Landau-Ginzburg description of boundary critical phenomena in 2D

(A.C., D'Appollonio, Zabzine, JHEP) '04

Outline

- conformal boundary conditions of Virasoro minimal models
- RG flows between boundary conditions
- Landau-Ginzburg theory with boundary: soliton solutions
- boundary multicriticality & boundary RG flows: Arnold's singularities
- $N=2$ superconformal minimal models
- perspectives

Algebraic description of conformal b.c.

- boundary condition: no flux of momentum
  - ≈ no conformal deformation of boundary
  - ≈ reality condition

$$(L_n - \bar{L}_{-n})\,|a\rangle = 0 \qquad \forall n$$

- standard solution for any pair of Virasoro representations $(\Delta, \bar{\Delta})$ in the bulk

$$|i\rangle\rangle = \sum_N \underbrace{|\Delta; N\rangle}_{\chi_\Delta} \otimes U\, \overline{\underbrace{|\bar{\Delta}; N\rangle}_{\overline{\chi_{\bar{\Delta}}}}}$$ (Ishibashi states)

- b. states are linear combinations of Ishibashi states having proper normalizations

$$|a\rangle = \sum_i B_{ai}\, |i\rangle\rangle$$

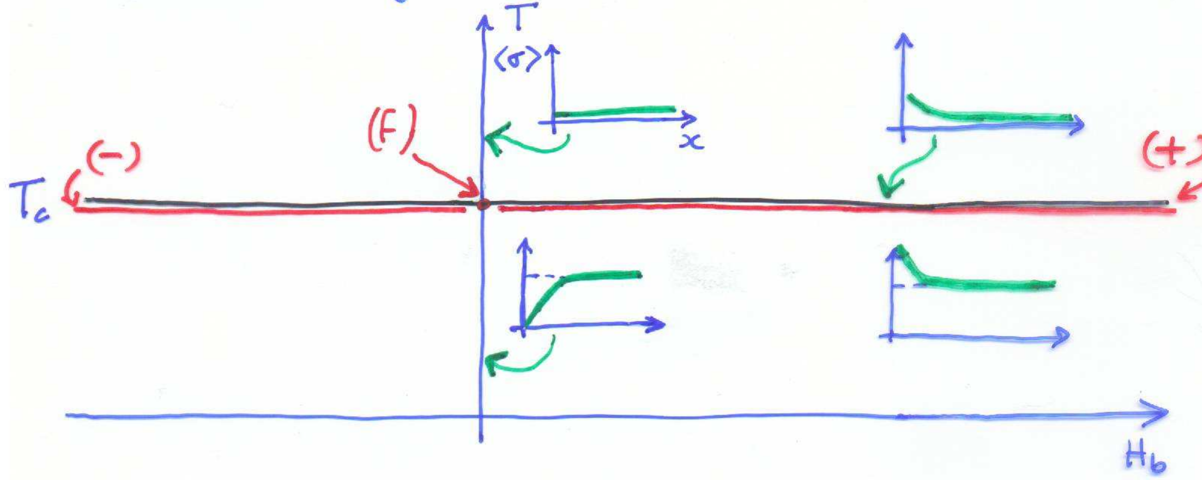
- Rational CFTs: finite # of bulk sectors
  - → finite # of conformal b.c.'s
  - → one-to-one correspondence with bulk sectors (if $\Delta = \bar{\Delta}$) (Cardy)
  - → b.c. are known (but not easy to visualize) (many people...)

- Rational CFTs are relevant in cond-mat;
- Describe "difficult" branes in compact space-times and non-trivial metrics

Ex: two-dimensional Ising model

$$Z = |\chi_0|^2 + |\chi_{1/2}|^2 + |\chi_{1/16}|^2$$ (periodic b.c.)

- 3 bulk sectors: Id, ϵ, σ $(\Delta = 0, \frac{1}{2}, \frac{1}{16})$
- 3 conformal boundaries:
  - (+), (−) $\sigma = +1, -1$ fixed b.c.
  - (f) free b.c.
- phase diagram



- boundary RG flow at bulk criticality $T = T_c$

$$(-) \longleftarrow\!\longleftarrow (f) \longrightarrow\!\longrightarrow (+)$$

driven by boundary magnetic field H_b;
(f) b.c. has one relevant boundary field

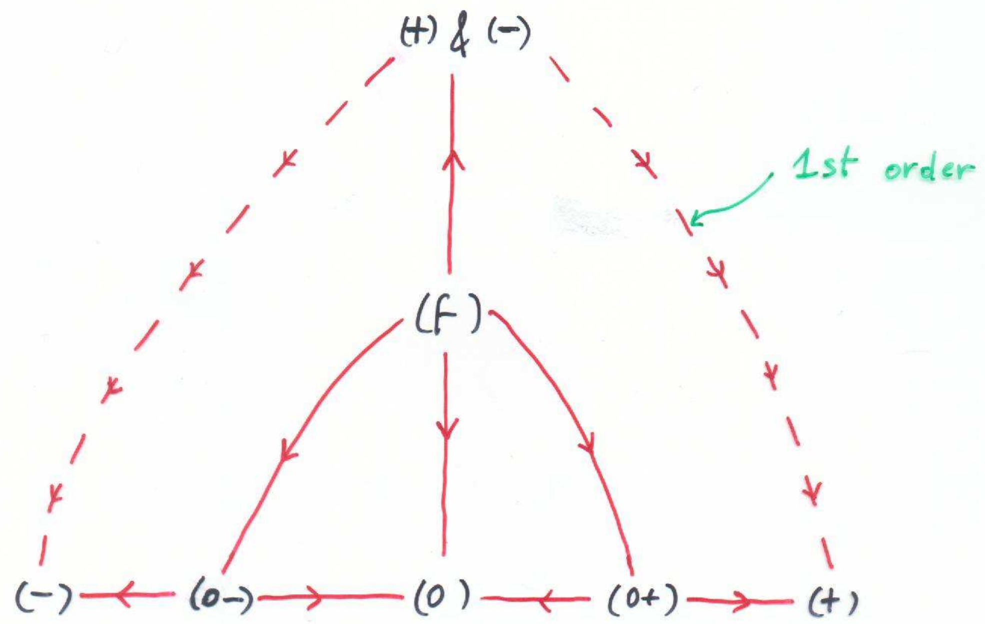
$$S = S_{CFT}\Big|_{(f)} + H_b \int d\tau\, \sigma_b(\tau)$$

Ex: boundary RG flow of tri-critical Ising

- Next Virasoro minimal model, first $N=1$ model
- 6 sectors, 6 Cardy boundaries
- Ising model with vacancies $\sigma_i = \pm 1, 0$
- boundaries : $|+\rangle, |-\rangle, |0\rangle$ fixed

  $|0+\rangle, |0-\rangle$ partially fixed

  $|f\rangle$ free



- compilation of results of integrable boundary interactions (Affleck)
- can flow from a Cardy state to a superposition of them (Recknagel et al.)

Virasoro Minimal Models

(Belavin, Polyakov, Zamolodchikov)

- central charge $c = 1 - \frac{6}{m(m+1)} < 1$

  $m = 3$ (Ising), 4 (Tricr. Ising), ...

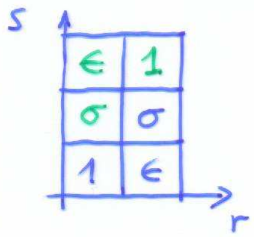
- A-D-E pattern: as for Lie algebras, each model is associated to a Dynkin diagram

  A_2  Ising , A_3  Tricr. Ising , ..., A_{m-1}

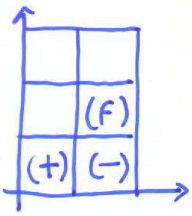
  D_4  Potts $(m=5)$, ...

- A_{m-1} series : bondary states and their stability (# relevant boundary fields)

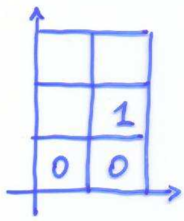
→ Kac table : $\phi_{r,s}$, $1 \leq r \leq m-1$, $1 \leq s \leq m$, $\Delta_{r,s}$ dim



bulk sectors

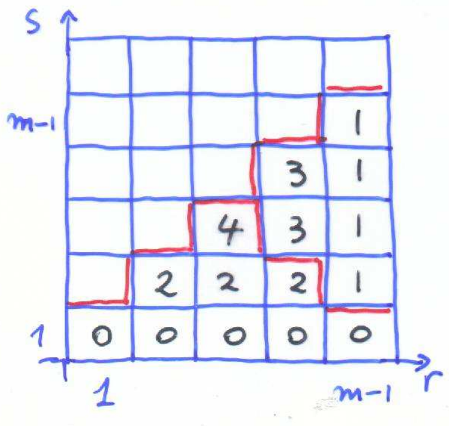


conformal b.c.



relevant b. fields

$(m=6)$



$\frac{m(m-1)}{2}$ b.c.

$m-1$ stable

$m-2$ once unstable

$m-3$ 2-fold "

⋮

Landau-Ginzburg description

Q. : How to determine the pattern of boundary RG flows ?

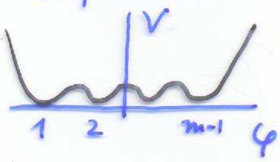
A. : Use Landau-Ginzburg (mean field) theory adapted to boundary problems

• Recall bulk L-G description (Zamolodchikov; Cardy, Ludwig)

$$S = \int d^2x \; \tfrac{1}{2}(\partial_\mu \varphi)^2 + V(\varphi)$$

$$V = \lambda \varphi^{2(m-1)}$$

(m-1)-fold critical point
≈ coexistence of (m-1) phases
≈ (m-1) minima



• LG description matches CFT of Minimal models

• (m-1)-critical point can RG flow to a lower multi-critical point by perturbing with
$\varphi^2, \varphi^4, \ldots \varphi^{2(m-2)}$

• RG flow ≈ detuning of high critical point

$$V_{UV} = \lambda \varphi^{2(m-1)} + \mu \varphi^{2k} \xrightarrow[RG]{} V_{IR} = \mu \varphi^{2k} \qquad k<m-1$$

• classical theory is correct at qualitative level

Assumptions:

- locality of eff. action w.r.t. order parameter φ (SSB)
- analyticity at $\varphi = 0$

Results: (2D!!)

- nice qualitative description of Virasoro minimal models and RG flows (A.B. Zamolodchikov; Cardy, Ludwig)
- exact description of $N=2$ superconformal minimal models (Martinec; Vafa, Warner; ....) Arnold's simple singularities; physical picture for A-D-E classification; ....

  many nice results

Extension with boundary:

- hope for nice picture of boundary RG flows and boundary physics

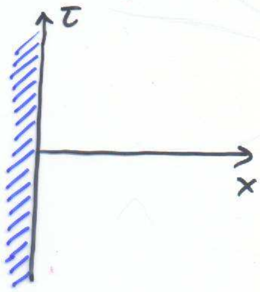
BUT :
- no SSB on 1D boundary
- no order parameter
- evidences of nonlocality and strong quantum effects $\langle \varphi_{Ren}(x \sim 0) \rangle = x^{-\Delta} \to \infty$

STILL ....

Boundary Landau-Ginzburg theory

(Binder; Cardy; ...; Diehl
A.C., D'Appollonio, Zabzine

$$S = \int_{x>0} d^2x \; \frac{1}{2}(\partial_\mu \varphi)^2 + V(\varphi) \; + V_b(\varphi_0)\,\delta(x)$$



$$0 = \delta S = \int \left(-\partial^2\varphi + \frac{\partial V}{\partial \varphi}\right)\delta\varphi + \partial_x\varphi\,\delta\varphi\Big|_0^\infty + \frac{\delta V_b}{\partial\varphi}\Big|_0 \delta\varphi_0$$

$$\begin{cases} \partial_x^2 \varphi = \dfrac{\partial V}{\partial \varphi} & \text{bulk eq. of motion} \\[2ex] \dfrac{\partial \varphi}{\partial x}\Big|_0 = \dfrac{\partial V_b}{\partial \varphi}\Big|_{\varphi_0} & \text{b.c. at } x=0 \end{cases}$$

- Integrate bulk eq. to get another b.c.

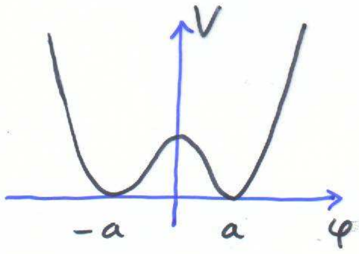
$$0 = \int_0^\infty dx \; \frac{\partial\varphi}{\partial x}\left(\frac{\partial^2}{\partial x^2}\varphi - \frac{\partial V}{\partial \varphi}\right)$$

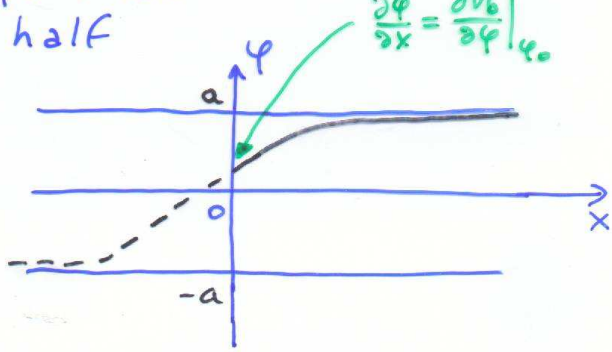
$$\rightarrow \quad \frac{\partial \varphi}{\partial x}\Big|_0 = \pm\sqrt{2(V(\varphi_0) - \cancel{V(\varphi_\infty)})}$$

- Remarks:

→ <u>Soliton equation</u> evaluated at $x=0$
At phase coexistence profiles are solitons cutted half

Ex:





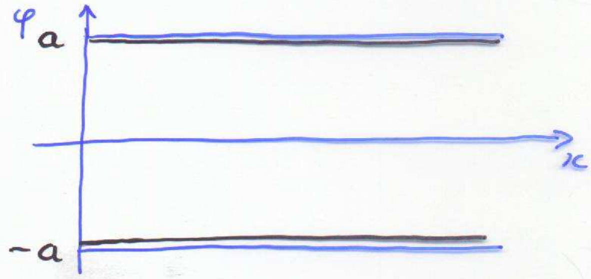
- Vafa, Iqbal, Hori: in the $N=2$ susy LG theory, the susy b.c. are one-to-one with the solitons of the massive phases;
  critical limit $\longrightarrow$ conformal b.c.'s

$\longrightarrow$ conformal b.c. $\approx$ universality class of ground-state profiles
$\approx$ universality class of soliton solutions

- Pearce et al.: local description of b.c. in integrable height models $h_i = 1, 2, \ldots, m-1$;
- stable b.c. $\approx$ $\varphi(x) = \text{const}$ in each phase

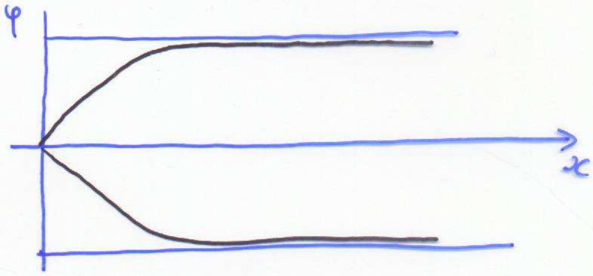
trivial solitons

Ising $\varphi = a$ $(+)$ b.c.
$\varphi = -a$ $(-)$ b.c.



- unstable b.c. $\approx$ mixtures of solitons

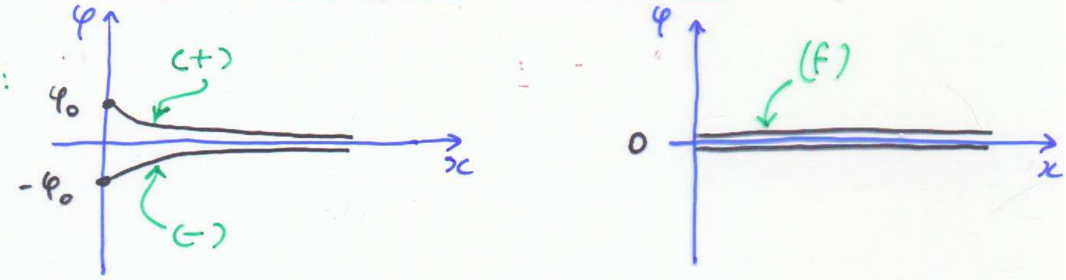
Ising (F) b.c.
$\mathbb{Z}_2$ symmetry



- take critical limit in the bulk $a \to 0$:
  - $\to$ solitons become degenerate;
  - $\to$ solutions completely determined by $V_b(\varphi_0)$

$$\left.\frac{\partial \varphi}{\partial x}\right|_0 = \underbrace{\frac{\partial V_b}{\partial \varphi_0} = \pm\sqrt{V(\varphi_0)} = \pm\, \varphi_0^{\,m-1}}_{\text{algebraic equation for } \varphi_0}$$

Ising:



$$m=3\,, \quad V_b = H_b \varphi_0\,, \qquad \varphi_0^2 = \pm H_b$$

$\to \; V_b = 0 \qquad\qquad \varphi_0^2 = 0 \qquad$ degenerate solution

$\to \; V_b = H_b \varphi_0 \qquad \varphi_0 = \pm\sqrt{|H_b|} \qquad$ non-degenerate solutions

- introduce $\quad W = \varphi_0^m + V_b = \varphi_0^m + a_1 \varphi_0^k + \cdots$

  study stationary points $\qquad 0 = \dfrac{\partial W(\varphi_0)}{\partial \varphi_0}$

non-deg. stat. point of W	$\longleftrightarrow$	stable b.c. (+), (-)
n-fold deg " " "	$\longleftrightarrow$	n-fold unstable b.c. (f)

$$S_{cl}[\varphi] = T\, E_{\text{soliton}}[\varphi] + T\, V_b = T\, W(\varphi_0)$$

boundary term

Arnold's Singularity Theory

Study stationary points of functions $W = W(x)$,
i.e. singular points of inverse map,
modulo reparametrizations $x \to x' = x + \epsilon(x)$

here: reparam. invariance $\approx$ universality

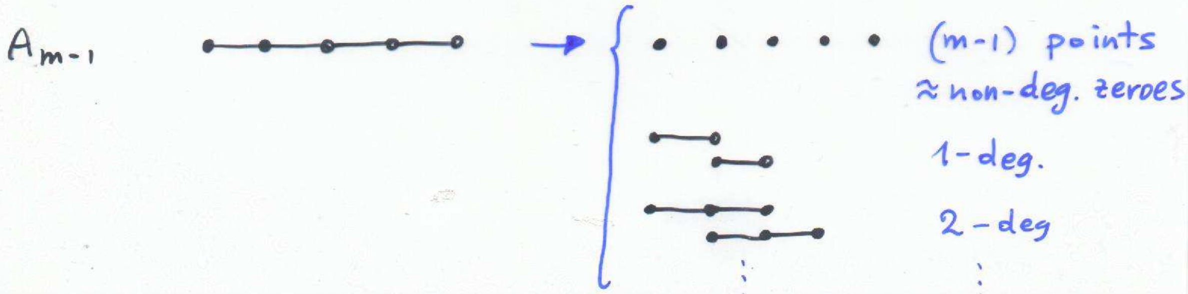
- Th: (Arnold) The "simple" singularities have the A-D-E pattern

A_{m-1}	$W = x^m$	$m = 2, 3, \dots$
D_k	$W = x^{k-1} + xy^2$	$k = 4, 5,$
E_6	$W = x^3 + y^4$	
E_7	$W = x^3 + xy^3$	
E_8	$W = x^3 + y^5$	

- Empirical Th: (A.C.) The pattern of degenerate stationary points of

$$W = x^m + a_1 x^{m-2} + a_2 x^{m-3} + \dots + a_{m-1} \quad (x \text{ Real})$$

is described by the sub-diagrams of the Dynkin diagram, as follows:

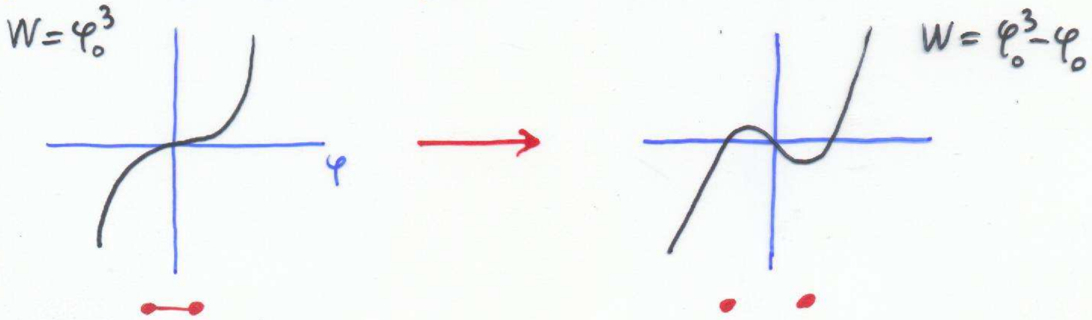


Ex: Ising $m=3$, A_2 →
- 2 non-deg. ≈ stable b.c. $(\pm)$
- 1 once deg. ≈ once unstable (f)

Tricrit. Ising A_3 →
- 3 stable $(0), (\pm)$
- 2 once unstable $(0+), (0-)$
- 1 twice unstable (f)

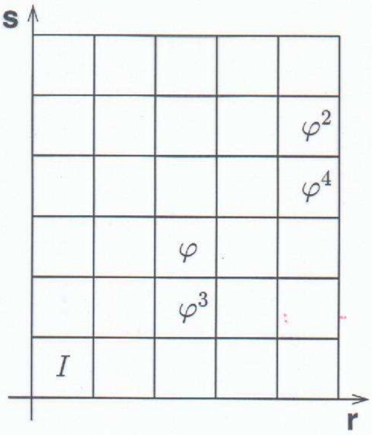
Results:

- It matches the pattern of conformal b.c.'s of A_{m-1} Virasoro minimal models;
- It extends earlier uses of catastrophy theory for bulk LG theory (Vafa, Warner; Martinec)
- It describes boundary RG flows of unstable b.c. as detuning of coincident stationary points of $W(\varphi_0) = \varphi_0^m + V_b(\varphi_0)$ upon varying the parameters in V_b



- Description of boundary fields of the most unstable b.c. is manifest $\approx$ W deformations

Ex : $m=6$



$$W = \varphi_0^6 + V_b(\varphi_0) = \varphi_0^6 + a_1 \varphi_0^4 + a_2 \varphi_0^3 + a_3 \varphi_0^2 + a_4 \varphi_0$$

$$Z_{(3,3)|(3,3)} = \chi_{(1,1)} + \chi_{3,3} + \chi_{5,5} + \chi_{3,2} + \chi_{5,4} + \text{irrel.}$$

parity symmetry matches $\varphi_0 \to -\varphi_0$ (P. Ruelle)

- boundary fields of other, less unstable b.c.'s given by deformations of lower order singularity φ_0^k, $k=5,4,\dots$ occurring in W when some of the stationary points coincide

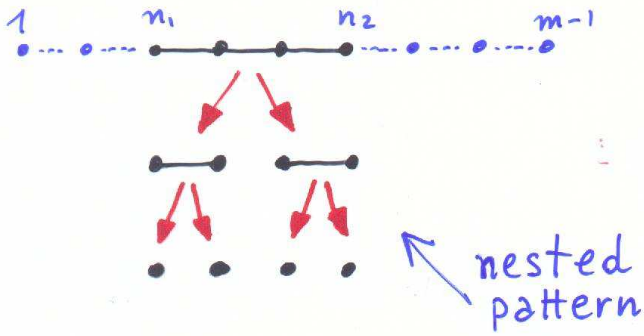
$\longrightarrow$ nested pattern

- ## Selection rule

Boundary RG flow:

$\approx$ detuning of higher stationary point

$\approx$ breaking the Dynkin diagram.



$(n_1, n_2) \leftrightarrow (r,s)$ of Kac's table

(cf. Graham)

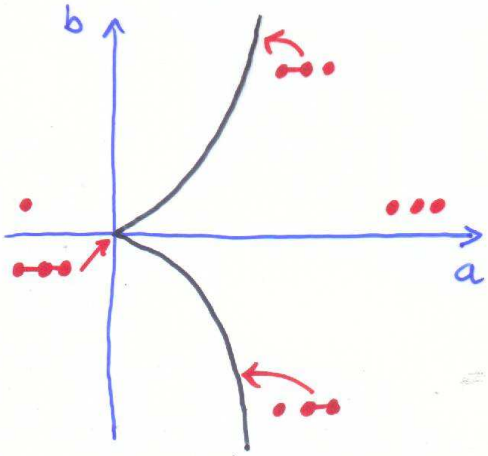
Graham, Watts

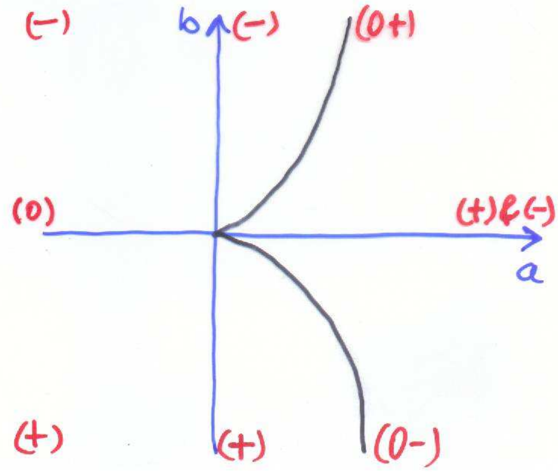
- ## RG space

Space of coupling constants with RG flows

$\approx$ space of parameters of deformations of Arnold's singularities

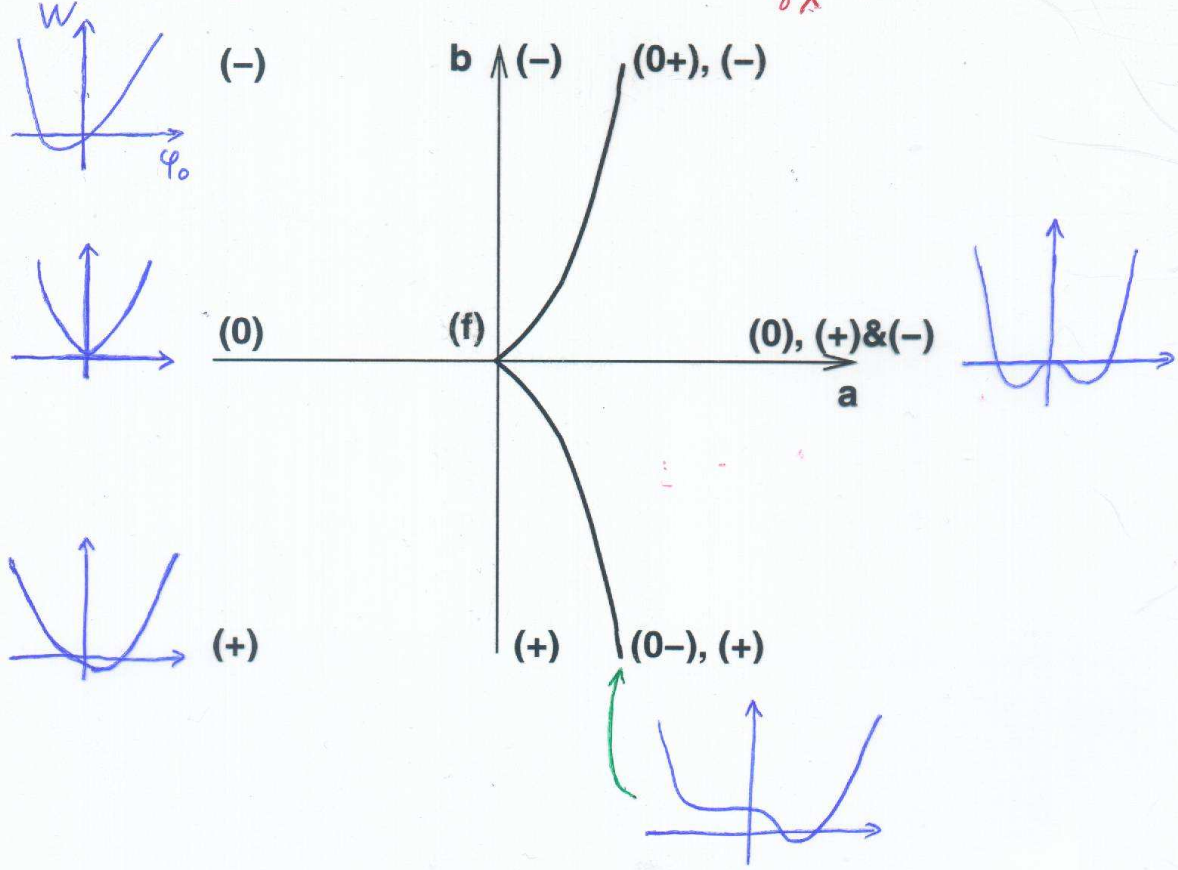
Ex: Tricritical Ising $\quad 0 = \frac{\partial W}{\partial x} = x^3 - a x + b$





Tricritical Ising

$$\frac{\partial W}{\partial x} = x^3 - a x + b = 0$$



- conformal b.c. :

  $x = x_0(a,b)$, $a, b \to \infty$ in one direction

  $x_0 \to \infty$

- RG flows $(F) \to (0), (+), (-)$, $(+) \oplus (-)$

- RG flow $(0-) \to (0), (-)$

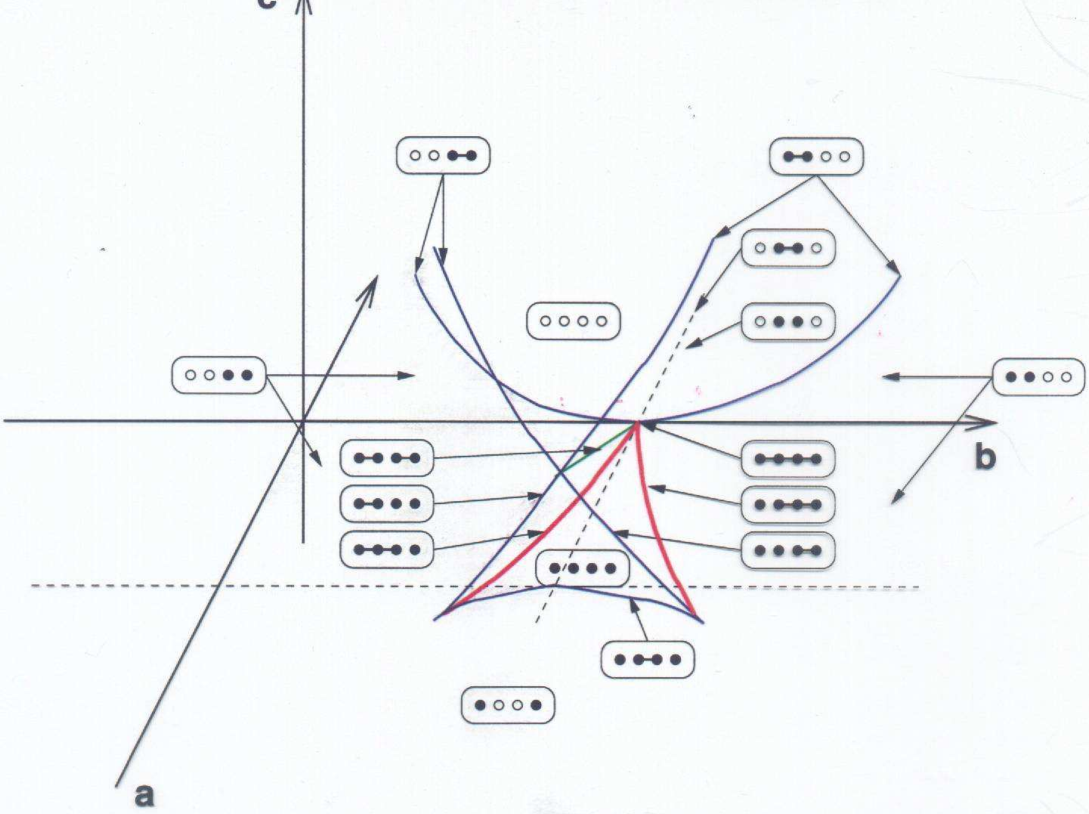
  expand around $x \sim x_{(0-)}$ on the wing

  $$W \sim (x - x_{(0-)})^3 + \tilde{b}\,(x - x_{(0-)})$$

  lower-order pattern (Ising)

Tetra-critical Ising

"swallow tail"



$$0 = x^4 + a x^2 + b x + c$$

$N=2$ minimal models

(Gepner; Vafa, Lerche, Warner, ------)

$$\frac{\widehat{SU(2)}_k}{\widehat{U(1)}_{2k+4}} \times \widehat{U(1)}_4 \qquad c = \frac{3k}{k+2}$$

$\ell = 0,1,\dots,k$ $\quad m = -k-1, -k, \dots, k+2 \mod 2k+4$ $\quad s = -1,0,1,2 \mod 4$

$(\ell, m, s) \sim (k-\ell, m+k+2, s+2)$ $\qquad s=0,2$ NS

$\ell + m + s = 0 \mod 2$ $\qquad s=-1,1$ R

- $N=2$ supersymmetric boundary conditions:

(A): $Q = \bar{Q}_+ + \eta Q_-$ $\qquad Q^+ = \bar{Q}_- + \eta Q_+$ $\qquad$ real $\qquad \eta = \begin{cases} -1 & NS \\ +1 & R \end{cases}$

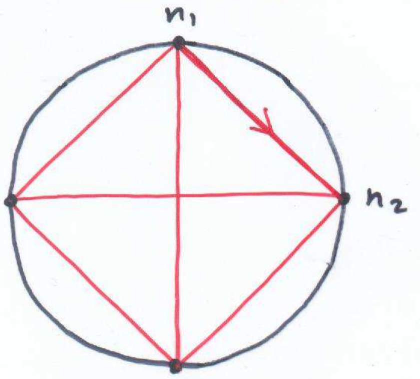
(B): $Q = \bar{Q}_+ + \eta \bar{Q}_-$ $\qquad Q^+ = Q_+ + \eta Q_-$ $\qquad$ holomorphic

- geometric interpretation: (Maldacena, Moore, Seiberg 2001)

(A)-type are D1 branes, (B)-type D0 and D2 branes

- (A)-type b.c.: Cardy's states represented by oriented chords (n_1, n_2)

Ex: $k+2=4$



$\ell + 1 = |n_1 - n_2|$

$m = n_1 + n_2$

$s = \text{sign}(n_1 - n_2)$

(R)

$|\ell, m, s=\pm 1\rangle = |n_1, n_2\rangle$

$(k+2)(k+1)$ boundaries

→ Dynkin diagram representation $\hat{A}_{k+1}$

$m \sim k+2$

$N=2$ L-G theory

(Hori, Iqbal, Vafa, 2000)

$$S = \int d^2x \left(d^4\theta \, K(\Phi, \bar{\Phi}) + d^2\theta \, W(\Phi) + d^2\bar{\theta} \, \overline{W(\Phi)} \right)$$

$$W_{crit} = \lambda \Phi^{k+2}$$ chiral ring $\{1, \Phi, \dots, \Phi^k\}$

- massive theory possesses BPS solitons (Cecotti, Vafa) (Fendley et al.)

$$Q\,|BPS\rangle = 0 \qquad Q^+\,|BPS\rangle = 0$$

- same as (A)-type supersymmetric boundary conditions

(A)-type b.c.: $\partial_t \phi = \pm \dfrac{\partial \overline{W}}{\partial \bar{\phi}} \quad \leftrightarrow \quad \mathrm{Im}\, W(\phi) = \mathrm{const.}$

- D1 brane: curve in complex ϕ plane stemming from a stationary point $W(a)$

$$\gamma_a : \quad \mathrm{Im}\, W(\phi) = \mathrm{Im}\, W(a), \quad \mathrm{Re}\, W(\phi) > \mathrm{Re}\, W(a)$$

"vanishing cycle" $\approx$ half time soliton

- in the critical limit $W \to W_{crit}$, $\mathrm{Im}\, \phi^{k+2} = 0$ are rays from the origin, labelled by $n = 1, \dots, k+2$
  vanishing cycles are pairs of $k+2$ rays (n_1, n_2)
  $\rightarrow$ match CFT description of conformal b.c.

$\rightarrow$ LG calculation of $\langle a | b \rangle_{RR} = I(a,b)$

$\rightarrow$ other nice results

but: rather formal; needs limit from non-critical bulk

Extend N=0 strategy: boundary potential

- $$S = S_{bulk} + \int_\gamma dt \, \mathrm{Im}\left(V_b(\phi)\right)$$
- consider bulk critical $\quad W = W_{crit} = \Phi^{k+2}$
- (A)-type b.c. (Lindström, Zabzine)

$$\gamma \; : \; \mathrm{Im}\,\widetilde{W} = \text{const}, \qquad \widetilde{W} = W_{crit} + V_b$$

- boundary potential allows non-trivial ground state solution; another, orthogonal BPS soliton, now time independent

$$\sigma \; : \; \mathrm{Re}\,\phi^{k+2} = 0, \quad \mathrm{Im}\,\phi^{k+2} > 0$$

other set of rays from the origin

- D1 boundary condition satisfied by orthogonality in $\widetilde{W}$ plane

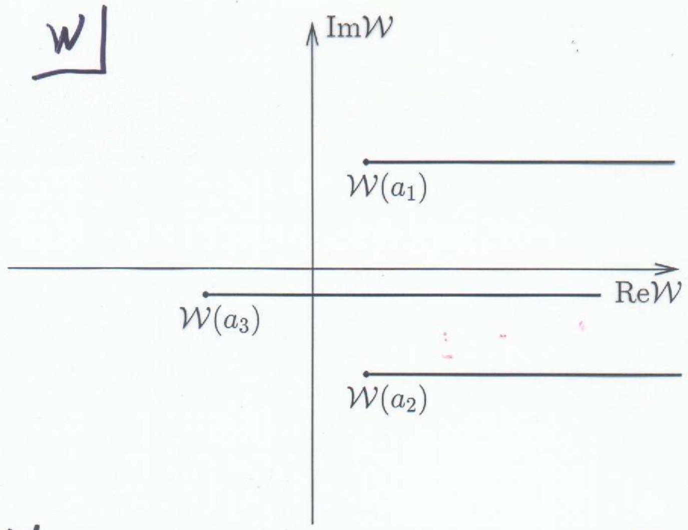
Conclusions

1) smooth description of b.c. at bulk criticality:
   - ground state profile → space soliton
   - boundary field values → time soliton

2) RG flow different from N=0 case:
   - all Cardy states are stable
   - critical points of $\widetilde{W}(\phi_0)$ are non-degenerate

$K+2=4$, $\mathrm{Im}\,\phi^4 = 0$, $\mathrm{Re}\,\phi^4 > 0 \to$ axis

$W|$



$\phi|$

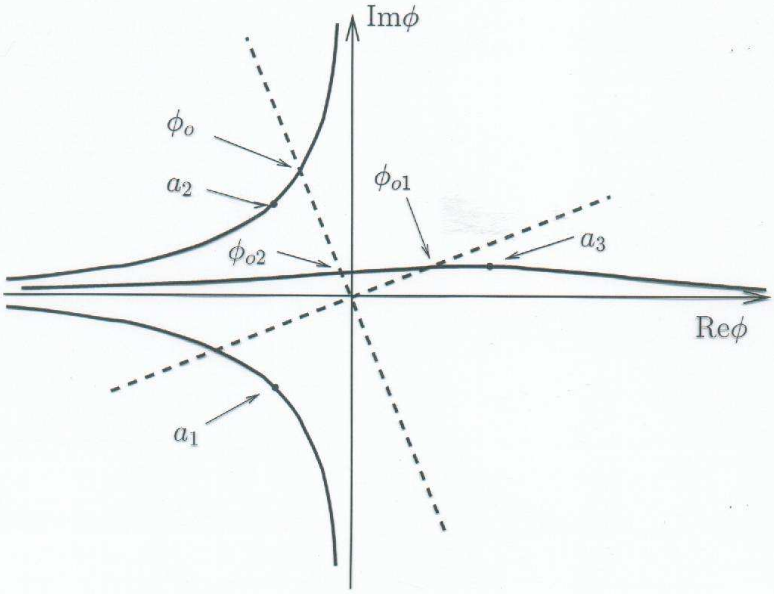
$\frac{\partial \tilde{W}}{\partial \phi} = 0$ at $\phi = a_i$

$\mathrm{Re}\,\phi^4 = 0$ $\mathrm{Im}\,\phi^4 > 0$

γ_{a_2}

$Z = (\ell+1)\, e^{-S_{cl}(\phi_{0i})}$, $S_{cl} = T\,\mathrm{Im}\,\tilde{W}(\phi)$

$\tilde{W} = \phi^4 + V_b(\phi)$


Imϕ
ϕ_o
a_2
ϕ_{o1}
ϕ_{o2}
a_3
Reϕ
a_1

Conclusions

- simple picture for boundary conditions of Virasoro minimal models; consistent with BCFT data & integrable models

- predictions for boundary RG flows:
  - nested pattern;
  - moduli space of (A_m) singularities;

  (consistent with naive expectations)

  → many things to verify (compute)

Other results

- D and E series of minimal models
  → bulk LG: missing relevant fields → reduced RG space

- N=2 LG models (⟷ Vafa et al.)
  → A-type b.c. are lines not points;
  → soliton solution implied by sosy (OK);
  → B-type requires boundary d.o.f. (Lerche etal)