Quarks, Strings, Anyons

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sub-title:

Two-dimensional Conformal Field Theorie[s]
& Applications

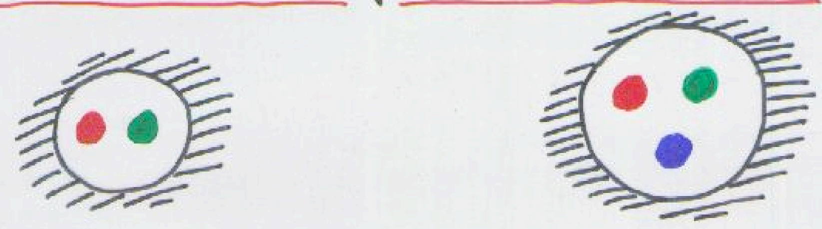
sub-sub-titles:

- A quick tour of the most theoretical theory (with happy end)
- Aesthetics-driven research with serendipitous discoveries......

Plan

- quarks are pulled by strings
- strings produced CFTs and many other things
- CFTs explain anyons, with fractional charge and statistics, in the quantum Hall effe[ct] and other low-dim cond-mat systems

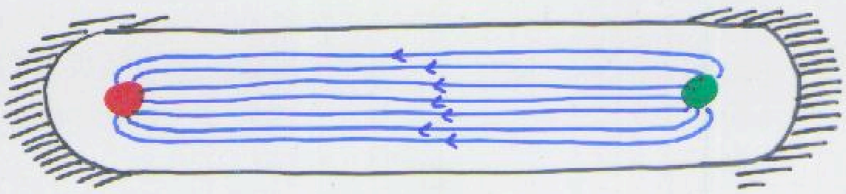
Quarks & QCD String



- quarks do exist : $S = \frac{1}{2}$ fermions

  $Q = \pm\frac{1}{3}, \pm\frac{2}{3}$ fractional

- but are confined inside hadrons



- flux lines are trapped : "dual superconductor" ('t Hooft Mandelstam)

  $\mathcal{E}(R) = \text{const}$, $V(R) = \sigma R$

- effective description by relativistic, open string (Polyakov)

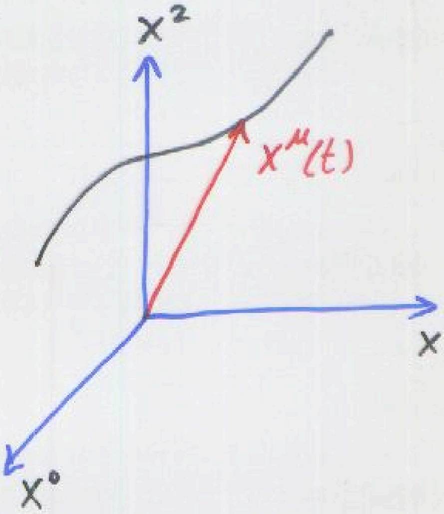
  $m_J^2 \sim \sigma J$ spin-J resonances

  $\sigma \sim 1\ \text{GeV}^2$

The String Action

- relativistic particle

$$X^\mu(t) : \mathbb{R} \to \mathcal{M} = \mathbb{R}^D$$



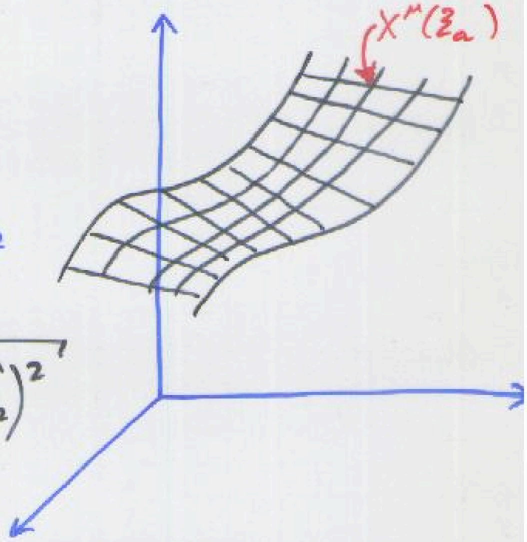
$$S = -m \int ds \quad = -m \int \sqrt{(\dot{X}^\mu)^2}\, dt$$

$$ds^2 = \frac{dX^\mu}{dt}\frac{dX^\mu}{dt}\, dt^2$$

$S \propto$ length

- relativistic string

$$X^\mu(\xi_1, \xi_2) : \mathbb{R} \times [0,1] \to \mathcal{M}$$



$S \propto$ area

$$ds^2 = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\mu}{\partial \xi^b}\, d\xi^a d\xi^b, \quad a,b \in 1,2$$

$$S = -\sigma \int d^2\xi \sqrt{\left(\frac{\partial X^\mu}{\partial \xi^1}\right)^2 \left(\frac{\partial X^\mu}{\partial \xi^2}\right)^2 - \left(\frac{\partial X^\mu}{\partial \xi^1}\frac{\partial X^\mu}{\partial \xi^2}\right)^2}$$

$$= -\sigma \int d^2\xi \sqrt{g_{IND.}}$$

- equivalent action of two-dimensional fields $X^\mu(\xi_1, \xi_2)$

$$S = -\frac{\sigma}{2} \int d^2\xi \sqrt{g}\, g^{ab} \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\mu}{\partial \xi^b}$$

conformal gauge choice $\quad g_{ab} = e^\phi \eta_{ab}, \ \sqrt{g} = e^\phi$

$$S = -\frac{\sigma}{2} \int d^2\xi\, \eta^{ab} \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\mu}{\partial \xi^b}$$

D massless scalars living on (ξ_1, ξ_2) plane

Conclusion:

string evolving in target space-time $\mathcal{M} = \mathbb{R}^D$

$$=$$

D-component scalar field theory on the world-sheet plane $(\xi^1, \xi^2) \in \mathbb{R} \times [0,1]$

- massless
- scale invariant $\xi^a \to \lambda \xi^a$ no scales
- conformal invariant $\begin{cases} \xi^a \to \eta^a(\xi) \\ ds^2 = d\xi^a d\xi^a = e^{\varphi} d\eta^a d\eta^a \end{cases}$

  $\phi \to \phi + \varphi$

$\Rightarrow$ $\boxed{\text{Two-dimensional CFT}}$

Remarks

- if space is compact, e.g. $\mathcal{M} = S^3 \times \mathbb{R}$ the CFT is interacting

  $$S = \frac{\sigma}{2} \int d^2\xi \, (\partial_a X^\mu)^2 \quad , \quad \sum_{\mu=1}^{3} (X^\mu)^2 = r^2 \quad \sigma\text{-model}$$

- Belavin, Polyakov, Zamolodchikov (1983):
  - study CFTs for their own sake;
  - exploit conformal symmetry (Virasoro algebra)
  - use math results (V. Kac) of affine Lie algebras to solve exactly large classes of CFTs – strongly-interacting massless field theories

Some String "revolutions"

- closed string excitations comprise the graviton

⟹ Fundamental String $\quad \sigma \sim M_{Planck}^2$

⟹ unification of gauge & gravity forces

⟹ "Theory of Everything" $\quad$ (Gross et al.)

- supersymmetry & superstring $\quad$ (Ademollo et al.) (Casalbuoni et al.

- string needs extra dimensions : $D=10$

$\mathcal{M} = \mathbb{R}^4 \times \Sigma \qquad$ 6 compact → interacting CF

- open strings end on hypersurfaces $\subset \mathcal{M}$ that can be dynamical :
  D-branes $\qquad$ → bounded world-sheet plane e.g. strip
  → boundary CFT

⟹ plenty of : • model building

• new math techiques (Witten)

• exactly solved CFTs

• spin-offs in other domains, math-phys, cond-mat phys, ...

About 1000 physicists are devoted to string theory ; like building of a gothic cathedral

2D Conformal field theories: Intro

Consistent formalism developed for String Theory (Belavin, Polyakov, Zamolodchikov; Cardy; ....) based on the math of ∞-dim Lie algebras

$$z = x + iv\tau \qquad \bar{z} = x - iv\tau$$

$z = f(w)$ analytic

$$ds^2 = dx^2 + d\tau^2 = dz\,d\bar{z} = \left|\frac{df}{dz}\right|^2 dw\,d\bar{w}$$ conformal

implied by scale and translation invariances

$z = \lambda w$ $\qquad z = w + c$

massless correlators are covariant

$$\langle \phi_\Delta(z,\bar{z})\,\phi_\Delta(0,0)\rangle = \frac{1}{|z|^{2\Delta}} = \frac{\lambda^{-2\Delta}}{|w|^{2\Delta}} \qquad z = \lambda w$$

infinitesimal transformations $\quad z = w + \epsilon(w)$

$$\phi(w + \epsilon(w)) = \left(1 + \epsilon(w)\frac{\partial}{\partial w}\right)\phi(w)$$

$$\epsilon(w) = \sum_n \epsilon_n w^{n+1} \longrightarrow L_n = -z^{n+1}\frac{\partial}{\partial z}$$

Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\,\delta_{n+m,0}$$

c = central charge, characteristic of each conformal theory

Hamiltonian density is part of stress tensor

$$\mathcal{H} = T_{00}(z,\bar z)$$

analytic, i.e. chiral, splitting

$$\begin{cases} \partial_\mu T_{\mu\nu} = 0 \\ T_\mu{}^\mu = 0 \end{cases} \longrightarrow \begin{cases} \frac{\partial}{\partial \bar z} T_{zz} = 0 \\ T_{z\bar z} = 0 \end{cases} \quad T_{zz} \text{ analytic}$$

$$T_{zz}(z) \quad , \quad T_{\bar z \bar z}(\bar z)$$

the same for fields, that carry a representation of the Virasoro algebra of dim h (and $\bar h$)

$$\phi_\Delta(z,\bar z) = \phi_h(z)\, \bar\phi_{\bar h}(\bar z) \qquad \Delta = h + \bar h$$

correlators factorize into chiral x antichiral blocks

CFT main data: $\begin{cases} c & \text{central charge} \\ \{\Delta_n\} & \text{set of scale dimensions} \end{cases}$

spectra determined by representation theory and some physical conditions

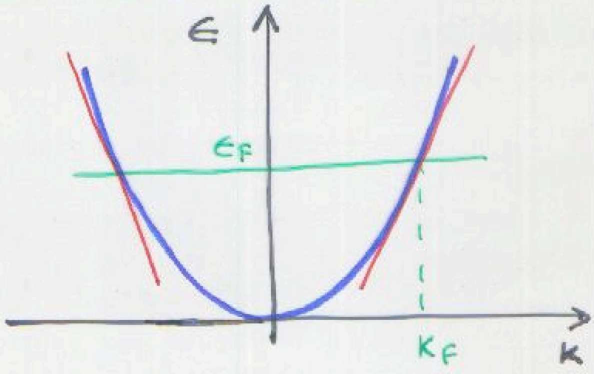
BIG ZOO of massless interacting theories

+ massive integrable "deformations"

- exact S-matrix bootstrap
- form factors expansion
- thermodynamic Bethe ansatz

Luttinger theory ≈ c=1 scalar field CFT

Interacting (spinless) fermion in one dim:



- expand around $k \sim \pm k_F$
  → massless "relativistic"
- bosonize the current

$$J_\mu = (J_z, J_{\bar z}) \quad , \quad \frac{\partial}{\partial \bar z} J_z = 0 \quad , \quad J_z = \psi_R^+ \psi_R$$

$$J_z = i \frac{\partial}{\partial z} \varphi \quad , \quad J_{\bar z} = -i \frac{\partial}{\partial \bar z} \varphi$$

φ free scalar field, compactified $\varphi \approx \varphi + 2\pi r$

stress tensor $T_{zz} = \frac{1}{2} : J_z J_z :$ (Sugawara)

$$H = \int dx\, \mathcal{H} = \int dx \left(T_{zz} + T_{\bar z \bar z} \right) = \frac{1}{2} \int dx : (\partial_\tau \varphi)^2 + (\partial_x \varphi)^2 :$$

(Tomonaga - Luttinger)

correlators $\langle : e^{i\beta\varphi(z)} : : e^{-i\beta\varphi(w)} : \rangle = |z-w|^{-2\beta^2}$ $\Delta = \beta^2$

radius $\sqrt{2} \leq r < \infty$ related to marginal coupling g
$c=1$ all along the line

$$\left\{ \Delta_{m,n} = h_{m,n} + \bar h_{m,n} \;\middle|\; h = \frac{1}{2}\left(\frac{m}{2r} + nr\right)^2, \bar h = \frac{1}{2}\left(\frac{m}{2r} - nr\right)^2, m,n \in \mathbb{Z}\right.$$

$\widehat{U(1)}$ current algebra: $J_z = \sum_n \rho_n z^{-n-1}$

$$[\rho_n, \rho_m] = n\, \delta_{n+m,0} \qquad \rho_0 = \beta \text{ charge}$$

elementary excitations have fractional charge

Applications of Luttinger theory & other CFTs

Main Features:

- anomalous scaling laws $\langle \phi(x)\, \phi(0) \rangle = |x|^{-2\Delta}$
- excitations with fractional charge
- spin-charge separation

Success stories:

- quantum spin chains (Haldane, ...)
- Kondo effect (Affleck, Ludwig, ...)
- quantum Hall effect (Read, M. Fisher, Saleur, ...)

Italian & European research groups

- INFN FI11 : FI (A.C., F. Colomo, ...), TS (Mussardo, ...
  GE (Magnoli, ...)
  other groups in TO (Caselle), BO (Ravanini), PG (Sodano),
  NA (Cristofano, ...)
- EC Network "EUCLID: Integrable models and applications"
  (2002 - 200
- ESF Programme "INSTANS" expected 2005 - 2010 (?)

Quantum Hall Effect

- Laughlin's phenomenological theory:
  - incompressible fluid of electrons
  - excitations (vortices) with fractional charge & fractional statistics

anyons $\quad Q = 1/3 \;, \; \frac{\theta}{\pi} = 1/3$

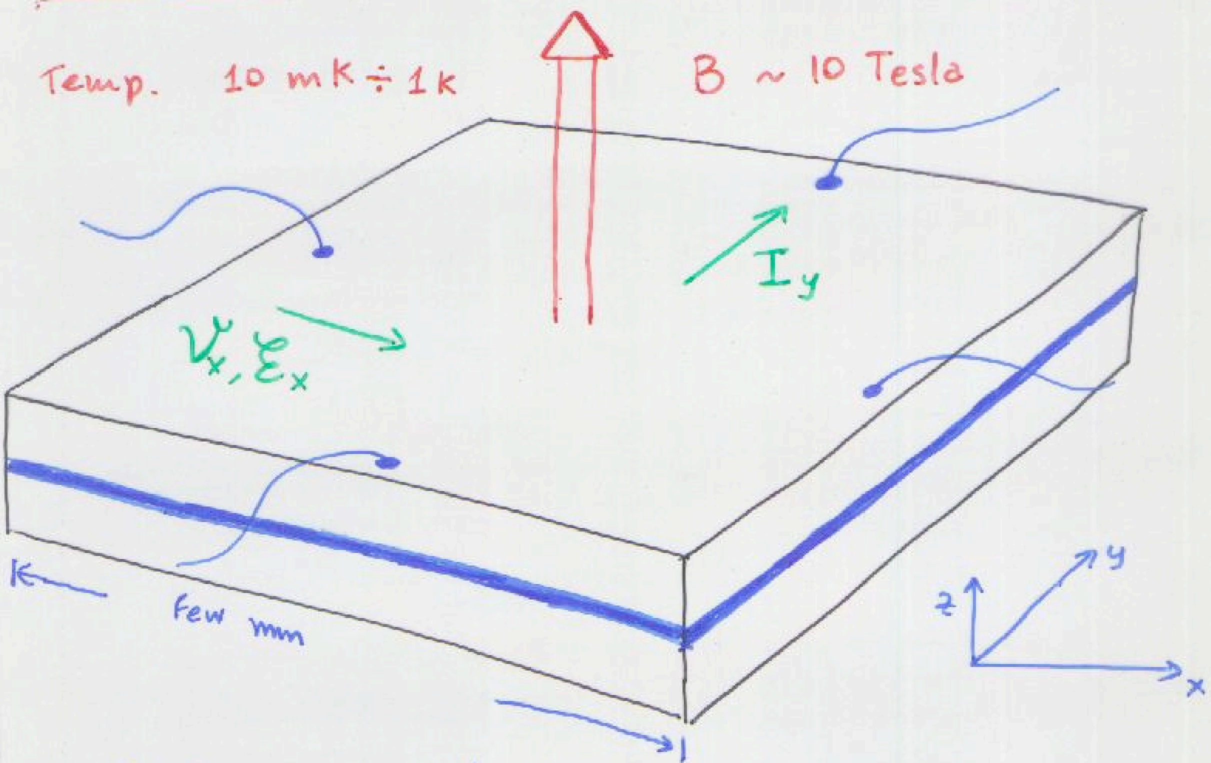
- are described as edge excitations by CFT of chiral Luttinger theory

- the experiment of resonant tunnelling: measure of $Q = 1/3$

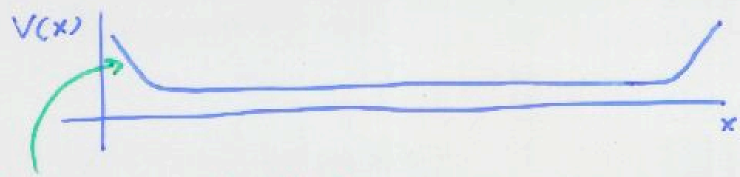
(• my work : CFT model building for general incompressible fluids; W_∞ symmetry)

TYPICAL EXPERIMENTAL SET UP

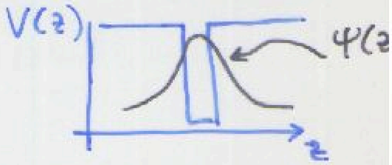
Temp. $10\ mK \div 1K$ $\qquad$ $B \sim 10$ Tesla



- GaAs compound
- $N \lesssim 10^{11}$ electrons $\qquad$ many-body problem
- two-dimensional problem

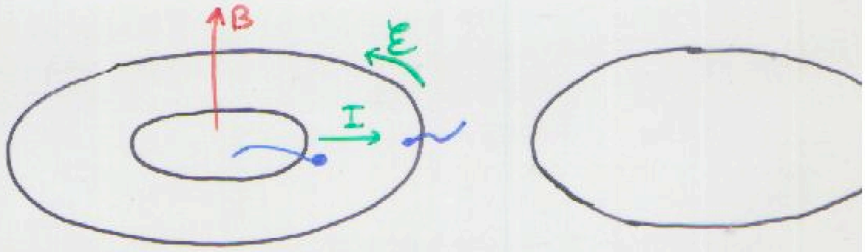


Confining potential



- Varying B , measuring $\mathcal{V}_i = R_{ij} I_j \quad , i,j = x,$

Equivalent simpler geometries

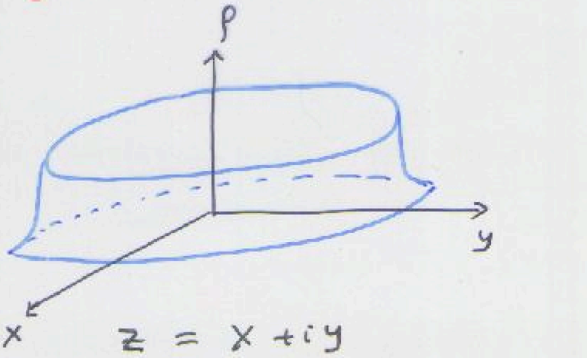


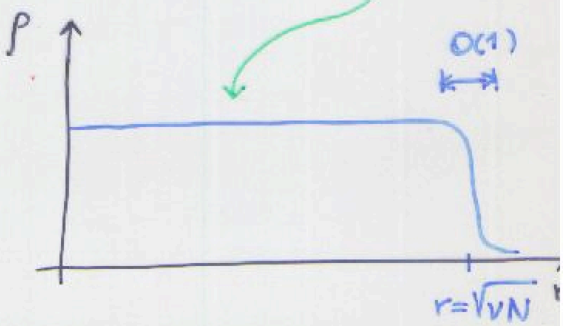
annulus $\qquad$ disk

The Laughlin incompressible fluid

Electrons behave as a droplet of liquid without sound waves

$$\begin{cases} \text{Incompressible} \equiv \text{density waves have a gap} \\ \text{Fluid} \equiv \rho(\vec{x}) = \text{const.} \end{cases}$$





$z = x + iy$

A = area of the droplet

N = # of electrons

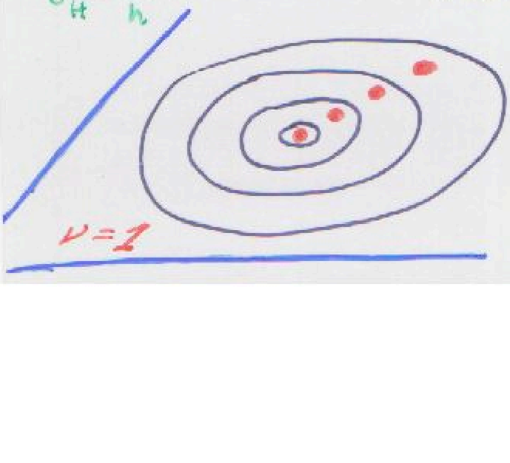
$\mathcal{D}_A = \dfrac{BA}{2\pi \frac{\hbar c}{e}}$ = # of degenerate Landau orbitals

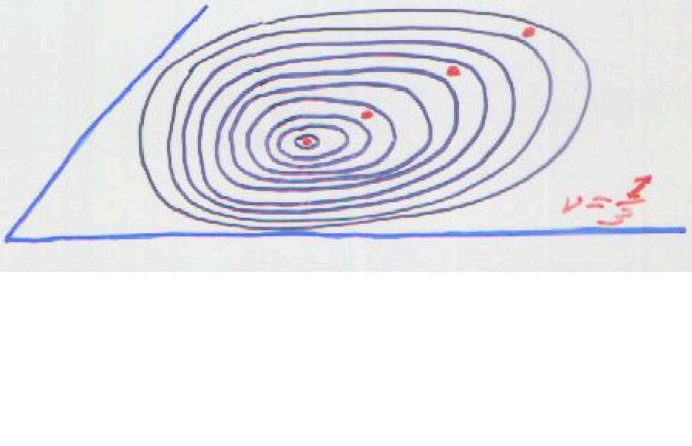
→ $\rho = \dfrac{N}{A}$ = electron density

→ $\nu = \dfrac{N}{\mathcal{D}_A} \propto \dfrac{N}{BA}$ = filling fraction = $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$

= density for quantum-mechanical proble

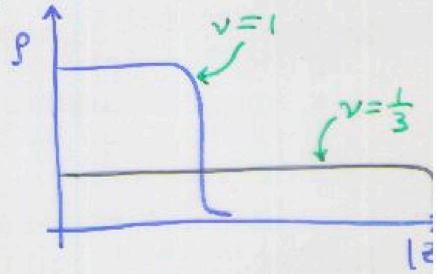
$\sigma_H = \frac{e^2}{h}\nu$





• Laughlin's wave function $\nu = \frac{1}{m} = 1, \frac{1}{3}, \frac{1}{5}, \ldots$

$$\Psi_{g.s.}(z_1, \ldots, z_N) = \prod_{1=i<j}^{N} (z_i - z_j)^m \, e^{-\frac{1}{2}\sum \frac{|z_i|^2}{\ell^2}}$$

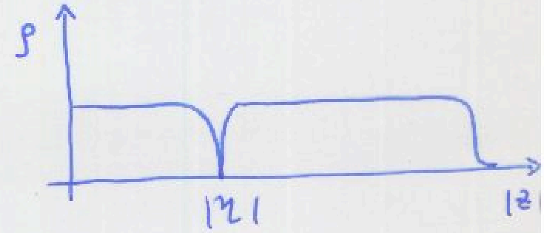


• $\nu = 1$ obvious
filled Landau level
$\text{gap} = \omega_c = \frac{eB}{mc}$

• $\nu = 1/3, 1/5, \ldots$ highly non-trivial
due to repulsive electron-electron interaction
$\text{gap} \sim O(e^2/\ell)$ $\quad \ell =$ magnetic length $\propto 1/\sqrt{B}$

• quasi-hole excitation $\simeq$ vortex

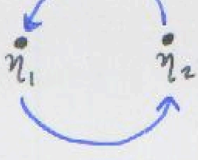
$$\Psi(\eta; z_1 \ldots z_N) = \prod_{i=1}^{N} (\eta - z_i) \prod_{i<j} (z_i - z_j)^m \, e^{-\frac{1}{2}\sum |z_i|^2/\ell^2}$$



– For $m > 1$, it has fractional charge $Q = \frac{e}{m}$

– For $m > 1$, it has fractional statistics $\frac{\theta}{\pi} = \frac{1}{m}$

$$\Psi(\eta_1, \eta_2; z_1 \ldots z_N) \sim (\eta_1 - \eta_2)^{\frac{1}{m}}$$



$$\Psi(\eta_1 - \eta_2 \to e^{i\pi}(\eta_1 - \eta_2)) = e^{\frac{i\pi}{m}} \Psi(\eta_1,$$

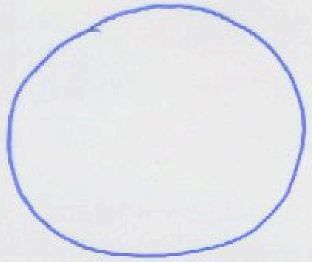
Fractiona statistics $\frac{\theta}{\pi} = \frac{1}{m} = \frac{1}{3}, \frac{1}{5}, \ldots$

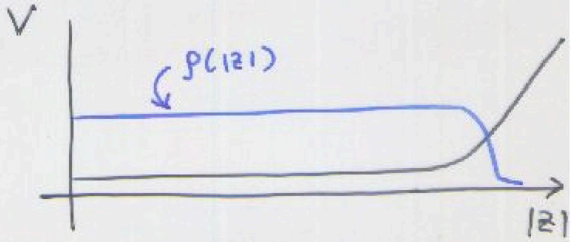
quasi-hole is an anyon (Wilczek et al

- fractional statistics is a long-range correlation of vortices, independent of distance, i.e. "topological"
  (nicely modelled by conformal field theory or Chern-Simons gauge field)

Edge excitations of the incompressible fluid

The sample has a boundary, with a confining potential; take e.g. a disk:



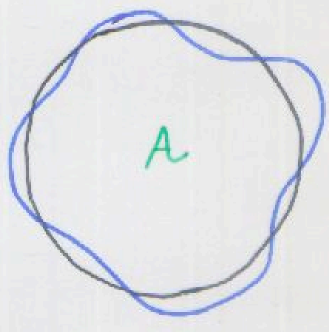


The incompressible fluid satisfies $\rho(|z|) = \rho_0$,
i.e. (2+1)-dimensional waves have a high gap and can be neglected.

But:

- the boundary shape can fluctuate:

→ "neutral" edge excitations



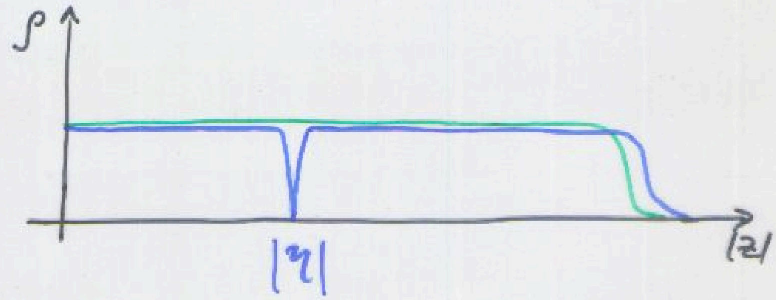
- almost gapless
- excitations satisfy $A = \text{cost}$

$$N = \int d^2x\, \rho(x) = \rho_0 \cdot A \quad , \quad N, \rho_0 = \text{cost}$$

→ area-preserving diffeomorphisms of the plane
$= W_\infty$ symmetry (my work with C. Trugenberger & G. Zemba

- $\nu = 1$ obvious:
the filled Landau level is like a Fermi sea
edge excitations = particle-hole excitations at the Fermi surface

• charged quasi-hole excitation also observed at the edge



depleted density is spilled at the edge
→ charged excitation at the edge

Conclusion: both excitations of the incompressible fluid can be detected at the edge

Idea: conformal field theory description of edge excitations $(\xi^1, \xi^2) = (t, R\theta)$

chiral Luttinger theory $(c=1, \bar{c}=0)$

Main features

• effective description valid at low energy, it can explain the conduction experiments

• spectrum at $\nu = 1/3$:

$Q = \frac{n}{3}$ $n = \pm 1, \pm 2, \ldots$ fractional charge

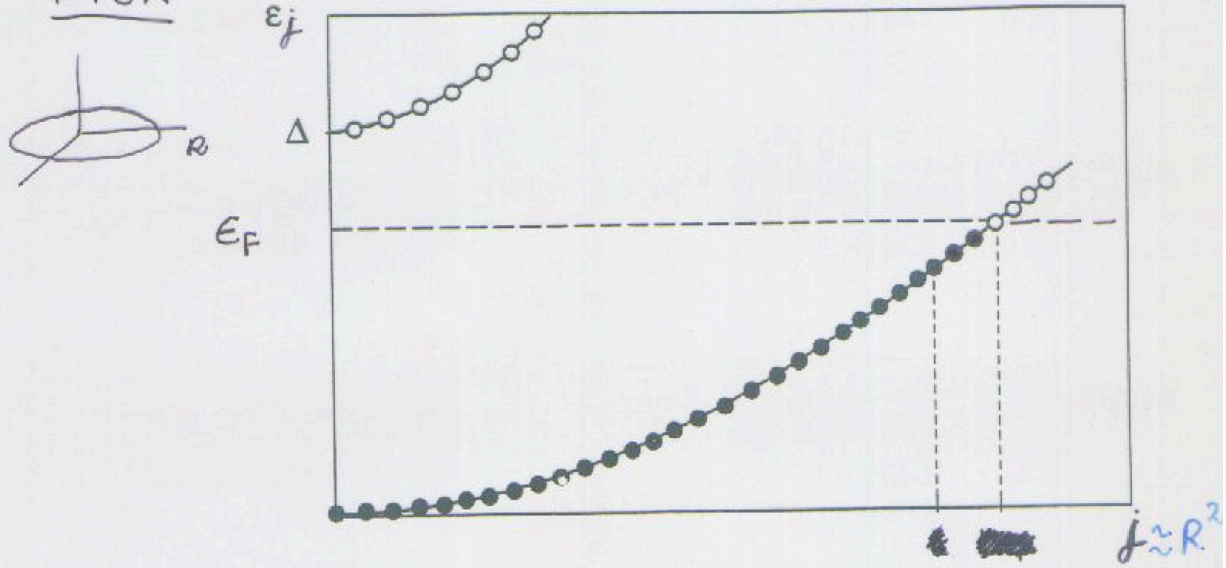
• vertex operators $:e^{in\varphi}:$ anyons at the ed

$\langle :e^{i\varphi(z_1)}::e^{i\varphi(z_2)}: \rangle = (z_1 - z_2)^{1/3}$, $|z_i| = 1$

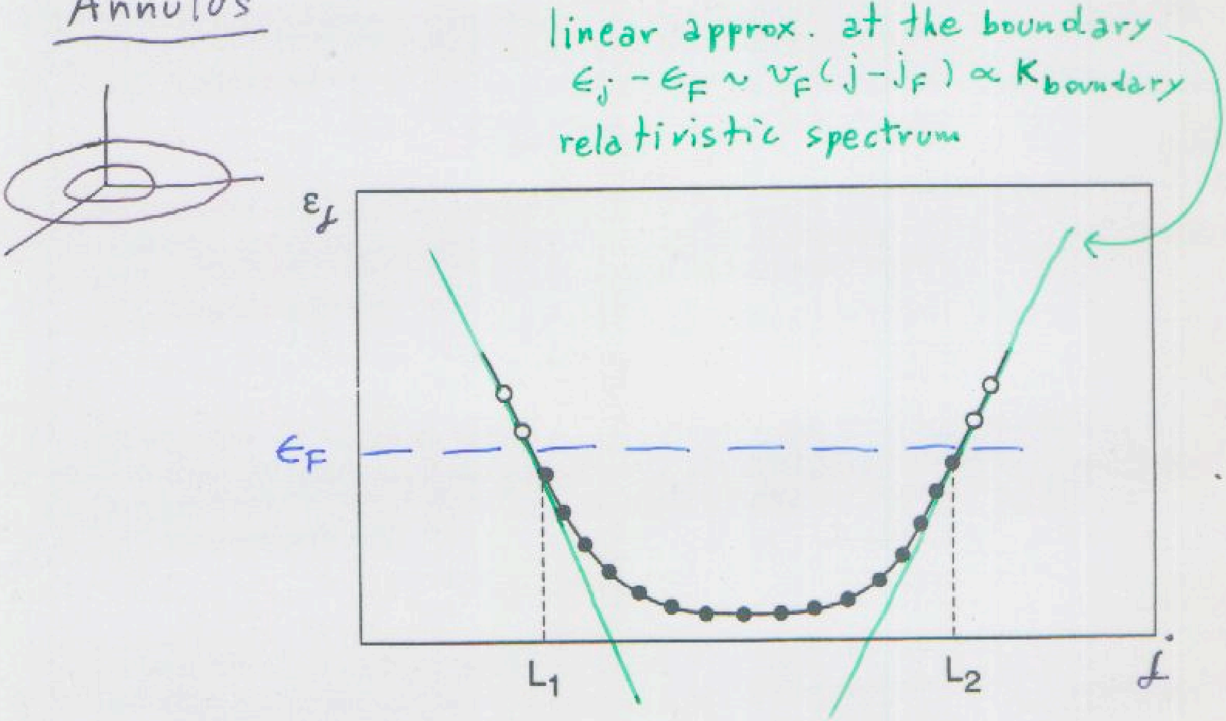
fractional statistics

The $\nu=1$ quantum incompressible fluid is like a Fermi sea in coordinate space

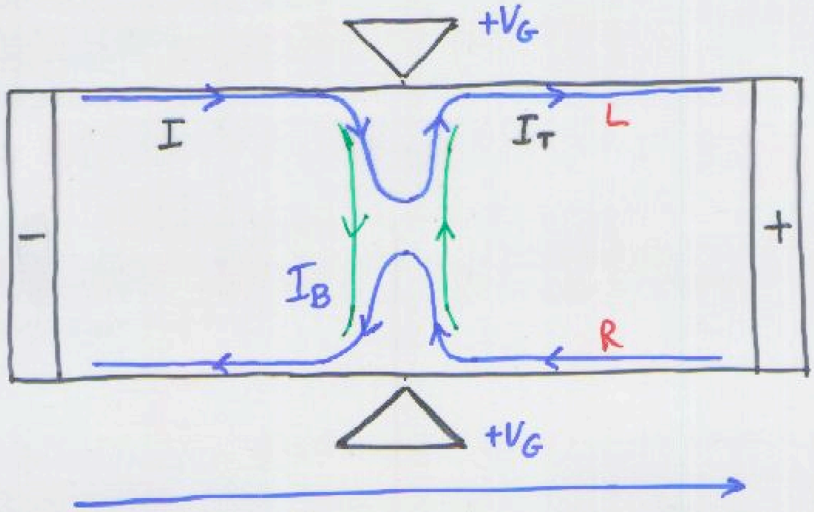
Disk



Annulus



Resonant tunnelling experiments



$$I = G\Delta V$$

- the electron fluid is squeezed at one point
- L & R chiral excitations interact
- CFT description + integrable massive theory (Fendley, Ludwig, Saleur)

$$G = \frac{e^2}{h}\,\frac{1}{3}\,F\left(\frac{V_G}{T^{2/3}}\right)$$ "anomalous" scaling

- Fluctuations of the current : Shot Noise (T=0
  quantum noise due to discrete nature of carriers

low current → uncorrelated tunnelling events
→ Poisson statistics

$$S_I = \langle |\delta I(\omega)|^2 \rangle_{\omega \to 0} = \frac{e}{3}\, I_B$$ weak constrict $I_B \ll I$

direct measure of $Q = \frac{1}{3}$ (Glattli et al., 1997 (De Picciotto et al., 1997

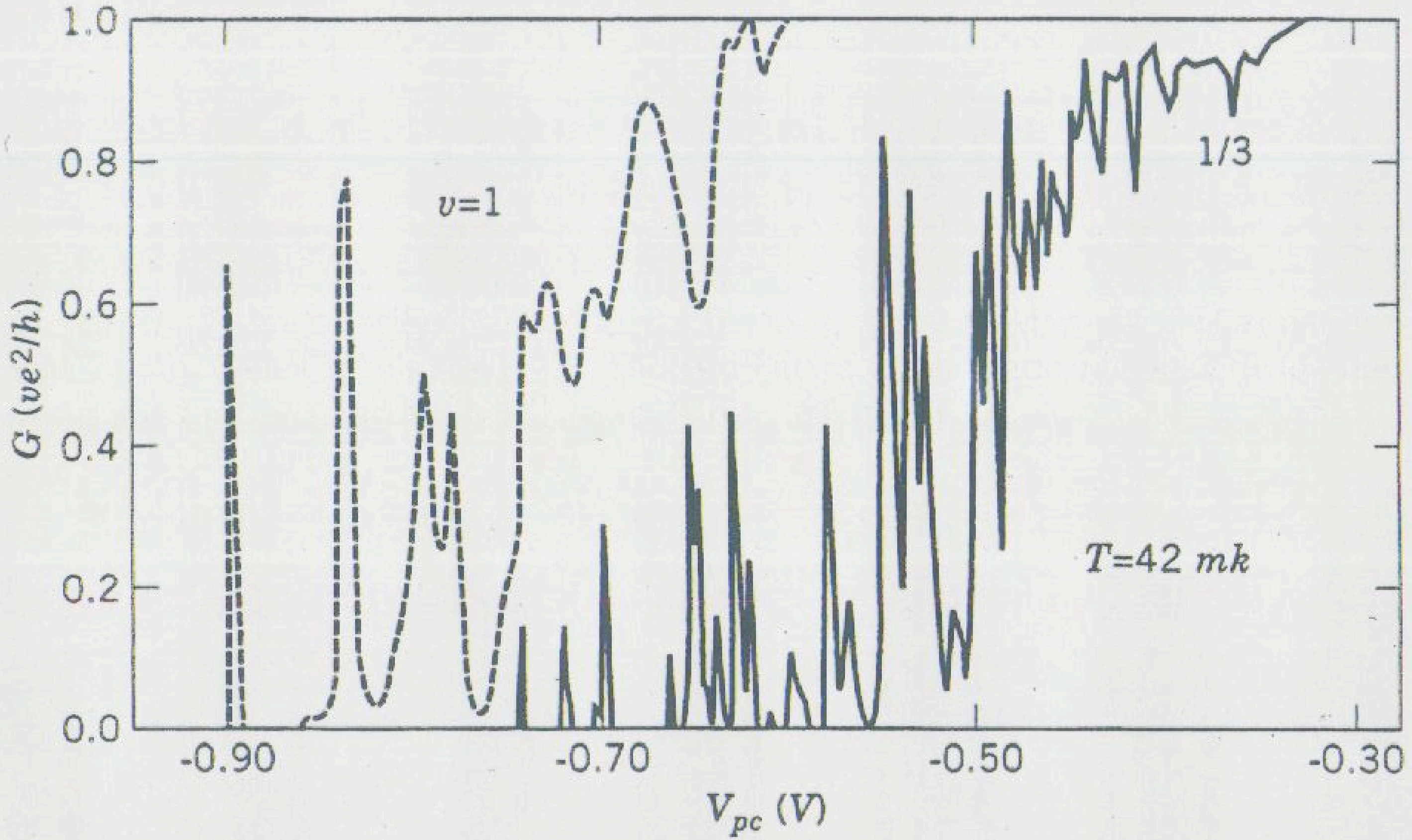


Figure 4.7. Two-terminal conductance as a function of gate voltage of a GaAs quantum Hall point contact taken at 42 mK. The two curves are taken at magnetic fields that correspond to $\nu = 1$ and $\nu = 1/3$ plateaus. (From Ref. [29].) (Milliken et al, S.S.C. 1996)

$\nu = \frac{1}{3}$

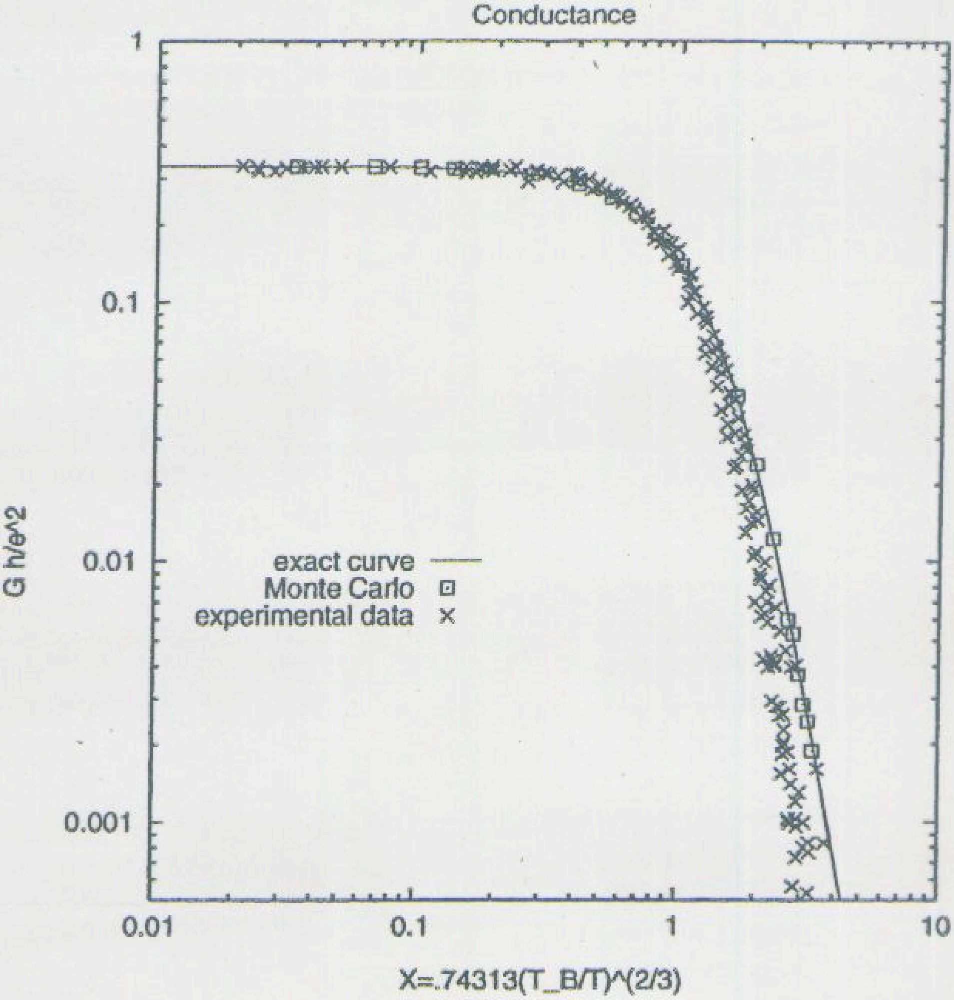


Figure 4.14. Log-log scaling plot of the lineshape of resonances at different temperatures from Ref. [29]. The x axis is rescaled by $T^{2/3}$. The crosses represent experimental data at temperatures between 40 and 140 mK. The squares are the results of the Monte Carlo simulation, and the solid line is the exact solution from Ref. [31].

[29] Milliken et al

[31] Fendley et al

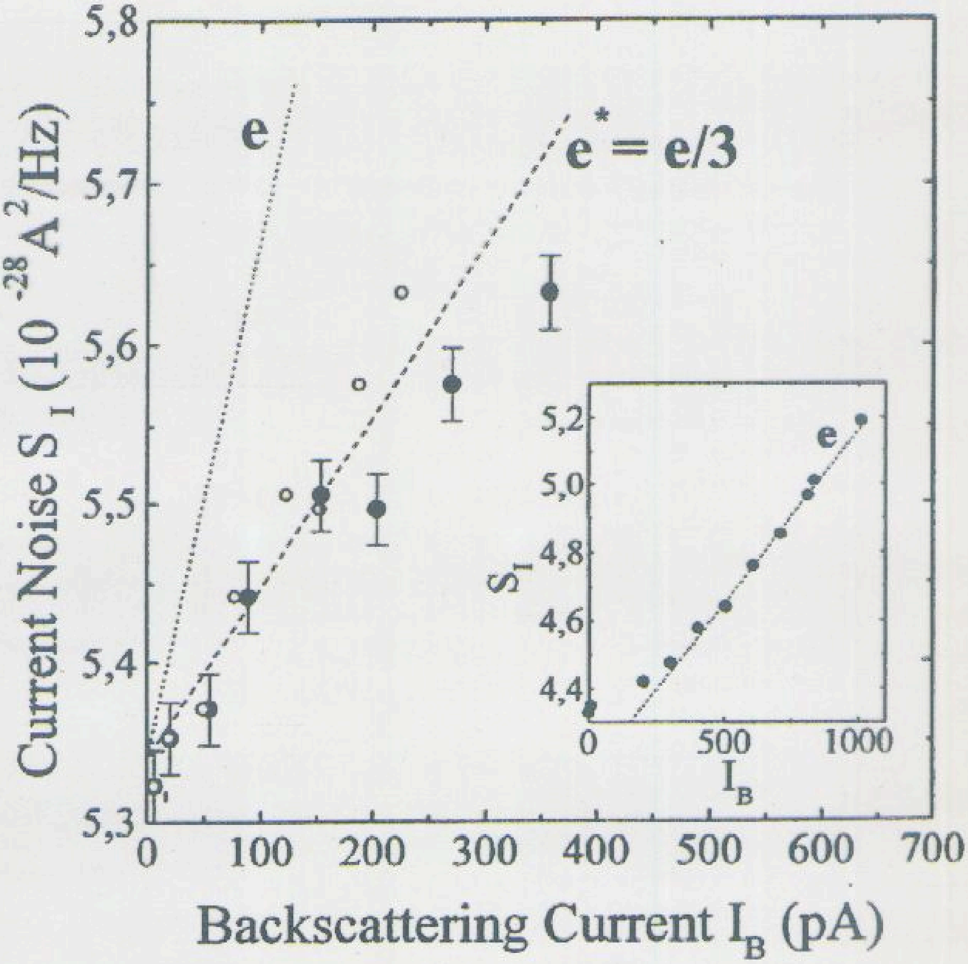


FIG. 2. Tunneling noise at $\nu = 1/3$ ($\nu_L = 2/3$) when following path A and plotted versus $I_B = (e^2/3h)V_{ds} - I$ (filled circles) and $I_B(1 - R)$ (open circles). The slopes for $e/3$ quasiparticles (dashed line) and electrons (dotted line) are shown. $\Theta = 25$ mK. Inset: data in same units showing electron tunneling for similar $G = 0.32e^2/h$ but in the IQHE regime ($\nu_L = 4$). The expected slope for electrons $2eI_B(1 - R)$ [$R = 0.68$, $I_B = (e^2/h)V_{ds} - I$] is shown. $\Theta = 42$ mK.

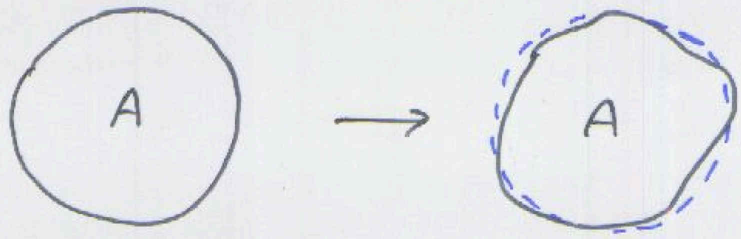
[Gattli et al. PRL ('97)]

W_∞ - symmetry of the incompressible fluid

- semiclassical symmetry

$$\int d^2z\, \rho(z) = N = \rho_0 \cdot \text{Area} \longrightarrow \text{Area} = \text{constant}$$

deformations of the droplet at fixed Area
= area-preserving diffeomorphism of the plane



- symmetry can be implemented in CFT of edge excitations
- its representations are completely known (Kac, Radu
- we can run a classification program
(A.C., C. Trugenberger, G. Zemba)

Results

- $\nu = 1, \frac{1}{3}, \frac{1}{5}, \ldots$ chiral Luttinger theory $c=1$ OK
- $\nu = \frac{2}{5}, \frac{2}{7}, \ldots$ Jain series of QH states $\nu = \frac{m}{mp \pm 1}$

$c = m$ W_∞ minimal models

- fractional charge has been measured
- detailed dynamics need 4-point functions