On the c-theorem above two dimensions

(A.C., G. D'Appollonio, hep-th/0005115
(A.C., R. Guida, N. Magnoli, hep-th/0103237
proceedings hep-th/0009119

Outline

- Introduction: the two-dimensional c-theorem
  → irreversibility of the renormalization-group flow
- Motivations for the c-theorem in $d>2$ ($d=4$)
  → recent activity: AdS/CFT correspondence
- Calculation of the Euler term of the trace anomaly in any $d=2K\geq 4$ for free fields
  → "a-theorem": "strange" counting of d.o.f.
- General expression of $\langle TTT\rangle$ in $d=4$
  → trace anomaly ⟷ finite amplitudes
  → sum rules for the RG flow of anomaly coefficients "a" & "c"
- No theorem ......yet....

The renormalization-group flow

Transformation of (dimensionless) coupling constants g^i under change of scale

$t \to t + dt \quad (t = -\log \Lambda)$

$$\frac{dg^i}{dt} = -\beta^i(g) \qquad \text{"beta-functions"}$$

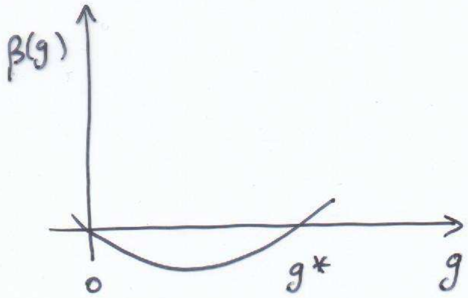
which leaves the field theory invariant (covariant

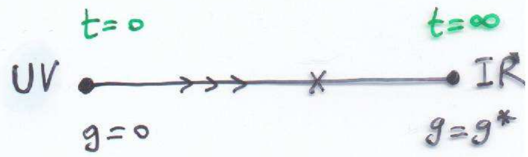
$$\Lambda \frac{d}{d\Lambda} S = \left(\Lambda \frac{\partial}{\partial \Lambda} + \beta^i \frac{\partial}{\partial g^i}\right) S(g, \Lambda) = 0 \qquad \text{effective action}$$

$$\rightarrow \quad S(g^i, \Lambda) = S(g^i + \beta^i dt, \Lambda(1 + dt))$$

change of scale $\approx$ tuning of parameters

Ex: 1 coupling





$g = 0 \rightarrow \beta = 0$ scale invariance

$g > 0 \rightarrow$ massive theory: no invariance, but still we can predict the change

Ex: $\lim_{|x| \to \infty} \langle \phi(x) \phi(0) \rangle_g$ given by $\langle \phi(x) \phi(0) \rangle_{g = g^*}$

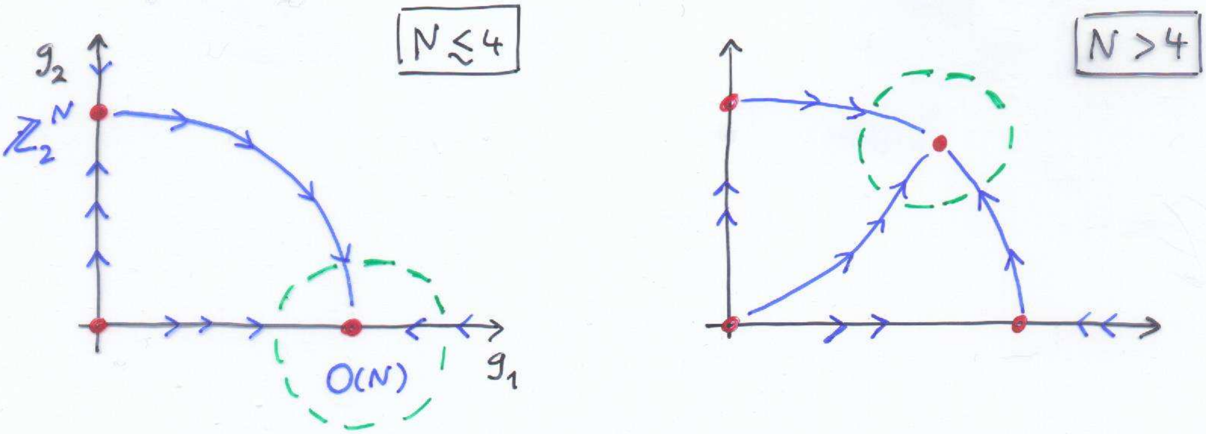
WHY such a reparametrization invariance?

Renormalization: (classical) scale-invariant field theory made finite by introducing one scale: the cutoff

Ex: 2 couplings (Brezin et al.)

$$S = \int d^d x \left[\frac{1}{2} \sum_{a=1}^{N} (\partial_\mu \varphi_a)^2 + g_1 \Big(\sum_{a=1}^{N} \varphi_a^2 \Big)^2 + g_2 \sum_a \varphi_a^4 \right] \qquad d=4-$$

Heisenberg ~ $O(N)$ $\qquad$ $(\mathbb{Z}_2)^N$ ~ Ising



→ The $O(3)$-symmetric theory is stable in $d<4$

Fixed points ↔ basins of attraction
universality

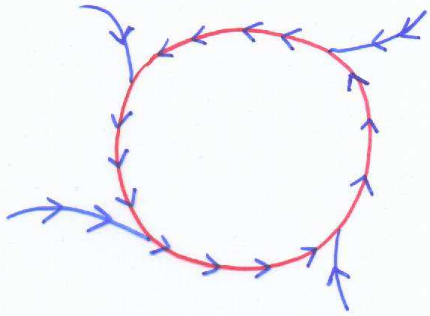
Conclusion: renormalization-group is a powerful tool for qualitative arguments, a "paradigm" (+ effective action)

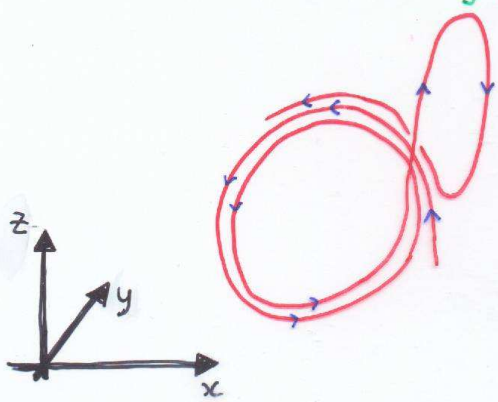
Key point: RG flow ends in fixed points only

$$\frac{dg^i}{dt} = -\beta^i(g)$$ non-linear differential equations

(i) recurrent behavior = limit cycles

(ii) chaos = strange attractors (no focussing)





These asymptotic flows have no interpretation in (unitary) field theory and should not be possible

Related issue: irreversibility of the RG flow

RG transformation = "integrating out" high-energy degrees of freedom (Wilson) (Kadanoff)

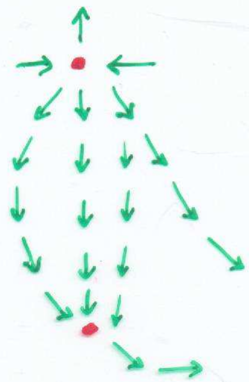
$$g^i(\Lambda_0) \leftrightarrow g^i(\Lambda_0 + d\Lambda_0) \quad \text{by} \quad \int_{\Lambda_0}^{\Lambda_0+d\Lambda_0} dk \, S_{eff}$$

cutoff

→ correlations of $\Lambda_0 < |k| < \Lambda_0 + d\Lambda_0$ are lost:
RG transformation cannot be exact, even in principle.

The 2-d c-theorem

(A.B. Zamolodchikov, '86)



- RG flow $\quad \frac{d}{dt} = -\beta^i(g)\frac{\partial}{\partial g^i}$

- Th: $\exists$ c-function $c(g^i)$:

  → $\frac{d}{dt}c \leq 0$ monotonic

  → $\beta^i = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial g^i}c = 0$ stationary at fixed points g^*

  → $c(g^*)$ = Virasoro central charge c

- consequences:
  → the RG flow has only fixed points
  no limit cycles or strange attractors

  → fixed points are classified by their "height"
  $c_{UV} > c_{IR}$

- remarks:
  → inputs of proof are kinematic $\begin{cases} \text{- unitarity} \\ \text{- Poincaré invariance} \end{cases}$

  → $c(g)$ is finite once g^i are renormalized;
  $c(g^*)$ is finite and uniquely defined
  ⇔ trace anomaly is universal
  $c(g)$ is globally (non-perturbatively) defined
  in the whole space of theories

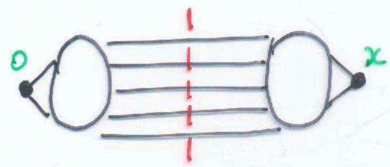
RG flow & irreversibility in $d=2$

$c \propto$ # of fields in the theory

$c_{UV} > c_{IR}$ $\rightarrow$ some fields become massive, are integrated out and desappear : irreversibility

- Use dispersive proof of the c-theorem (Friedan, 89) to show this explicitly:

$\rightarrow$ spectral density



i) $\rho(\mu, g^i(\Lambda)) = c_{IR}\,\delta(\mu) + \rho_s(\mu, g^i(\Lambda))$

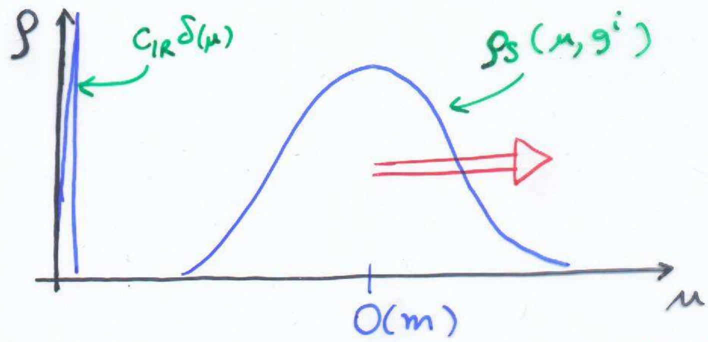
ii) $\rho(\mu, g^i)$ is positive definite

$$\sum_n \langle T^\mu_\mu(0)|n\rangle\langle n|T^\mu_\mu(x)\rangle$$

iii) $c_{UV} = \int_0^\infty d\mu\, \rho(\mu, g^i)$ sum rule (Cardy, 88)

theorem : $c_{UV} = c_{IR} + \int_\epsilon^\infty d\mu\, \rho(\mu, g^i) > c_{IR}$

loss of d.o.f. :



$\lim_{\Lambda\to\infty} \rho(\mu, g^i(\Lambda)) \searrow c_{IR}\,\delta(\mu)$ infrared weak limit

smooth part goes to ∞ and does not contribute to any correlator at finite distance : massive states are lost at the new IR fixed point.

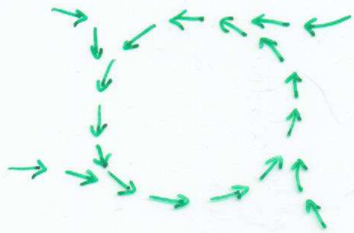
Motivations for the theorem in $d>2$

$\boxed{\text{It has to be true}}$

c = measure of degrees of freedom of field theory

RG = "integrating out" high-energy d.o.f.

no "zombie" d.o.f.
as in limit cycles



BUT : no proof since more than ten years

- straightforward extension of $d=2$ proof not possible
  (A.C., J. Latorre, D. Friedan, '90)
- $d = 3, 5, ..$ no trace anomaly
  → no obvious global definition of $c(g)$
- big idea is missing maybe

HOPE : new ingredient for the proof pops out;
this will tell us something new on
field theory and how field theory
interacts with gravity ($c \leftrightarrow$ stress tensor $T_{\mu\nu}$)

Evidences for the theorem in $d=4$

Discuss the terms in the trace anomaly

Euler char. $\chi = \int \sqrt{g}\, G$ — $(\text{Weyl})^2$ — local, scheme-dependent $\frac{\delta}{\delta\sigma} \int \sqrt{g}\, \mathcal{R}^2 \propto \mathcal{D}^2 \mathcal{R}$

$$\langle T_\mu^\mu \rangle = a\, G \quad - 3c\, F \quad + a' \mathcal{D}^2 \mathcal{R}$$

- "a"-theorem (Cardy, 88) : non-trivial evidences in $N=1$ gauge theories (Seiberg dualities) for $a_{UV} > a_{IR}$ (Anselmi, Freedman, et al)
- "c"-theorem : not true in general ($\exists$ counterex.) but evidences for the theorem by the AdS/CFT correspondence say $c_{UV} > c_{IR}$ if $c_i = a_i$ $(i = UV, IR)$
- "a′"-theorem : cannot be true, scheme dependence $a' \to a' + \text{const.}$
  $a'(g)$ would not be globally defined
  (Anselmi : dynamical hypothesis of $a' \propto a$ would cure this)

LOTS OF ACTIVITY

(Earlier work by Cardy, Jack, Osborn, Shore, ...)

Seiberg dualities

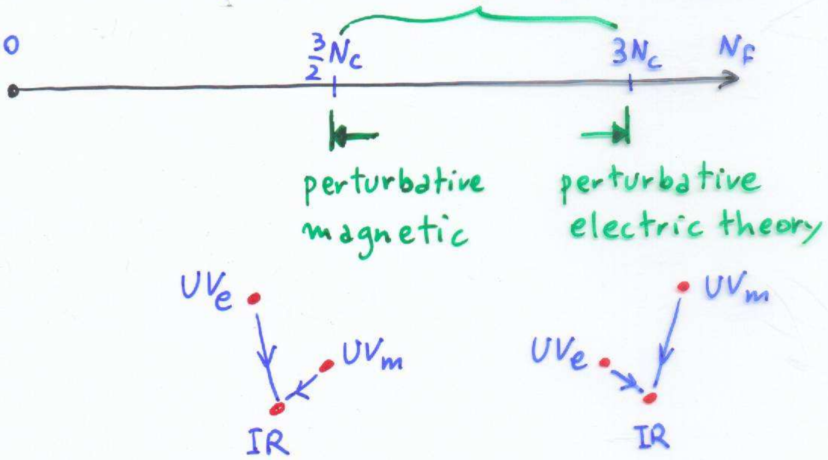
(Seiberg, 95)

$\mathcal{N}=1$ Susy gauge theory with N_f quarks and $SU(N_c)$ colour group:

i) UV fixed point is free for $N_f < 3N_c$ (asymptotic freedom

ii) IR fixed point is interacting (non-trivial gauge theory) for $\frac{3}{2} N_c < N_f < 3 N_f$ ("conformal window")

iii) same IR f.p. is found in the dual "magnetic" gauge theory with N_f quarks and group $SU(N_f - N_c)$



Anselmi, Freedman et al, 97:

- $a_{UV} - a_{IR} > 0$ for both flows over the whole range; tests of a as a measure of d.o.f. are OK
- $c_{UV} - c_{IR}$ not always positive throughout (as known already in non-Susy theories)

Bastianelli, 96: other range $0 \le N_f \le \frac{3}{2} N_c$ is also checked, but flows are simpler there.

AdS/CFT correspondence

(Girardello et al.; Freedman et al., 98–01)

- Supergravity on AdS_5 corresponds to $\mathcal{N}=4$ gauge theory living on the $d=4$ Minkowsky boundary

- Correspondence can be extended to more general five-geometries asymptotically matching AdS_5

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu - dr^2 , \qquad A(r) \underset{\substack{r\to\infty \\ r\to-\infty}}{\longrightarrow} \frac{r}{L}$$

- Evolution in r can be interpreted as RG flow on the boundary (relevant perturbation)

- Correspondence can be used to compute the $d=4$ trace anomaly at fixed points (Henningson, Skenderis, 98)

$$a = c = \frac{\text{const}}{A'(r)^3}$$

- Einstein's equations for supergravity + positive-energy condition $\rightarrow$ c-theorem

$$-3A''(r) = R^t_t - R^r_r \propto T^{(5)}_{\alpha\beta} \xi^\alpha \xi^\beta \geq 0 \qquad (\xi^2 = 0)$$

($\rightarrow$ holographic RG flow and other ideas)

"a" as a measure of degrees of freedom

(A.C., G. D'Appollonio, hep-th/0005115
Phys Lett B

- a-theorem is the most promising.

HOW MUCH IS a ?

$d=4$

spin	0	1/2	1
a	1	11	62

free fields, conformal (Weyl) invariant

- could it be a sensible measure of d.o.f. ?

i) compute it in any $d = 2k > 4$ for free fields: scalar (S), Dirac fermion (F) and antisymmetric tensor (AT) $\left(p = \frac{d-2}{2} \text{ form} \right)$

ii) compare it to the # of field components

iii) take limit $d \to \infty$ (semi-classical limit)

— o —

$$\int_M \sqrt{g} \langle T_\mu^{\ \mu} \rangle = a\,\chi + \int \sqrt{g} \sum_i I_i(\text{Weyl})$$

(Deser, Schwimmer) 93

- on the sphere S^{2k} only Euler term
  → use ζ-function to compute explicitly the scale variation of $\det(\Delta_\sigma)$ $\quad \sigma = S, F, AT$

$$\frac{d}{d\alpha} \log Z = \zeta_\Delta(s=0) = -\int \sqrt{g} \langle T_\mu^{\ \mu} \rangle \qquad \zeta_\Delta(s) = \sum_n \frac{1}{\lambda_n^s}$$

(Hawking; Copeland, Toms)

- natural normalization for a (call it $\hat{a}$)

$$\frac{d}{d\alpha} \log Z[m] = \hat{a} \frac{\chi}{2}, \qquad \hat{a} = \zeta_\Delta(0)_{\text{sphere}}$$

\# of "effective zero modes" topological \#

- in this normalization

$$\frac{\hat{a}(\sigma)}{n(\sigma)} \xrightarrow[d\to\infty]{} 0 \qquad \text{for } \sigma = S, F, AT$$

\# field components

$\longrightarrow$ OK with the semiclassical limit of $d\to\infty$

———— o ————

- another normalization for the a-theorem
  a = quantum measure of d.o.f.

$\rightarrow$ $a(B) \equiv 1$ any d

$\rightarrow$ $\frac{a(\sigma)}{n(\sigma)}$ grows with d ($\sigma = F, AT$)
$\longrightarrow$ table

- The weight per field component does not go to 1
- The AT field is overwelming: $O(d^3)$
- counter-intuitive but not manifestly
  inconsistent with $a_{UV} > a_{IR}$ on known examples
  (as far as I could see)
- Conclusion .... quantum $\neq$ classical

d	4	6	8	10	12	14		$2k$
$a(S)$	1	1	1	1	1	1		
$a(F)$	11	$\frac{191}{5}$	$\frac{2497}{23}$	$\frac{73985}{263}$	$\frac{92427157}{133787}$	$\frac{257184319}{157009}$	$\cdots$	
$a(AT)$	62	$\frac{3978}{5}$	$\frac{161020}{23}$	$\frac{13396610}{263}$	$\frac{44166621324}{133787}$	$\frac{310708060404}{157009}$	$\cdots$	
$r(S)$	1	1	1	1	1	1		1
$r(F)$	2.75	4.77	6.79	8.79	10.79	12.80	$\cdots$	$\simeq 2k$
$r(AT)$	31	132.6	350.0	727.7	1310.	2142.	$\cdots$	$\simeq (2k)^3$

able 1: Counting function a and the corresponding weight per field component r arious dimensions d, with asymptotic behaviours for $d \to \infty$.

$$r(\sigma) = \frac{a(\sigma)}{n(\sigma)} \qquad \sigma = S, F, AT$$

\# of polarizations of the field σ

d	4	6	8	10	12	14		$2k$
$c(S)$	1	1	1	1	1	1		
$c(F)$	6	20	56	144	352	832	$\cdots$	
$c(AT)$	12	90	560	3150	16632	84084	$\cdots$	
$r'(S)$	1	1	1	1	1	1		1
$r'(F)$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\cdots$	$\simeq k$
$r'(AT)$	6	15	28	45	66	91	$\cdots$	$\simeq 2k^2$

le 2: Counting function c and the corresponding weight per field component r' in ous dimensions d, with asymptotic behaviours for $d \to \infty$.

Field content of "c = a" theories

in $d = 4$ (AdS/CFT) and higher d,
as proposed by Anselmi et al. (hep-th/0002066)

$$
\begin{array}{llllll}
d=4: & 2\,N_S + & 7\,N_F - & 26\,N_{AT} & = & 0\,; \\
d=6: & 13\,N_S + & 169\,N_F - & 2358\,N_{AT} & = & 0\,; \\
d=8: & 67\,N_S + & 2543\,N_F - & 110620\,N_{AT} & = & 0\,; \\
d=10: & 817\,N_S + & 81535\,N_F - & 9994610\,N_{AT} & = & 0\,,
\end{array}
$$

Susy, $\mathcal{N}=4$ vector multiplet

$$
\begin{array}{lll}
d=4: & (6,2,1), & (13,0,1); \\
d=6: & (161,169,13), & (265,161,13); \\
d=8: & (835,65,2), & (2524,64,3); \\
d=10: & (11950,248,3), & (47095,141,5)\,.
\end{array}
$$

(N_S, N_F, N_{AT})

$\langle T\,T\,T \rangle$ in $d=4$

(A.C., R. Guida, N. Magnoli, hep-th/0103237

$$\langle T_\mu^\mu \rangle = a\, G \;-3c\, F \;+ a'\, \mathcal{D}^2 R$$

- physical properties of each anomaly term are worth studying
- a, c anomalies are finite, scheme-independent
  → relation to finite amplitudes of correlators
  → $d=4$ → three-point function

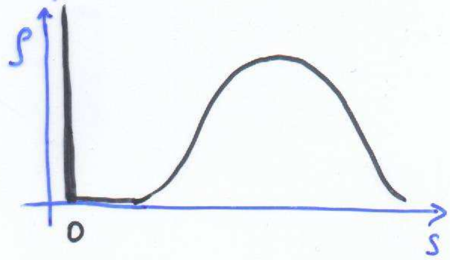
Recall $d=2$

$$\langle \underbrace{T_{\mu\nu}(p)}_{\dim=2}\, T_{\rho\sigma}(-p) \rangle = \underbrace{A(p^2)}_{\dim=-2} \, \underbrace{(p_\mu p_\nu - \delta_{\mu\nu} p^2)(p_\rho p_\sigma - \delta_{\rho\sigma} p^2)}_{\dim=4}$$

transverse tensor

- $A(p^2)$ is finite and scheme-independent
- $A(p2) \longrightarrow \dfrac{c}{p^2}$ at fixed point
- $A(p^2) = \displaystyle\int ds \, \frac{\rho(s)}{s+p^2}$ $\qquad \rho(s) \sim \mathrm{Im}\, A(p^2=-s)$ positive spectral density

$$\rho(s) \longrightarrow c\,\delta(s) \quad \text{at fixed point}$$



- Cardy's sum rule

$$c_{UV} - c_{IR} = \int_\varepsilon^\infty ds\, \rho(s) > 0$$

$$\sim \int_{|x|>\varepsilon} d^2x \, x^2 \langle T_\mu^\mu(x)\, T_\rho^\rho(0) \rangle$$

$d=4$ analogs

transverse, $s=0,2$

$$\langle T_{\mu\nu}(p)\, T_{\rho\sigma}(-p)\rangle = A_0(p^2)\, \mathcal{P}^{(0)}_{\mu\nu,\rho\sigma} + A_2(p^2)\, \mathcal{P}^{(2)}_{\mu\nu,\rho\sigma}$$

dim $=4$ dim $=0$ dim $=4$

$$A_0(p^2) \longrightarrow a'$$
$$A_2(p^2) \longrightarrow c \log\left(\frac{p^2}{\mu^2}\right)$$

at fixed point

- both scheme-dependent by $A_\alpha(p^2) \to A_\alpha(p^2) + \text{const.}$
  (but c is scheme-independent)

→ consider $\langle TTT\rangle$

$$\langle T_{\mu_3\nu_3}(-k_1-k_2)\, T_{\mu_1\nu_1}(k_1)\, T_{\mu_2\nu_2}(k_2)\rangle \sim \sum_i A_i \; T^{(i)}_{\mu_1\nu_1,\mu_2\nu_2,\mu_3\nu_3}$$

dim $=4$ dim $=-2$ dim $=6$

- straightforward but cumbersome ($i=1,\dots,137$)
- general expression off-criticality contains 17 independent amplitudes solution of the Ward identities of Diff invariance ($\partial^\mu T_{\mu\nu}=0$)
- two of them match the anomalies at criticality by solving the Ward id. for Weyl invariance

$$A_G(q^2,k^2) \longrightarrow \frac{a}{q^2}$$

$k_1^2 = k_2^2 = k^2$, $q^2 = (k_1+k_2)^2$

$$A_F(q^2,k^2) \longrightarrow \frac{c}{q^2}$$

- singled out by solving a linear system over the 15-dim. basis of scheme-indep. tensors $\mathcal{T}^{(i)}$

Diff & Weyl Ward identities for $\langle TTT \rangle$

- definition of $T_{\mu\nu}$ and of the two symmetries :

$$T_{\mu\nu}(x) = \frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}(x)} W[g] \, , \qquad W[g] = \log Z[g]$$

diff $\quad \delta_\epsilon W \equiv \int dx \, D^{(\mu}\epsilon^{\nu)}_{(x)} \frac{\delta}{\delta g^{\mu\nu}_{(x)}} W = 0$

Weyl $\quad \delta_\sigma W \equiv -2 \int \sigma(x) \, g^{\mu\nu}(x) \frac{\delta}{\delta g^{\mu\nu}(x)} W = -2 \int \sigma(x) \sqrt{g} \, \mathcal{A}(x;g)$

$$g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu} \qquad \langle T^\mu_\mu(x) \rangle \equiv \mathcal{A}(x;g)$$

- index-free notation

$h_1^{\mu\nu} T_{\mu\nu}(x_1) \to (h_1 \cdot T_1)$ $\qquad h^{\mu\nu}$ polarization

$p^\mu h_{\mu\nu} q^\nu \to (p.h.q) \; ; \qquad \eta^{\mu\nu} h_{\mu\nu} \to (h)$

- Diff and Weyl operations (conservation & trace)

$h_{\mu\nu} = \frac{1}{2} k_{(\mu} v_{\nu)} \qquad v^\nu k^\mu T_{\mu\nu}(k) \to (v \otimes k . T(k))$
$\qquad\qquad = (v . T(k) . k)$

$h_{\mu\nu} = \eta_{\mu\nu} \qquad (\eta . T(k)) \equiv \Theta(k)$

- Diff WI

$$\begin{aligned} 0 = \; & 2 \langle (k_3 . T(k_3) . v)(h_2 . T(k_2))(h_1 . T(k_1)) \rangle \\ & + 2 \langle (k_3 . h_2 . T(-k_1) . v)(h_1 . T(k_1)) \rangle - (v . k_2) \langle (h_2 . T(-k_1))(h_1 . T(k_1)) \rangle \\ & + (1 \leftrightarrow 2) \end{aligned}$$

d

- Weyl WI

$$\langle \Theta(k_3)\,(h_1\cdot T(k_1))(h_2\cdot T(k_2))\rangle$$
$$+\langle (h_1\cdot T(-k_2))(h_2\cdot T(k_2))\rangle + \langle (h_1\cdot T(k_1))(h_2\cdot T(-k_1))\rangle$$
$$= \left(h_1\cdot\frac{\delta}{\delta g(k_1)}\right)\left(h_2\cdot\frac{\delta}{\delta g(k_2)}\right)A[g]$$
$$-\frac{1}{2}\left[\left(h_1\cdot\frac{\delta}{\delta g(k_1)}\right)A[g] + \left(h_2\cdot\frac{\delta}{\delta g(k_2)}\right)A[g]\right]$$

- Scheme dependence

WI involve local (polynomial) terms, they make sense in a specific (consistent) renormalization scheme.
Some of the solutions are scheme-independent (a, c anomalies), because the corresponding $\tau^{(i)}_{\mu\rho\nu\sigma,\dots}$ have enough free-index momenta
→ "tensorial dimension" δ_T

Ex.

$$\mathcal{T}^{(1)} = (k_1.h_1.k_2)(k_2.h_2.k_1)(k_1.h_3.k_2)+\dots \qquad \delta_T = 6$$

$$\mathcal{T}^{(2)} = (k_1.h_1.h_2.k_2)(k_1.h_3.k_2)+\dots \qquad \delta_T = 4$$

$$\langle TTT\rangle = \underset{-2}{A_1}\ \underset{6}{\mathcal{T}^{(1)}} + \underset{0}{A_2}\ \underset{4}{\mathcal{T}^{(2)}} + \dots$$

dim = 4

polynomial scheme-dependent terms have $\delta_T \le 4$
→ $A_2 \to A_2 + \text{const}$, A_1 is unchanged

Basic polynomials of six indices

<table>
<tr><th>N</th><th>N+1</th><th>N+2</th><th>N+3</th><th>N+4</th></tr>
<tr><td>0</td><td>(1|1|1) (1|2|1) (1|3|1)</td><td>(1|3|1) (2|1|2) (2|2|2)</td><td>(1|1|1) (1|2|1) (1|3|2)</td><td>(1|1|1) (1|3|1) (2|2|2)</td></tr>
<tr><td>4</td><td>(1|1|1) (1|3|2) (2|2|2)</td><td>(1|1|1) (1|3|1) (1|2|2)</td><td>(1|3|1) (1|1|2) (2|2|2)</td><td>(1|1|1) (1|2|2) (1|3|2)</td></tr>
<tr><td>8</td><td>(1|2|1) (1|3|1) (2|1|2)</td><td>(1|2|1) (1|3|2) (2|1|2)</td><td>(1|3|1) (1|2|2) (2|1|2)</td><td>(1|2|1) (1|3|1) (1|1|2)</td></tr>
<tr><td>12</td><td>(1|2|1) (1|1|2) (1|3|2)</td><td>(1|3|1) (1|1|2) (1|2|2)</td><td>(1|1|2) (1|2|2) (1|3|2)</td><td>(1|1|1) (1|32|1)</td></tr>
<tr><td>16</td><td>(1|31|1) (2|2|2)</td><td>(1|1|1) (1|32|2)</td><td>(1|1|1) (2|32|1)</td><td>(1|32|1) (2|1|2)</td></tr>
<tr><td>20</td><td>(1|2|1) (1|31|1)</td><td>(1|2|1) (2|31|1)</td><td>(1|2|1) (1|31|2)</td><td>(1|32|1) (1|1|2)</td></tr>
<tr><td>24</td><td>(1|31|1) (1|2|2)</td><td>(1|1|2) (1|32|2)</td><td>(1|2|2) (1|31|2)</td><td>(1|3|1) (1|21|1)</td></tr>
<tr><td>28</td><td>(1|21|1) (2|3|2)</td><td>(1|3|1) (1|21|2)</td><td>(1|3|1) (2|21|1)</td><td>(1|21|1) (1|3|2)</td></tr>
<tr><td>32</td><td>(1|3|2) (1|21|2)</td><td>(1|3|2) (2|21|1)</td><td>(3) (1|1|1) (1|2|1)</td><td>(3) (1|1|1) (2|2|2)</td></tr>
<tr><td>36</td><td>(3) (1|1|1) (1|2|2)</td><td>(2) (1|1|1) (1|3|1)</td><td>(1) (1|3|1) (2|2|2)</td><td>(2) (1|1|1) (1|3|2)</td></tr>
<tr><td>40</td><td>(3) (1|2|1) (2|1|2)</td><td>(3) (1|2|1) (1|1|2)</td><td>(2) (1|3|1) (2|1|2)</td><td>(1) (1|2|1) (1|3|1)</td></tr>
<tr><td>44</td><td>(1) (1|2|1) (1|3|2)</td><td>(3) (1|1|2) (1|2|2)</td><td>(2) (1|3|1) (1|1|2)</td><td>(1) (1|3|1) (1|2|2)</td></tr>
<tr><td>48</td><td>(2) (1|1|2) (1|3|2)</td><td>(1|312|1)</td><td>(1|321|1)</td><td>(1|231|1)</td></tr>
<tr><td>52</td><td>(1|312|2)</td><td>(1|321|2)</td><td>(1|231|2)</td><td>(2|231|1)</td></tr>
<tr><td>56</td><td>(32) (1|1|1)</td><td>(31) (1|2|1)</td><td>(32) (1|1|2)</td><td>(1) (1|32|1)</td></tr>
<tr><td>60</td><td>(2) (1|31|1)</td><td>(1) (1|32|2)</td><td>(2) (1|31|2)</td><td>(21) (1|3|1)</td></tr>
<tr><td>64</td><td>(21) (1|3|2)</td><td>(3) (1|21|1)</td><td>(3) (1|21|2)</td><td>(3) (2|21|1)</td></tr>
<tr><td>68</td><td>(2) (3) (1|1|1)</td><td>(1) (3) (1|2|1)</td><td>(2) (3) (1|1|2)</td><td>(1) (2) (1|3|1)</td></tr>
<tr><td>72</td><td>(1) (2) (1|3|2)</td><td>(321)</td><td>(2) (31)</td><td>(3) (21)</td></tr>
<tr><td>76</td><td>(1) (2) (3)</td><td></td><td></td><td></td></tr>
</table>

Table 1: The basis of $(1 \leftrightarrow 2)$ symmetric six-index polynomials of k_1^μ, k_2^ν and $\eta_{\alpha\beta}$: $\mathcal{P}_i = \mathcal{P}_i(k_1, h_1, k_2, h_2, h_3)$, $i = 1, \ldots, 77$. We use the short-hand notations: $(i|abc|j) \equiv k_i.h_a.h_b.h_c.k_j, \ldots$, $(i|j) \equiv k_i.k_j$ $(abc) \equiv \mathrm{tr}(h_a.h_b.h_c), \ldots$, $(a) \equiv \mathrm{tr}(h_a)$ and we omit the $(1 \leftrightarrow 2)$ exchanged term that must be added to all non $(1 \leftrightarrow 2)$ symmetric polynomials of the table. The tensorial dimension of the polynomial $\mathcal{P}_i$ is: $\delta_T = 6$ for $i = 1, \ldots, 15$, $\delta_T = 4$ for $i = 16, \ldots, 49$; $\delta_T = 2$ for $i = 50, \ldots, 73$ and $\delta_T = 0$ for $i = 74, \ldots, 77$.

$$h_1^{\mu\nu}\, T_{\mu\nu}^{(k_1)} \sim h_1^{\mu\nu}(k_{1\mu} k_{1\nu} + \eta_{\mu\nu}) = k_1.h_1.k_1 + \mathrm{tr}(h_1)$$
$$= (1|1|1) + (1)$$

Tensors solving the Diffeomorphism Ward id.

		$\perp k_{1,2}$	$\perp k_3$	$\mathrm{tr}_{1,2}=0$	$\mathrm{tr}_3=0$	δ_T	Δ
7 traceless & transverse	$\mathcal{T}_1$	y	y	y	y	6	>6
	$\mathcal{T}_2$	y	y	y	y	6	>6
	$\mathcal{T}_3$	y	y	y	y	6	>6
	$\mathcal{T}_4$	y	y	y	y	6	>6
	$\mathcal{T}_5$	y	y	y	y	6	>6
	$\mathcal{T}_6$	y	y	y	y	6	>6
	$\mathcal{T}_7$	y	y	y	y	6	>6
null →	$\mathcal{T}_8$	y	y	y	y	4	4
anomalous trace & transverse	$\mathcal{T}_E$	y	y	n	n	6	6
	$\mathcal{T}_W$	y	y	n	n	6	8
7 classical trace & transverse → 0	$\mathcal{T}_{11}$	y	y	n	n	4	6
	$\mathcal{T}_{12}$	y	y	n	n	4	6
	$\mathcal{T}_{13}$	y	y	n	n	6	6
	$\mathcal{T}_{14}$	y	y	n	n	6	6
	$\mathcal{T}_{15}$	y	y	n	n	6	6
	$\mathcal{T}_{16}$	y	y	n	n	6	6
	$\mathcal{T}_{17}$	y	y	n	n	6	>6
4 2-point function	$\mathcal{T}_{0+}$	n	n	n	n	4	4
	$\mathcal{T}_{0-}$	n	n	n	n	6	>6
	$\mathcal{T}_{2+}$	n	n	n	n	4	4
	$\mathcal{T}_{2-}$	n	n	n	n	6	6
1-p. function →	$\mathcal{T}_\Lambda$	n	n	n	n	0	4

Table 2: Properties of the tensors solving the Diff Ward Identity; ($\perp$):transversality w.r.t. $k_1(k_2)$ and k_3 (yes/no); (tr $=0$): tracelessness w.r.t. $T_1(T_2)$ and T_3; δ_T: tensorial dimension; Δ: scale dimension of the tensor in polynomial form.

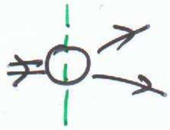
	transverse					non-transverse					total	
$k^2 \neq 0$	$\delta_T = 6$			$\delta_T = 4$		$\delta_T = 6$		$\delta_T \leq 4$			$\delta_T = 6$	$\delta_T \leq 4$
Diff	14			2 (3)		2		3			16	5
Weyl	7	2	5	(1)	2	1	1	1	1	1	10	2
value	$(tr = 0)$	a, c	0	null	0	f_{2-}	0	f_{2+}	a'	0		
label i	1,..,7	9,10	13,..,17	8	11,12	2−	0−	2+	0+	Λ		
$k^2 = 0$	$\delta_T = 6$			$\delta_T = 4$		$\delta_T = 6$		$\delta_T \leq 4$			$\delta_T = 6$	$\delta_T \leq 4$
Diff	12			4 (5)		1		3			13	7
Weyl	7	2	3	(1)	4	1		1	1	1	10	1
value	$(tr = 0)$	a, c	0	null	0	$f_2(q^2)$		0	a'	0		
label i	1,..,7	9,10	15,..,17	8	11,..,14	2		0−	0+	Λ		

Table 3: Summary of the properties of the solutions to the Ward identities for $k^2 \neq 0$ and $k^2 = 0$. Using the notations of Table 2, we report the number of solutions, split by the δ_T value, both off criticality (*Diff* line) and at criticality (*Weyl*); we also indicate the critical limit of the amplitudes (*value*) and their labels.

Results

- trace anomaly from dispersive renormalization as well-Known analysis of $\langle AVV \rangle$ (Frishman et al.) '81

- anomaly related to finite amplitudes of physical processes



decay $\propto a, c$

(practically useful for anomaly calculations)

- disentangle Kinematics (Poincaré, Weyl symmetries) from dynamics

- sum rules for the RG flow of a, c

$$A_G(q^2, k^2) = \int ds \, \frac{\rho_G(s, k^2)}{s + q^2} \quad \longrightarrow \quad \frac{a}{q^2} \quad \text{fixed point}$$

$$\rho_G \longrightarrow a\,\delta(s)$$

$$a_{UV} - a_{IR} = \int_{\varepsilon}^{\infty} ds \, \rho(s, k^2 = 0) \sim \int_{\varepsilon}^{\infty} ds \, \mathrm{Im} \langle TTT \rangle \Big|_{\rho_G}$$

- the same for $c_{UV} - c_{IR}$

- N. B. ρ_G, ρ_F are not (obviously) positive

otherwise I would have the theorem $a_{UV} - a_{IR} > 0$ (and $c_{UV} - c_{IR} > 0$!?!)

Conclusions

- lots of activity (but no solid proof yet)
- lots of evidences for the a-theorem (Euler anomaly)
- a counts field components in a "non-classical" manner
- $\langle TTT \rangle$ structure : mandatory analysis for further investigations, e.g. :

  i) conditions from $\mathcal{N}=1$ Susy

  ii) positivity conditions from positive-energy condition $T_{\alpha\beta}\, \xi^\alpha \xi^\beta > 0$ (Osborn, Latorre, 98)

  $\langle TTT \rangle \sim \langle \psi | T | \psi \rangle \geq 0$