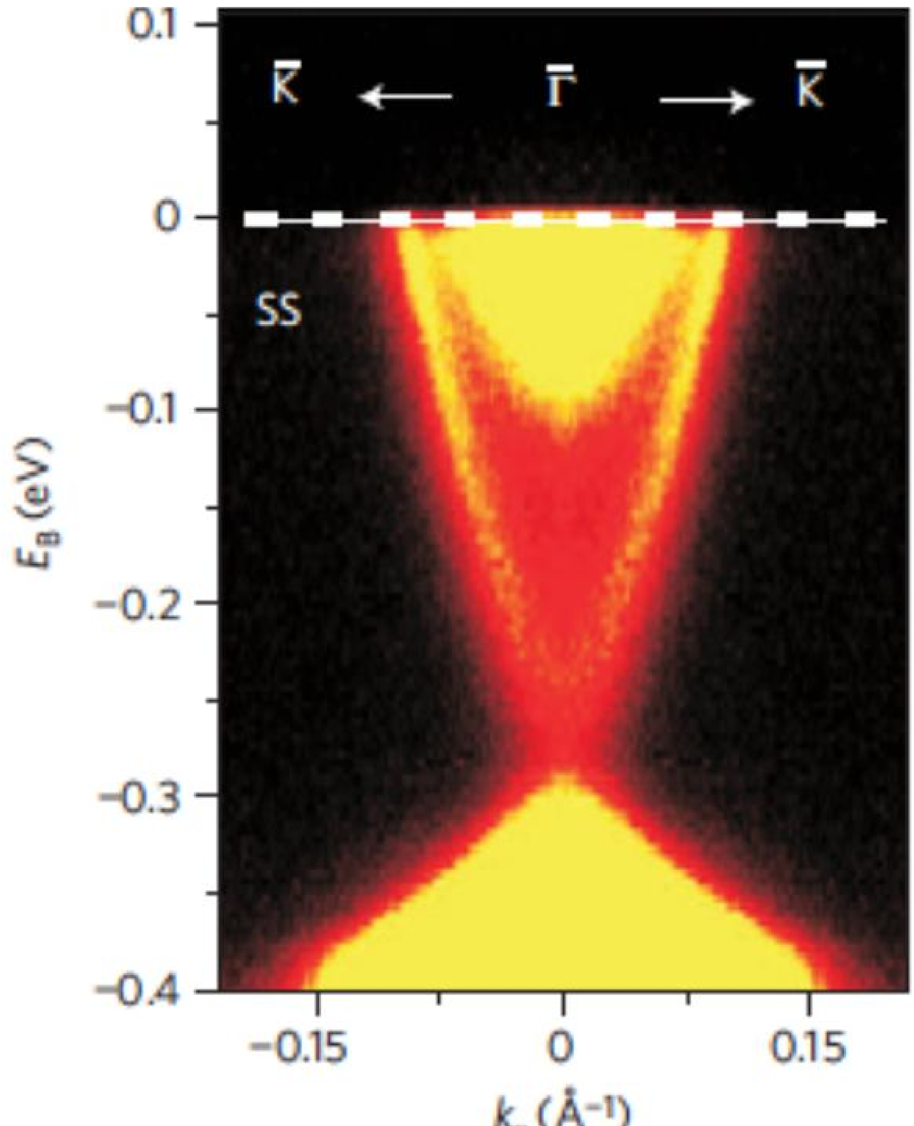
- Induced action to quadratic order

$$S_{ind}[A] = \frac{1}{8\pi} \int AdA + \int F_{\mu\nu} \frac{1}{\Box^{1/2}} F_{\mu\nu} + O(A^3)$$

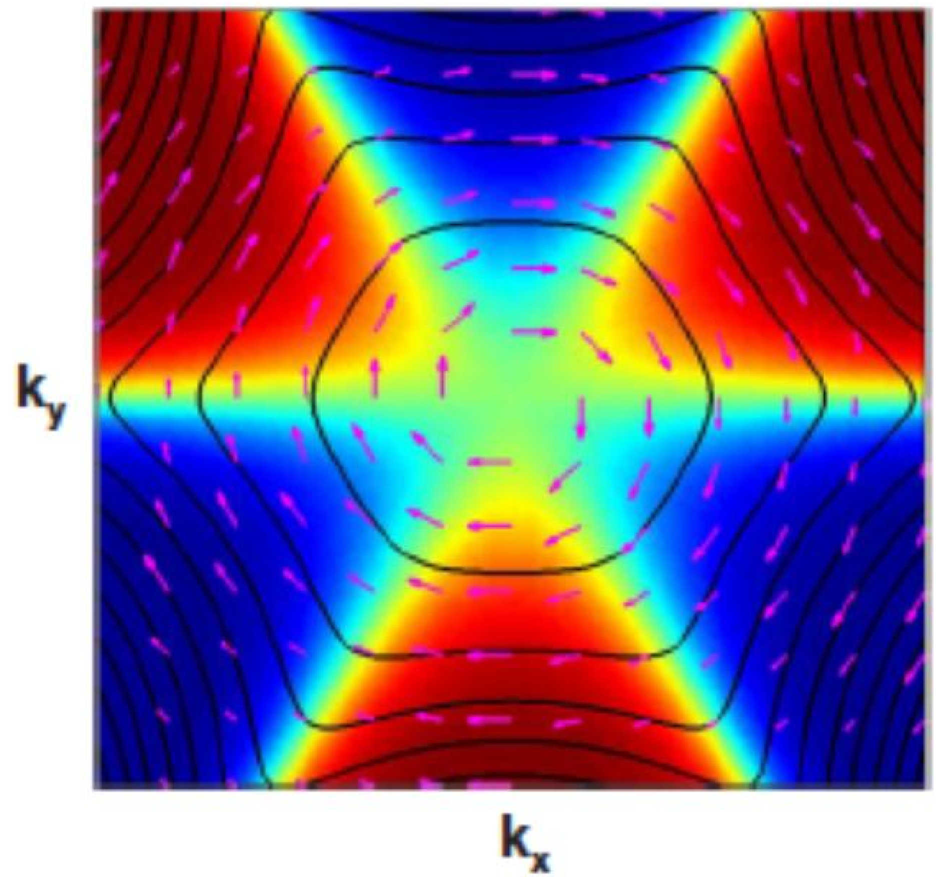
$\mathcal{P}, \mathcal{T}$ breaking, cancelled by bulk θ-term, $\theta = \pi$





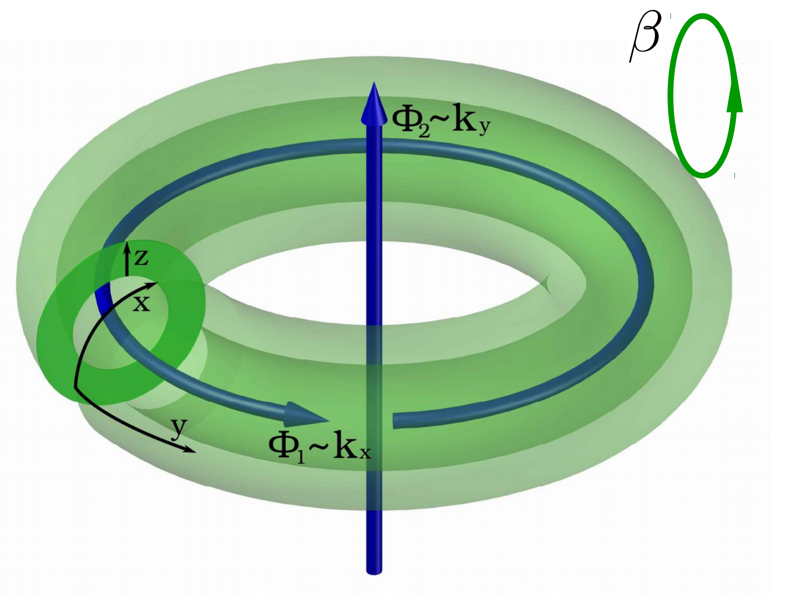


0.1
0
−0.1
−0.2
−0.3
−0.4
E_B (eV)
$\bar{K}$
$\bar{\Gamma}$
$\bar{K}$
SS
−0.15
0
0.15
k_x (Å^{-1})



k_y
k_x

Fermionic partition function on (2+1)-dim torus



- There are 8 spin sectors:
  (A,AA) ~ NS, (A,AP),... , (A,PP) ~ R, (P,PP)
- Straightforward calculation:

  i) Analyze the responses to background changes:

  - add half fluxes accross the the two spatial circles:

    (A) $\xrightarrow[\frac{\Phi_0}{2}]{}$ (P)

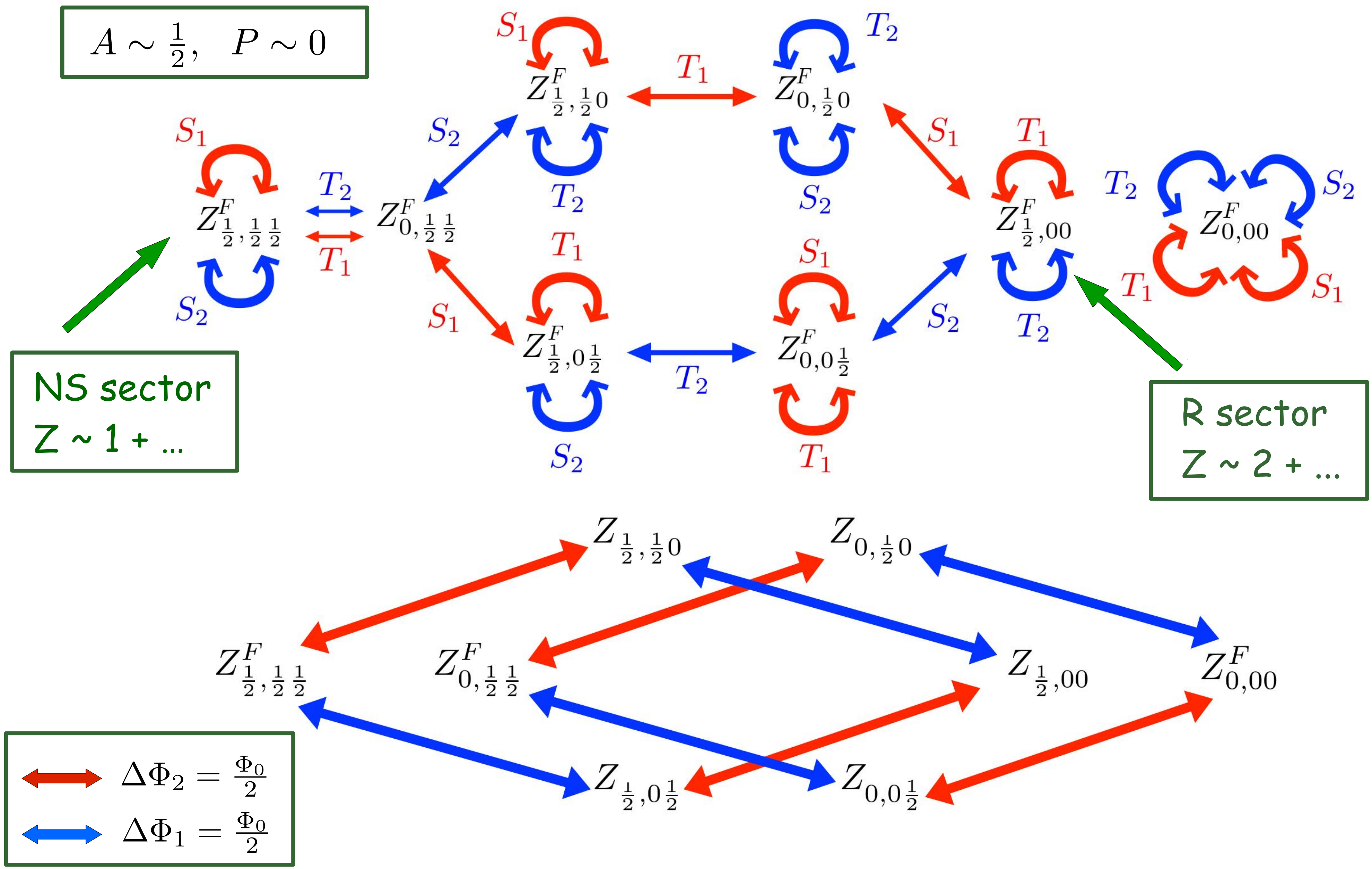
  - modular transf.: twisting circles T_1, T_2 and exchanging space-times S_1, S_2

  ii) Check degenerate Kramers pair in R sector: $(-1)^{2\Delta S} = -1$ (Fu-Kane stability index)

  iii) Check against (1+1)-d expressions by dimensional reduction:

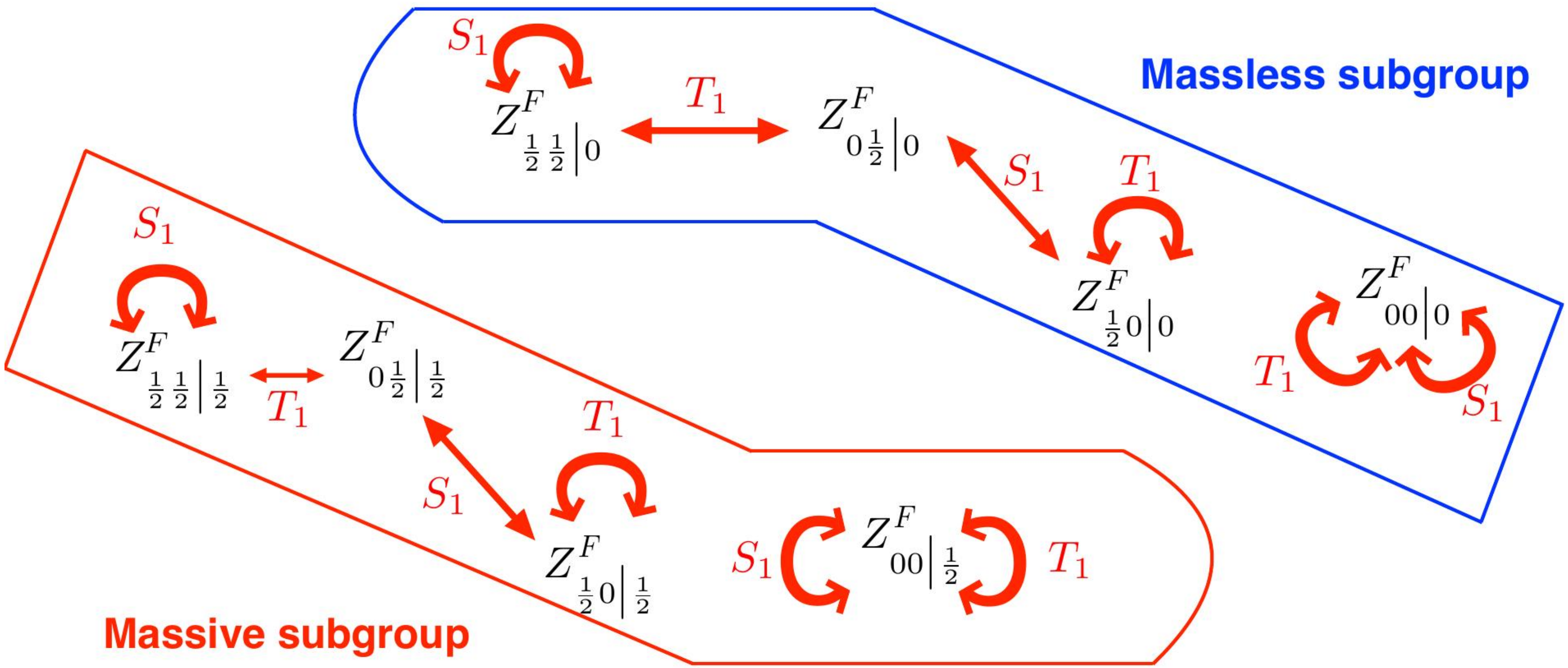
$$R_1 \to 0, \qquad p_1 = \frac{n_1}{R_1} \to \infty$$

Modular and flux transformations of Z^F



Dimensional reduction to (1+1) d

$$A \sim \tfrac{1}{2}, \quad P \sim 0$$



$(\mu = \frac{1}{2R})$

Bosonic effective action in 3D

- Particle and vortex currents: $J^\mu = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu b_{\rho\sigma}, \qquad V^{\mu\nu} = \frac{1}{2\pi}\varepsilon^{\mu\nu\rho\sigma}\partial_\rho a_\sigma$
- Simplest topological theory is BF gauge theory (Cho, J. Moore '11)

$$S_{eff}[a,b,A] = \int_{\mathcal{M}} \frac{k}{2\pi} bda + \frac{1}{2\pi} bdA + \frac{\theta}{8\pi^2} dadA \qquad a = a_\mu dx^\mu, \quad b = \frac{1}{2} b_{\mu\nu} dx^\mu dx^\nu$$

$$S_{ind}[A] = -\frac{\theta}{8\pi^2 k}\int_{\mathcal{M}} dAdA = -\frac{\theta}{8\pi^2 k}\int_{\partial\mathcal{M}} AdA$$

- For $k=1$ and $\theta=\pi$ it matches the anomalous term of the edge fermion
- For $k>1$ sources of a_μ and $b_{\mu\nu}$ describe braiding of particles and vortices in 3D
- Gauge invariance requires a boundary term: massless bosonic d.o.f. on the edge

$$S_{eff}[a,b,0] = \frac{k}{2\pi}\int_{\mathcal{M}} bda + \frac{k}{2\pi}\int_{\partial\mathcal{M}} \zeta da$$

gauge inv. $\quad b \to b + d\lambda, \quad \zeta \to \zeta - \lambda$

Bosonic theory on (2+1)-d boundary

- gauge choice $a_0 = b_0 = 0$, longitudinal and transverse d.o.f. $a_i = \partial_i \phi, \quad \zeta_i = \varepsilon_{ij}\partial_j \chi$
  they are Hamiltonian conjugate: $\phi, \ \pi_\phi \sim \Delta\chi$

$$S_{bound} = \frac{k}{2\pi}\int \zeta da = \frac{k}{2\pi}\int \Delta\chi\,\dot\phi \longrightarrow \int \pi\dot\phi - H(\phi,\pi)$$

- Bosonization exists, "flux attachment" idea, but cannot be described exactly
- Introduce quadratic relativistic dynamics, $\pi \sim \dot\phi$, and compute some quantities

→ bosonic partition function and spin sectors

- Canonical quantization of the compactified boson in (2+1) d (Ryu et al. '15-16)

  - Only need on-shell data: $\Box\phi = 0$

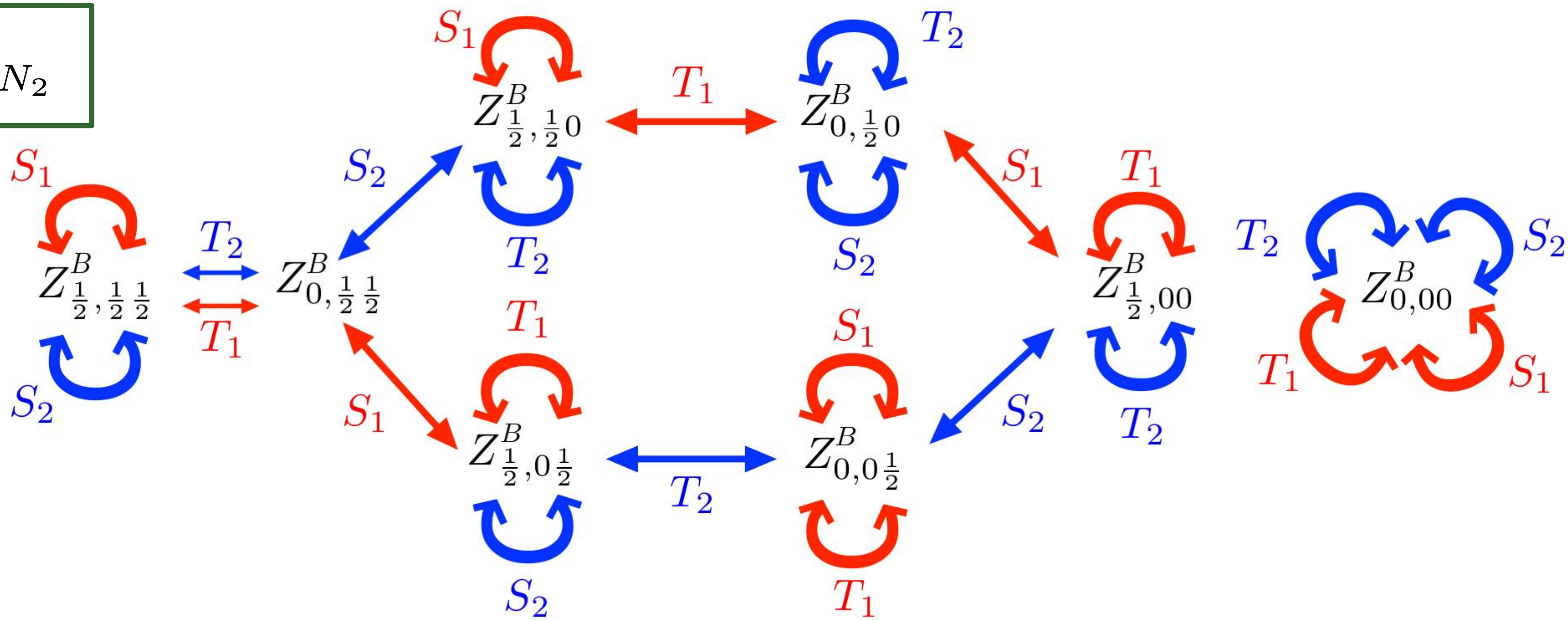
  - quantization of zero modes N_1, N_2 (of ϕ) and N_0 (of π):

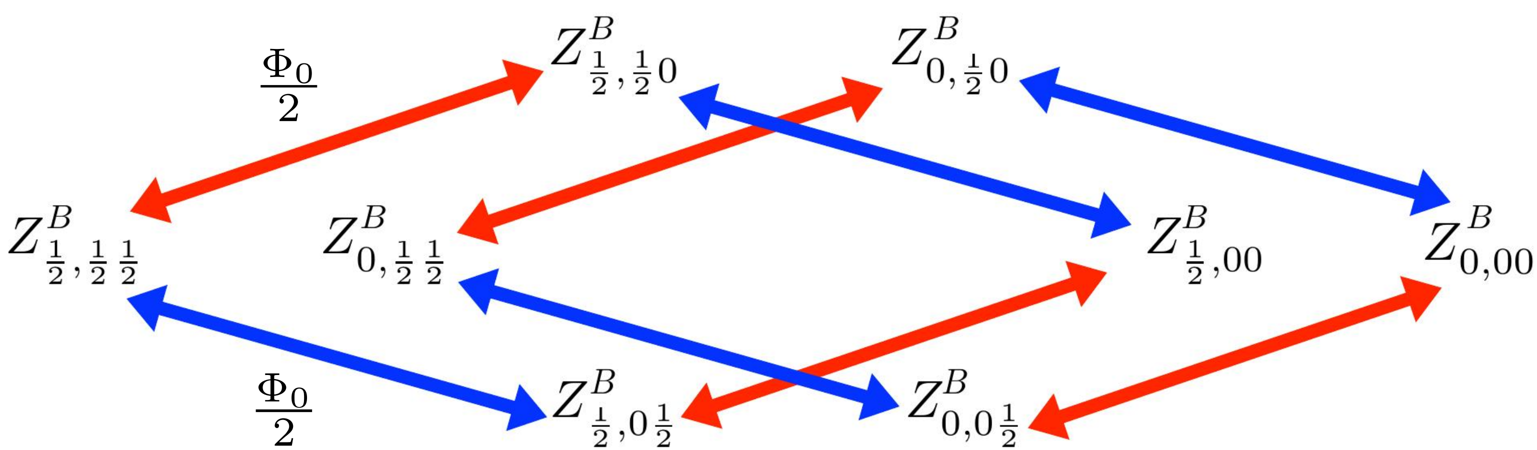
$$(P): \ N_i \in \tfrac{1}{k}\mathbb{Z}, \qquad (A): \ N_i \in \tfrac{1}{2} + \tfrac{1}{k}\mathbb{Z}, \qquad i = 0,1,2$$

→ eight sectors

Modular and flux transformations of Z^B

$Z^B_{N_0,N_1 N_2}$





Bosonic partition function: comments

- $Z^B_{N_0,N_1N_2}$ are different from $Z^F_{N_0,N_1N_2}$ but transform in the same way
- They become equal under dimensional reduction where they reproduce (1+1)-d bosonization formulas
- Spins sectors and spin-half states are identified:

  -The bosonic "NS" sector contains the ground state $Z^B_{\frac{1}{2},\frac{1}{2}\frac{1}{2}} \sim 1 + \cdots$

  -The bosonic "R" sector contains the $S = \frac{1}{2}$ Kramers pair $Z^B_{\frac{1}{2},00} \sim 2 + \cdots$

  → exact bosonization & Fu-Kane stability

- The compactified free boson in (2+1)-d describes a (yet unknown) theory of interacting fermions
- The bosonic spectrum contains further excitations that could be non-local

Conclusions

- Exact results for interacting Topological Insulators can be obtained in 3D using effective actions and partition functions
- Bosonization of relativistic fermions in (2+1) dimensions is proven
- Operator formalism (vertex operators) is semiclassical (Luther, Aratyn, Fradkin et al…..)
- Many more dualities are being discussed in (2+1)-d field theories (Senthil, Metlitski, Seiberg, Witten, Tong,…)

Readings

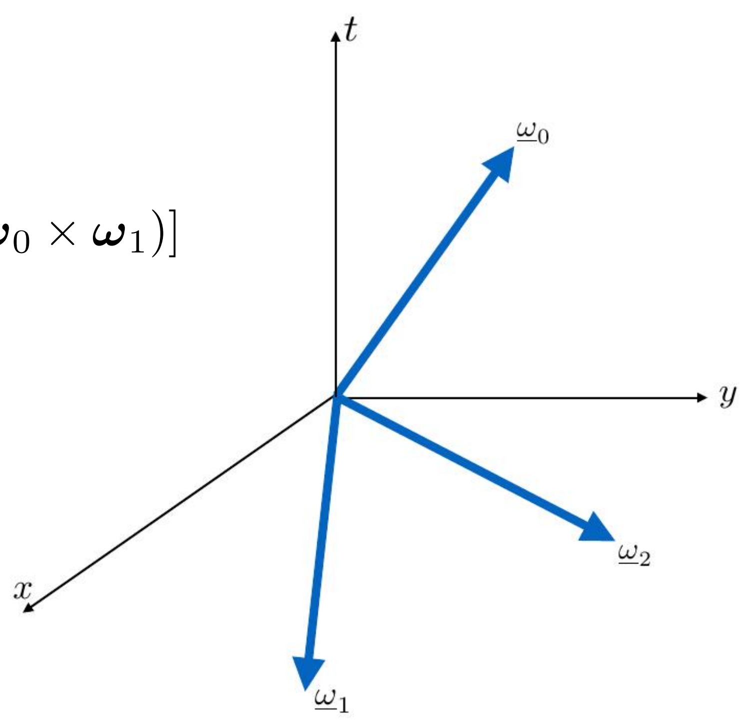
- M. Franz, L. Molenkamp eds., “Topological Insulators”, Elsevier (2013)
- X. L. Qi, S. C. Zhang, Rev. Mod. Phys. 83 (2011) 1057
- C. K. Chiu, J. C. Y. Teo, A. P. Schnyder, S. Ryu, arXiv:1505.03535 (to appear in RMP)

Fermionic Z

$$Z^F_{\alpha_0,\alpha_1\alpha_2} = e^{F_0} \prod_{n_1,n_2\in\mathbb{Z}} \left[1-\exp\left(-2\pi\mathcal{E}^{\alpha_1,\alpha_2}_{n_1,n_2} + 2\pi i\mathcal{P}^{\alpha_1,\alpha_2}_{n_1,n_2} - 2\pi i\mathcal{A}\right)\right][h.c.]$$

$$A \sim \alpha_i = \tfrac{1}{2}, \quad P \sim \alpha_i = 0$$

$$\begin{aligned}
\mathcal{A} &= \alpha_0 - i\frac{V^{(3)}A_0}{2\pi|\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2|} \\
\mathcal{E}^{\alpha_1,\alpha_2}_{n_1,n_2} &= \frac{V^{(3)}}{|\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2|^2}|(n_1+\alpha_1)\boldsymbol{\omega}_2-(n_2+\alpha_2)\boldsymbol{\omega}_1| \\
\mathcal{P}^{\alpha_1,\alpha_2}_{n_1,n_2} &= \frac{(\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2)}{|\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2|^2}\left[(n_1+\alpha_1)(\boldsymbol{\omega}_0\times\boldsymbol{\omega}_2)-(n_2+\alpha_2)(\boldsymbol{\omega}_0\times\boldsymbol{\omega}_1)\right] \\
F_0 &= -\frac{V^{(3)}}{2\pi}\sum_{n_1,n_2}{}' \frac{e^{-2\pi i(\alpha_2 n_1-\alpha_1 n_2)}}{|n_1\boldsymbol{\omega}_2-n_2\boldsymbol{\omega}_1|^3} \\
V^{(3)} &= \det(\boldsymbol{\omega})
\end{aligned}$$



Bosonic Z

$$Z^B_{\alpha_0,\alpha_1\alpha_2} = Z_{HO} Z^{(0)}_{\alpha_0,\alpha_1\alpha_2}, \qquad \alpha_0, \alpha_1, \alpha_2 = 0, \frac{1}{2}$$

$$\begin{aligned}
Z_{HO} &= e^{F_0} \prod_{n_1 n_2}{}' \left(1 - \exp\left(-2\pi\mathcal{E}_{\{\vec{n}\}} + 2\pi i \mathcal{P}_{\{\vec{n}\}}\right)\right)^{-1} \\
\mathcal{E}_{\{\vec{n}\}} &= V^{(3)} \frac{|n_1 \boldsymbol{\omega}_2 - n_2 \boldsymbol{\omega}_1|}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2}, \\
\mathcal{P}_{\{\vec{n}\}} &= \frac{(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} \left(n_1 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_2 - n_2 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_1\right), \\
F_0 &= \frac{V^{(3)}}{4\pi} \sum_{n_1,n_1}{}' \frac{1}{|n_1 \boldsymbol{\omega}_2 - n_2 \boldsymbol{\omega}_1|^3},
\end{aligned}$$

Bosonic Z

$$Z^B_{\alpha_0,\alpha_1\alpha_2} = Z_{HO} Z^{(0)}_{\alpha_0,\alpha_1\alpha_2}, \qquad \alpha_0, \alpha_1, \alpha_2 = 0, \frac{1}{2}$$

$$Z^{(0)}_{\alpha_0\alpha_1\alpha_2} = \sum_{n_0,\ n_1,\ n_2=0}^{k-1} Z^{n_0 n_1 n_2}_{\alpha_0\alpha_1\alpha_2}$$

$$Z^{n_0 n_1 n_2}_{\alpha_0\alpha_1\alpha_2} = \sum_{N_0,\ N_1,\ N_2 \in \mathbb{Z}} (-1)^{2\alpha_0 N_0}$$

$$\exp\Big(- \frac{k^2 \Lambda_0^2}{2\hat{R}_c^2} \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} - \frac{(2\pi \hat{R}_c)^2}{2} \frac{V^{(3)}}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} \left|\Lambda_1 \boldsymbol{\omega}_2 - \Lambda_2 \boldsymbol{\omega}_1\right|^2$$

$$- \frac{i 2\pi k \Lambda_0}{|\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2|^2} (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \cdot (\Lambda_1 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_2 - \Lambda_2 \boldsymbol{\omega}_0 \times \boldsymbol{\omega}_1) + i\pi n_0 \Big)$$

$$\Lambda_i = N_i + \frac{n_i}{k} + k\alpha_i, \qquad i = 1, 2$$

k odd, topological order = k^3