Hydrodynamics with Anomaly Inflow and Bosonization

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Outline

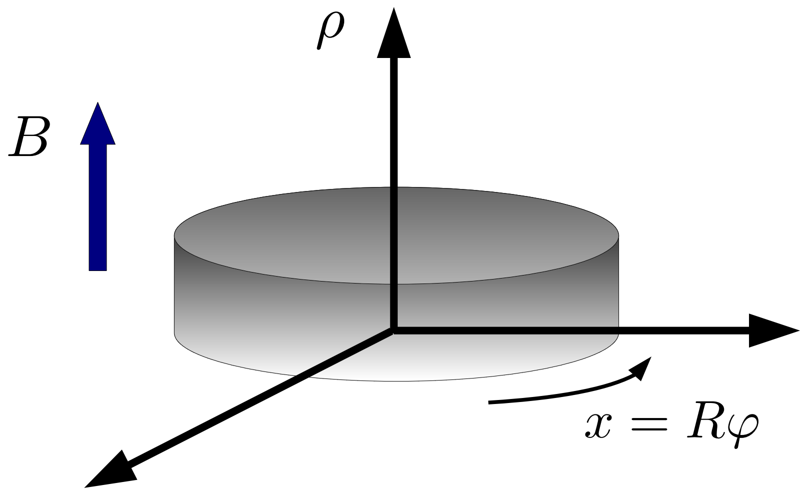
- Topological phases of matter and anomaly inflow
- Bosonization in 1+1d and anomalies
- Hydrodynamic of perfect fluid in 3+1d
- Including 3+1d anomalies via anomaly inflow
- Bosonization in 3+1d

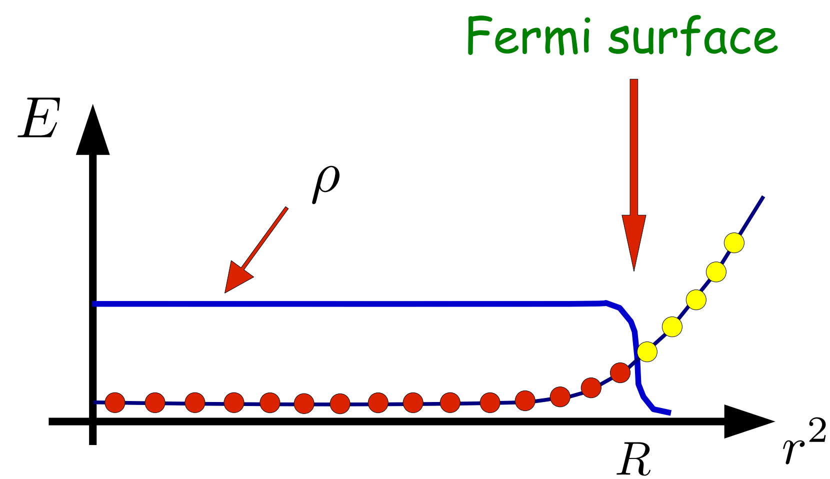
Effective field theory, anomalies & hydrodynamics

- Topological phases of matter provide us new insights:
  - bulk gap and massless boundary excitations
  - both bosonic and fermionic field theory descriptions → bosonization
  - topological gauge theories & anomalies
  - well understood in 1+1d, e.g. the Quantum Hall Effect
- Extension to 3+1d using hydrodynamics:
  - perfect fluid: T=0, s=const, dynamics from pressure & density
  - Lagrangian formulation
  - hydrodynamics with anomalies in 3+1d
  - bosonic `geometric' description of anomalies: universal response
  - many potential applications to interacting fermions in fluids phases

Quantum Hall effect: bulk and edge

Filled Landau level: bulk gap, massless edge fermion





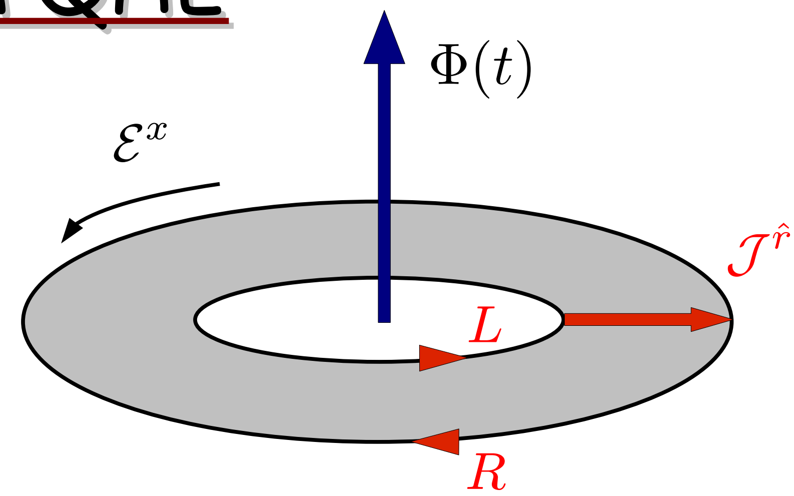
edge ~ Fermi surface: linearize energy $\varepsilon(k) = vk, \quad k = \frac{1}{R}(n - n_F), \quad n_F = N$

➡ set $r = R = \ell\sqrt{N}$; massless chiral fermion in (1+1) dimensions $\psi(r, \varphi, t)|_{r=R}$

➡ fractional fillings $\nu = \frac{1}{3}, \frac{1}{5}, \ldots$ ➡ interacting fermion ➡ bosonization

scalar field $\theta(x - t)$ (c=1 chiral conformal field theory/Luttinger theory)

Anomaly inflow in QHE



- Add flux $\Phi(t)$; tangential electric field $\mathcal{E}^x$
- Hall current $\mathcal{J}^{\hat{r}} = \sigma_H \mathcal{E}^x$
- Hall current = chiral anomaly of 1+1d edge theory

$$\oint dx\, \mathcal{J}^{\hat{r}} = \dot{Q}_{\text{edge}} = \oint dx\, \partial_\alpha J^\alpha, \qquad \partial_\alpha J^\alpha = -\frac{e}{2\pi}\varepsilon^{\beta\gamma}\partial_\beta A_\gamma, \quad \alpha = 0, 1,$$

- Hall current given by Chern-Simons topological gauge theory in 2+1d

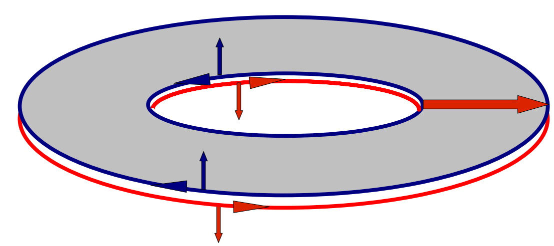
J^α 1+1d current

$\mathcal{J}^\mu$ 2+1d current

$$S_{CS}[A] = -\frac{1}{4\pi}\int d^3x\, AdA, \qquad A = A_\mu dx^\mu, \quad \mu = 0, 1, 2$$

$$\mathcal{J}^{\hat{r}} = \frac{\delta S_{CS}}{\delta A_{\hat{r}}} = -\frac{1}{2\pi}\varepsilon^{\hat{r}\alpha\beta}\partial_\alpha A_\beta = \partial_\alpha J^\alpha$$

- extends to non-chiral case (topological insulators)



bulk+boundary system is gauge invariant

1+1d bosonic theory and anomalies

$$S = \frac{1}{2}\int d^2x\ (\partial_\mu \theta)^2$$

- U(1) symmetry $\theta \to \theta + \text{const.}$ but two conserved currents

$$J^\mu = \partial_\mu \theta, \qquad \partial_\mu J^\mu = 0 \qquad \text{Noether current}$$

$$\tilde{J}^\mu = \varepsilon^{\mu\nu}\partial_\nu \theta, \qquad \partial_\mu \tilde{J}^\mu = 0 \qquad \text{topological (axial) current}$$

- like Dirac fermion: $J^\mu = \bar{\psi}\gamma^\mu\psi,$ $\tilde{J}^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$
- couple to corresponding backgrounds $A_\mu, \tilde{A}_\mu$

$$S = \int d^2x \frac{1}{2}\left(\partial_\mu\theta - A_\mu\right)^2 + \tilde{A}_\mu \varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right)$$

- gauge-invariant currents ("covariant currents") are anomalous

$$J^\mu = \partial_\mu\theta - A_\mu, \qquad \partial_\mu J^\mu = -\varepsilon^{\mu\nu}\partial_\mu \tilde{A}_\nu, \qquad (e/\pi \to 1)$$

$$\tilde{J}^\mu = \varepsilon^{\mu\nu}\left(\partial_\nu\theta - A_\nu\right), \qquad \partial_\mu \tilde{J}^\mu = -\varepsilon^{\mu\nu}\partial_\mu A_\nu$$

→ Anomalies reproduced at classical level in the bosonic effective theory

Checking the anomaly inflow

- Topological theory in the 2+1d: `hydrodynamic' gauge fields $p_\mu, \tilde{q}_\mu$ expressing conserved bulk currents, e.g. $\mathcal{J}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{q}_\rho, \quad \partial_\mu \mathcal{J}^\mu = 0,$

$$S_{BF}[p, \tilde{q}, A, \tilde{A}] = \int_{\mathcal{M}_3} p d\tilde{q} + p d\tilde{A} + \tilde{q} dA, \qquad \mathcal{J} = *d\tilde{q}, \quad \tilde{\mathcal{J}} = *dp$$

- equations of motion

$$dp + dA = 0 \quad \to \quad p = d\theta - A$$

$$d\tilde{q} + d\tilde{A} = 0 \quad \to \quad \tilde{q} = d\psi - \tilde{A}$$

$$S_{BF}[p, \tilde{q}, A, \tilde{A}] \underset{eom}{\to} S[\psi, A, \tilde{A}] = \int_{\mathcal{M}_3} (d\psi - \tilde{A}) dA$$

- check anomaly inflow

$$\tilde{\mathcal{J}}^{\hat{r}} \underset{eom}{=} \varepsilon^{\hat{r}\alpha\beta}\partial_\alpha \left(\partial_\beta \theta - A_\beta\right) = \partial_\alpha \tilde{J}^\alpha = -\varepsilon^{\alpha\beta}\partial_\alpha A_\beta, \qquad \alpha, \beta = 0, 1$$

$$\mathcal{J}^{\hat{r}} \underset{eom}{=} \varepsilon^{\hat{r}\alpha\beta}\partial_\alpha (\partial_\beta \psi - \tilde{A}_\beta) = \partial_\alpha J^\alpha = -\varepsilon^{\alpha\beta}\partial_\alpha \tilde{A}_\beta$$

→ hydrodynamic fields $p, \tilde{q}$ express the edge currents, once reduced to 1+1d θ, ψ

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\left(\partial_\beta \theta - A_\beta\right)$$

$$J^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta \psi - \tilde{A}_\beta)$$

Checking the anomaly inflow

- reduction to 1+1d edge

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\left(\partial_\beta\theta - A_\beta\right)$$

$$J^\alpha = \varepsilon^{\alpha\beta}(\partial_\beta\psi - \tilde{A}_\beta)$$

gauge invariant

- compare with bosonic theory

$$\tilde{J}^\alpha = \varepsilon^{\alpha\beta}\left(\partial_\beta\theta - A_\beta\right)$$

$$J^\alpha = \partial_\alpha\theta - A_\alpha$$

- bulk topological theory + inflow introduces `compensating' scalar fields θ, ψ ensuring gauge invariance of currents $J^\alpha, \tilde{J}^\alpha$ (also appearing in WZW terms):

- θ earlier scalar field

- ψ is actually the dual field: $\partial^\mu\theta - A^\mu = \varepsilon^{\mu\nu}(\partial_\nu\psi - \tilde{A}_\nu)$

Summing up so far

→ bosonic theory reproduces 1+1d chiral anomalies

→ anomaly inflow from topological theory (QHE setting, reservoir, etc) actually suggests the need of 1+1d `hydrodynamic' fields $\theta,\ \psi$

Variational principle for perfect fluid

- long history (Lichnerowitz, Carter, Arnold, Marsden, Holm...)
- recently made very explicit and simple $(T=0, s=\text{const.})$ (Abanov, Wiegmann, '22)
- can describes both relativistic and non-relativistic fluids

P - pressure
μ - chemical potential
ρ - fluid density
p_α- fluid momentum

$$S[p] = \int d^n x\; P(p_\alpha), \qquad dP = \rho\, d\mu, \quad \mu = \mu(p_\alpha)$$

- Euler hydrodynamics is a constrained system

- restricted variations: $\delta S[p] = 0$ for $\delta_\epsilon p_\nu = \epsilon^\alpha \partial_\alpha p_\nu + p_\alpha \partial_\nu \epsilon^\alpha, \quad \delta_\epsilon p = \mathcal{L}_\epsilon p$

- Equations of motion take the Lichnerowicz-Carter form

$$\hat{J}^\nu (\partial_\nu p_\mu - \partial_\mu p_\nu) = 0, \qquad \text{`particle current'} \quad \hat{J}_\nu = -\frac{\delta S}{\delta p_\nu}, \quad \partial_\nu \hat{J}^\nu = 0$$

in form notation $i_{\hat{J}} dp = 0$

- solution in 1+1d: $dp = 0 \quad \rightarrow \quad p = d\theta$ ($\hat{J}$ independent)

→ completely equivalent to earlier bosonic theory for $P(p_\alpha) = \frac{1}{2} p_\alpha^2 = \frac{1}{2}(\partial_\mu \theta - A_\mu)^2$
which is actually 1+1d hydrodynamics!

Hydrodynamics with anomalies in 3+1d

$$S[p] = \int d^4x\, P(p)$$

- Equation of motion $\quad i_{\tilde{J}} dp = 0 \quad \to \quad i_{\tilde{J}} dpdp = 0$
- solution in 3+1d $\quad dpdp = 0, \qquad \tilde{J}^\mu = \varepsilon^{\mu\nu\rho\sigma} p_\nu \partial_\rho p_\sigma, \qquad \partial_\mu \tilde{J}^\mu = 0$
- $\tilde{J}$ helicity current:

$$\tilde{Q} = \int d^3x\, \tilde{J}^0 \sim \int \vec{v}\cdot\vec{\omega}, \qquad \vec{\omega} = \nabla \times \vec{v}$$

- idea: identify it as axial current; include backgrounds (Abanov, Wiegmann '22)

$$S[\pi, A, \tilde{A}] = \int d^4x P(\pi - A) + \tilde{A}(\pi - A)d(\pi + A), \qquad p = (\pi - A) \ \text{ gauge inv.}$$

- obtain (some of) the anomalies

$$\partial_\mu \tilde{J}^\mu = *d[(\pi - A)d(\pi + A)] = - * dAdA \qquad \partial_\mu \tilde{J}^\mu = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \qquad (e/2\pi = 1)$$

$$\partial_\mu J^\mu = -2 * dAd\tilde{A}, \qquad \partial_\mu J^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} \tilde{F}_{\rho\sigma},$$

$$(dpdp = 0 \ \to d\pi d\pi = 0) \qquad \partial_\nu T^\nu_\mu = F_{\mu\nu} J^\nu + \tilde{F}_{\mu\nu} \tilde{J}^\mu$$

→ anomaly is `geometric', i.e. independent of specific dynamics $P(p_\alpha)$

→ hydrodynamics can describe interacting fermionic fluids

Anomaly inflow from 4+1d

- 4+1d topological theory: introduce hydrodynamic gauge fields dual to bulk currents

$$S[c,\tilde{c},p,\tilde{q},A,\tilde{A}] = \int_{\mathcal{M}_5} \tilde{c}(dA+p) + cd(\tilde{q}+\tilde{A}) + \tilde{q}dpdp + \frac{1}{3}\tilde{q}d\tilde{q}d\tilde{q}, \qquad \mathcal{J} = *d\tilde{c},\ \tilde{\mathcal{J}} = *dc, \quad c, \tilde{c} \ \ 3-\text{forms}$$

- equations of motion

$$\tilde{q} = d\psi - \tilde{A}, \qquad \tilde{c} = -2pd\tilde{q} + d\tilde{b}$$

$$p = d\theta - A, \qquad c = -pdp - \tilde{q}d\tilde{q} + db$$

$$S[c,\tilde{c},p,\tilde{q},A,\tilde{A}] \underset{eom}{\rightarrow} S[\psi,A,\tilde{A}] = \int_{\mathcal{M}_5} (d\psi - \tilde{A})(dAdA + \frac{1}{3}d\tilde{A}d\tilde{A})$$

- obtain 3+1d anomalies by inflow

$$\partial_\mu J^\mu = -2 * dAd\tilde{A}$$

$$\partial_\mu \tilde{J}^\mu = - * dAdA - * d\tilde{A}d\tilde{A} \qquad (e/2\pi = 1)$$

- match hydrodynamic formulation:

➡ identify fluid momentum $p = (d\theta - A) \quad \rightarrow \quad p = (\pi - A)$

➡ complete form of anomalies needs additional d.o.f.: pseudoscalar field ψ

3+1d Hydrodynamics from inflow

Extended hydro action including terms given by inflow

$$S[\pi, \psi, A, \tilde{A}] = \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A)d(\pi + A) + \psi(d\pi d\pi + \frac{1}{3}d\tilde{A}d\tilde{A}) - \int_{\mathcal{M}_5} \tilde{A}dAdA + \frac{1}{3}\tilde{A}d\tilde{A}d\tilde{A}$$

→ bulk + boundary system is gauge invariant

$$p = (\pi - A), \quad \tilde{q} = (d\psi - \tilde{A}) \quad \text{gauge inv.}$$

- Results

- ψ is Lagrange multiplier enforcing C-L equation: can do free variations of S

- `geometric' bosonic theory for 3+1d interacting fermions with anomalies

- ψ additional d.o.f. of the fluid (pion-like); can also add dynamics for it, not affecting anomalies

- can also describe mixed axial-gravitational anomaly

Mixed axial-gravitational anomaly

$$D_\mu \tilde{J}^\mu = -*dAdA - 3\alpha * d\tilde{A}d\tilde{A} - \beta * \mathrm{Tr}(R^2) \qquad R_{ab} = \frac{1}{2} R_{\mu\nu,ab} dx^\mu dx^\nu, \quad \beta = 1/24$$

- No truly gravitation anomaly in 3+1d, i.e. no violation of stress-tensor conservation
- additional term to 4+1d action known from anomaly literature

$$\Delta S \underset{eom}{=} \beta \int_{\mathcal{M}_5} (d\psi - \tilde{A}) \mathrm{Tr}(R^2)$$

- no extra hydro fields needed
- results: $D_\mu J^\mu = -2 * dAd\tilde{A}$

$$D_\nu T^\nu_\mu = F_{\mu\nu} J^\nu + \tilde{F}_{\mu\nu} \tilde{J}^\mu + \mathrm{Tr}(R_{\mu\nu} \Sigma^\nu), \qquad \Sigma^{\mu,ab} \quad \text{spin current}$$

- spin current is related to axial current (axial background equivalent to torsion bkgd)

- free Dirac identity $\Sigma^{\mu,ab} = \frac{1}{4} \bar{\Psi} \{\gamma^\mu, \sigma^{ab}\} \Psi = \frac{1}{2} \varepsilon^{\mu\nu,ab} \tilde{J}_\nu$

- ψ (axial gauge transf.) corresponds to spinor rotation (local-Lorentz transf.)

Conclusions

- Topological phases of matter: topological bulk and massless boundary

→ anomaly inflow gives a new view on anomalies

- anomaly inflow helps writing bosonic effective field theory/Euler hydrodynamics:

→ obtain bosonization/perfect fluid with anomalies in 3+1d

→ clearly see that anomalies parameterize universal, `geometric' effects/responses

Perspectives/extensions

- 3+1d hydrodynamics with 2-form field is suggested by topological theory: purely chiral fluids (Weyl fermions) ?
- 2+1d hydrodynamics (global anomaly)
- add temperature and entropy; extend to many species & non-Abelian symmetries
- applications to interacting fermions: topological phases, heavy-ion collisions, cosmology,….