Critical Ising Model in Varying Dimension by Conformal Bootstrap

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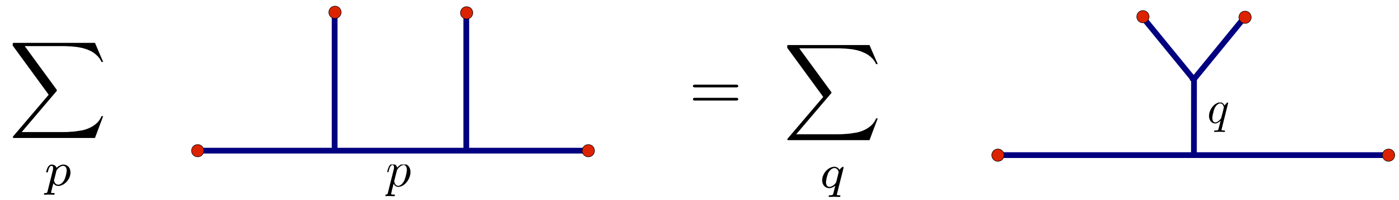
Outline

- Conformal bootstrap in varying dimension
- Precise exponents and structure constants
- Leading twists and d=2 limit
- Decoupling of states at Ising point

Conformal bootstrap



- Polyakov old idea, recently revived (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin, Vichi '14)
- Conformal partial waves in d>2 vs. d=2 (Virasoro blocks) (Dolan, Osborn '01-'04)

$$\langle \sigma(0)\sigma(1)\sigma(\eta)\sigma(\infty)\rangle \sim \sum_{p=1}^{2} |F_p(\eta)|^2, \qquad F_p(\eta) \sim \eta^a + (1-\eta)^b \qquad d=2$$

$$\langle \sigma(0)\sigma(1)\sigma(\eta)\sigma(\infty)\rangle \sim \sum_{p'=1}^{\infty} |\widehat{F}_{p'}(\eta)|^2, \qquad \widehat{F}_{p'}(\eta) \sim \eta^{a+n} + \log(1-\eta) \qquad d \geq 2$$

- Needs to resum infinite logarithms, i.e. many tiny contributions
- It can be done! Routines have been implemented and are available:

  SDPB (Simmons-Duffin '15), Extremal Functional (El-Showk, Paulos '13)

$$\Delta_\sigma = 0.5181489(10), \qquad \Delta_\varepsilon = 1.412625(10), \qquad d=3$$

Motivations for CFT 4>d>2

- Understand CFT in d>2 (a dream…)
- Ising CFT = singular point on the boundary of unitary bootstrap

→ reduced bootstrap

→ "minimal model"?

- The unitary boundary at d=2 corresponds to the minimal model relation

$$\Delta_{13}(c) = \frac{8}{3}\Delta_{12}(c) + \frac{2}{3}, \qquad \frac{1}{2} \leq c \leq 1 \qquad\qquad \Delta_{r,s} \in \text{Kac table}$$

- For d>2 it approximatively extends as

$$\Delta_\varepsilon \sim \frac{8}{3}\Delta_\sigma + \text{const}(d) \qquad 2 \leq d \leq 3$$

(El-Showk et al. '14)

→ let us have a look

Precise conformal data for 4>d>2

- Extend routines to $d \neq 3$ and study $13\ d$ values (single-correlator bootstrap)
- Analyze $\Delta_{\mathcal{O}}, \quad f_{\sigma\sigma\mathcal{O}}$ of six low-lying fields $\mathcal{O} = \sigma, \varepsilon, \varepsilon';\ T';\ C, C' \quad (\ell = 0, 2, 4)$ checked against best $d = 3$ data (three-correlator bootstrap, one extra digit)

(Simmons-Duffin '17)

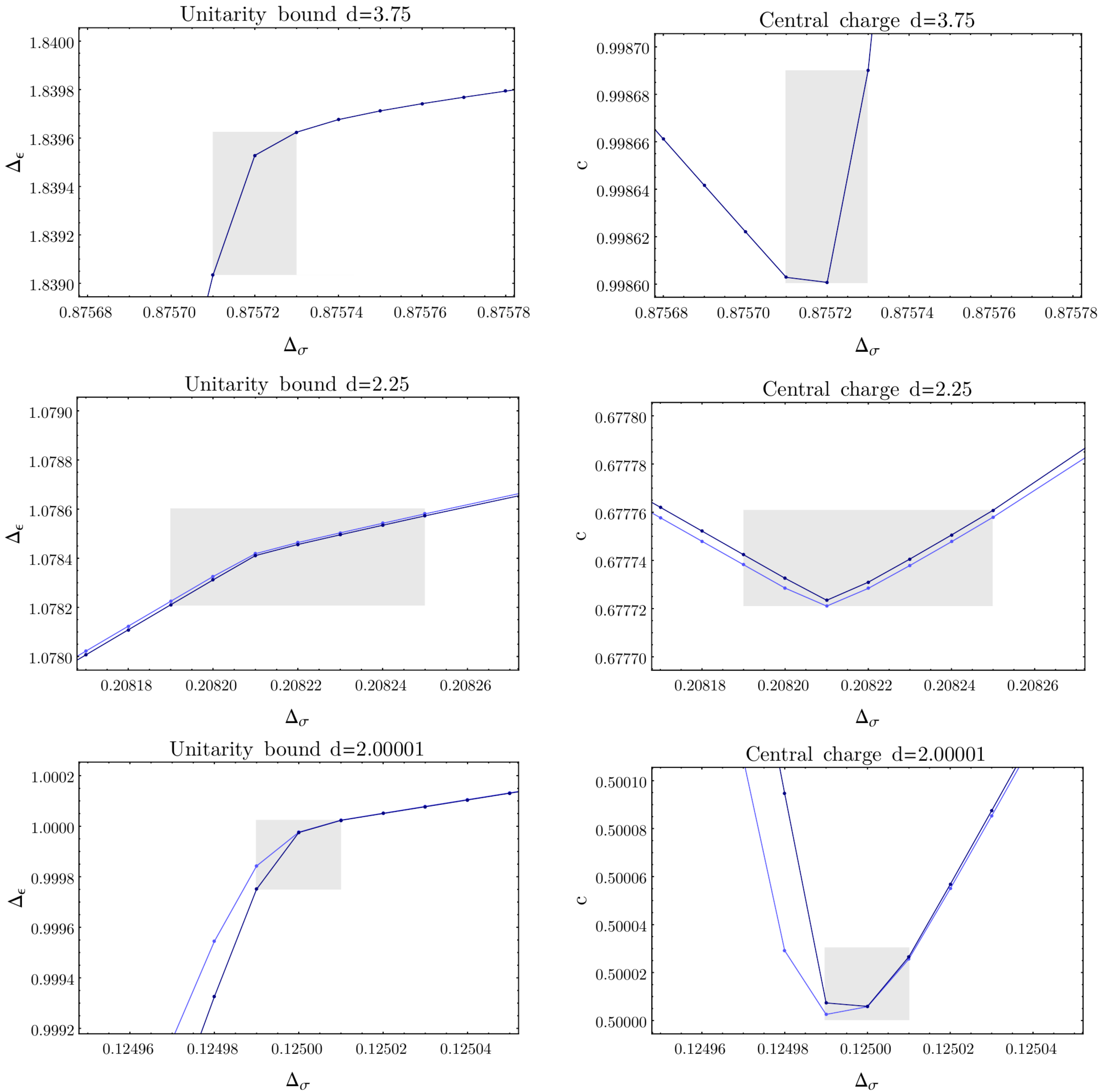
- Polynomial fit in $y = 4 - d$
- Examples:

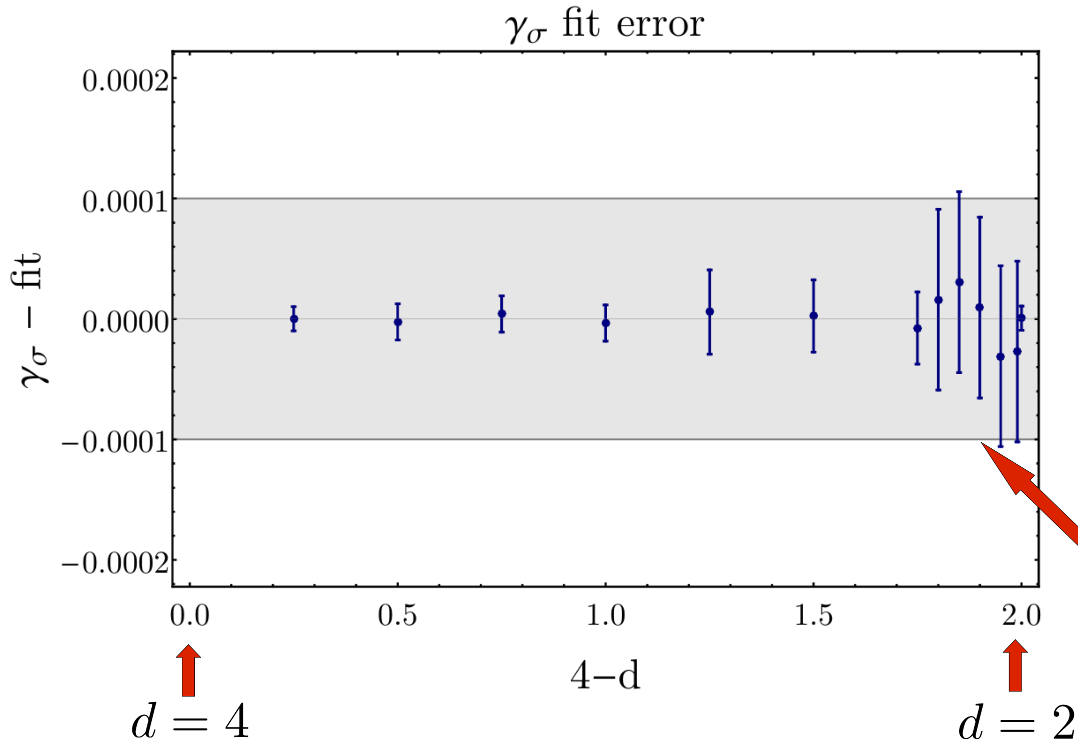
$$\Delta_\sigma = \frac{d-2}{2} + \gamma_\sigma, \qquad \Delta_\varepsilon = d - 2 + \gamma_\varepsilon$$

$$\gamma_\sigma(y) = 0.00955001 y^2 + 0.00764826 y^3 + 0.00091284 y^4$$
$$- 0.00024948\ \mathrm{y}^5 + 0.000296768 y^6, \qquad \mathrm{Err}(\gamma_\sigma) < 0.0001$$

$$\gamma_\varepsilon(y) = 0.336000 y + 0.0914812 y^2 - 0.0229152 y^3 + 0.00729869 y^4$$
$$+ 0.000890045 y^5, \qquad \mathrm{Err}(\gamma_\varepsilon) < 0.001$$

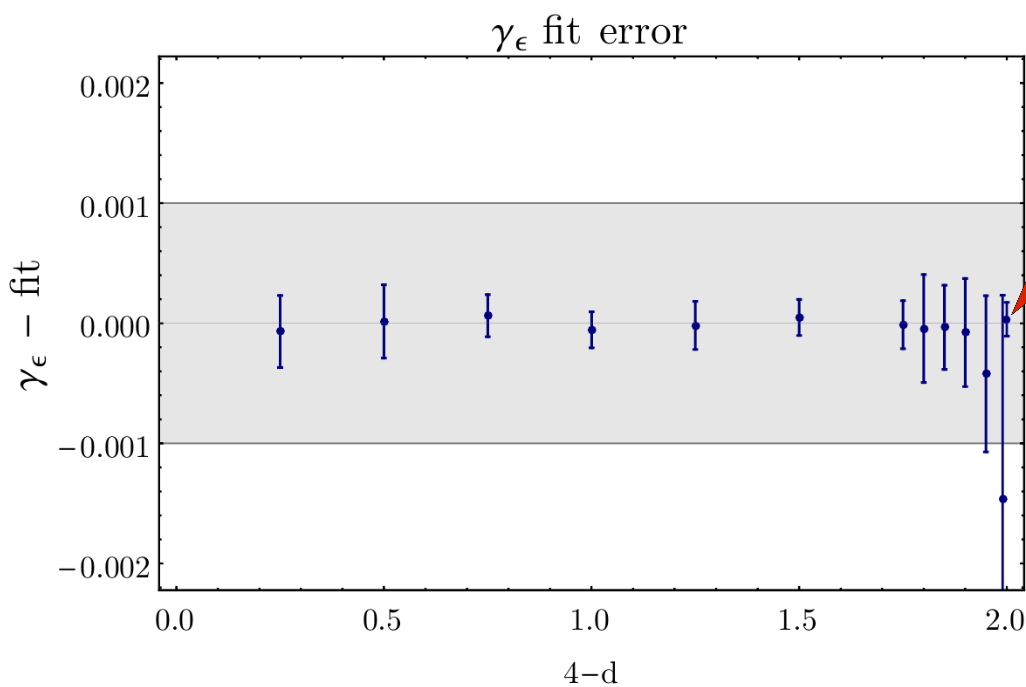
$\mathrm{Err}()$ estimated from uncertainty of Ising point in parameter space


Unitarity bound d=3.75
Δ_ϵ
1.8400
1.8398
1.8396
1.8394
1.8392
1.8390
0.87568
0.87570
0.87572
0.87574
0.87576
0.87578
Δ_σ
Central charge d=3.75
c
0.99870
0.99868
0.99866
0.99864
0.99862
0.99860
0.87568
0.87570
0.87572
0.87574
0.87576
0.87578
Δ_σ
Unitarity bound d=2.25
Δ_ϵ
1.0790
1.0788
1.0786
1.0784
1.0782
1.0780
0.20818
0.20820
0.20822
0.20824
0.20826
Δ_σ
Central charge d=2.25
c
0.67780
0.67778
0.67776
0.67774
0.67772
0.67770
0.20818
0.20820
0.20822
0.20824
0.20826
Δ_σ
Unitarity bound d=2.00001
Δ_ϵ
1.0002
1.0000
0.9998
0.9996
0.9994
0.9992
0.12496
0.12498
0.12500
0.12502
0.12504
Δ_σ
Central charge d=2.00001
c
0.50010
0.50008
0.50006
0.50004
0.50002
0.50000
0.12496
0.12498
0.12500
0.12502
0.12504
Δ_σ



overall relative error $O(10^{-3})$

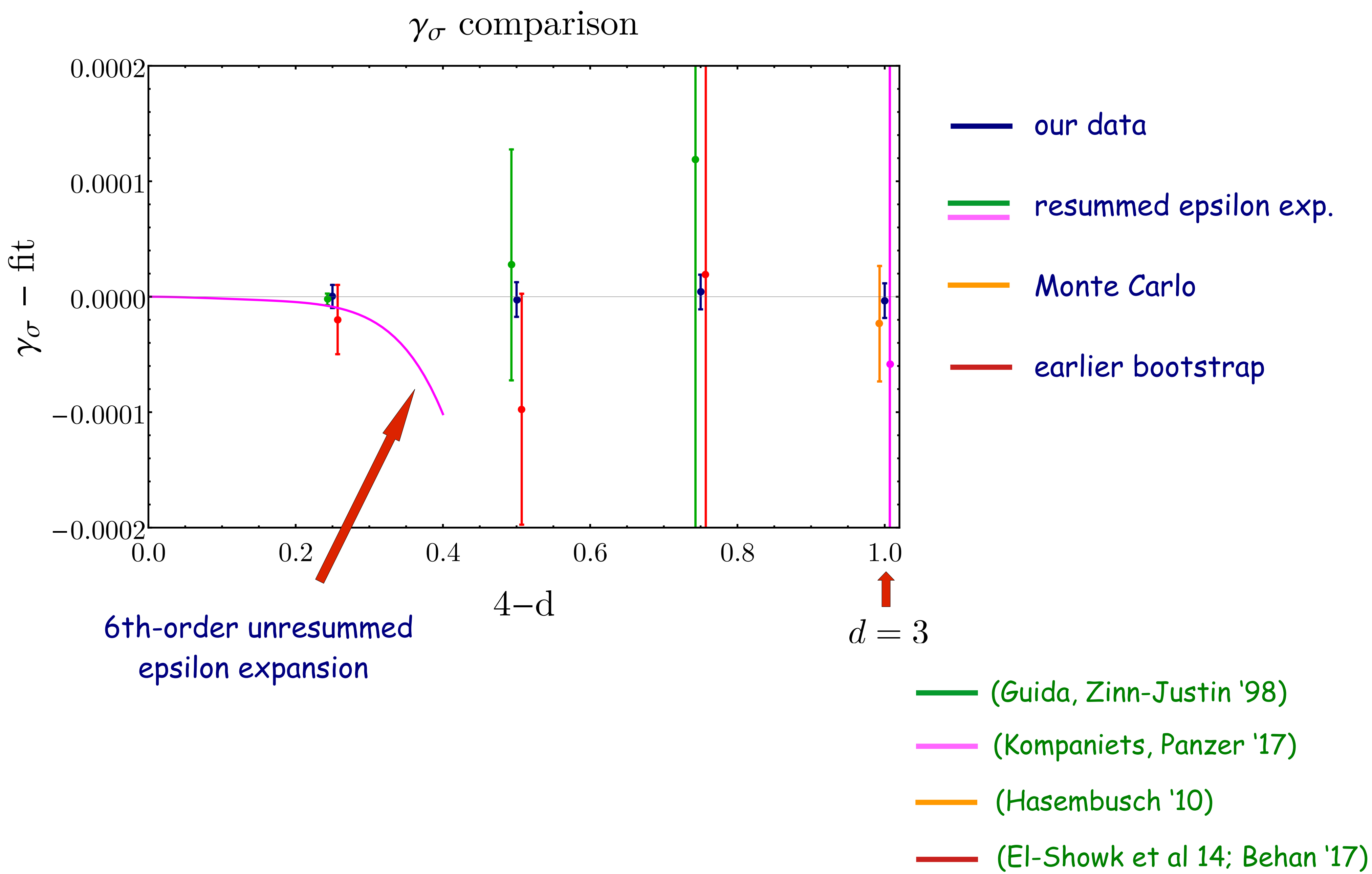
less data for 2.25>d>2



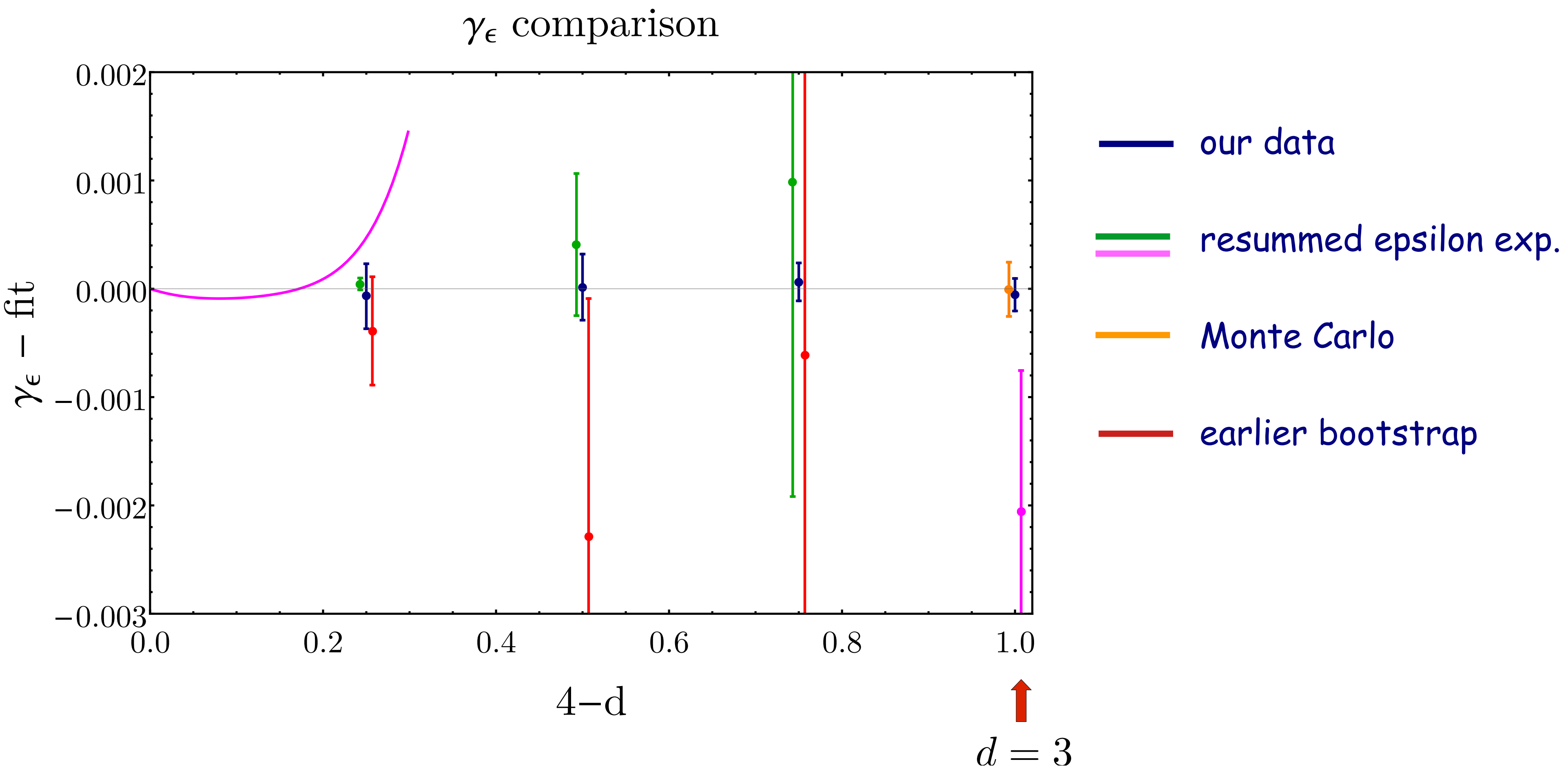
precise match at d=2

<u>no sign of non-unitarity</u>

Comparison with other methods



Comparison with other methods

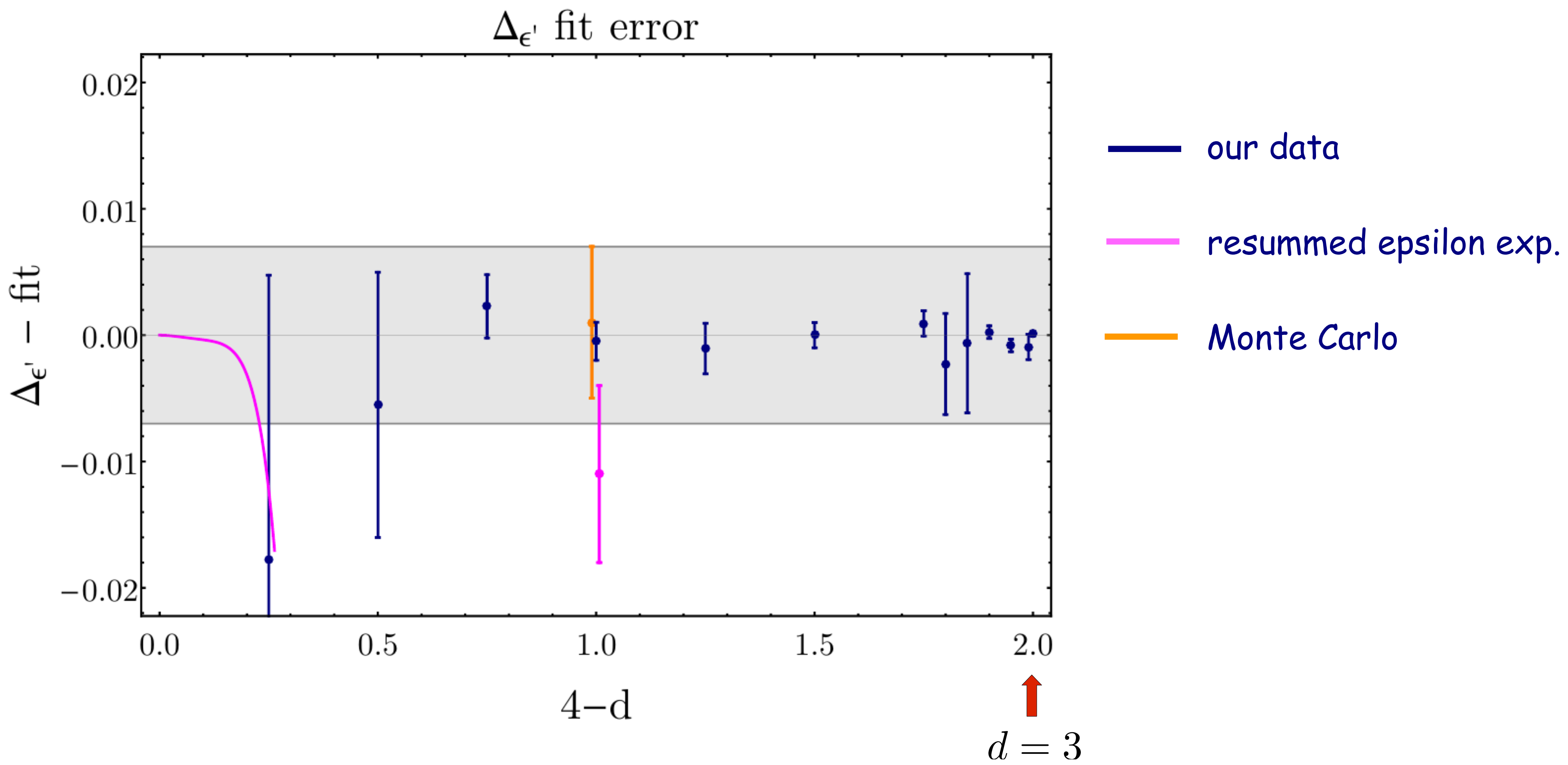


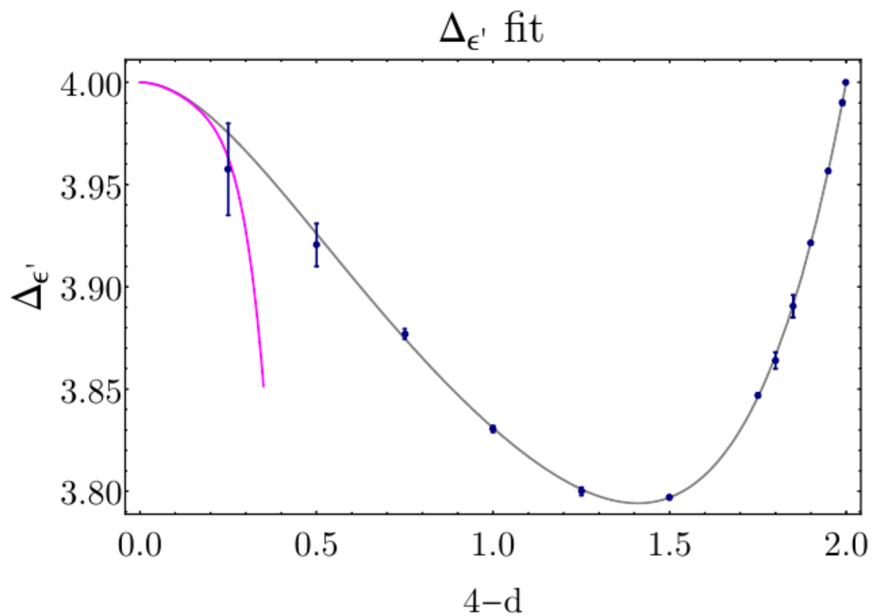
Interest:

- test of epsilon expansion and other analytic methods
- other universality classes related to Ising(d), e.g. long-range Ising

(Behan et al. '17; Defenu et al. '17, '20)

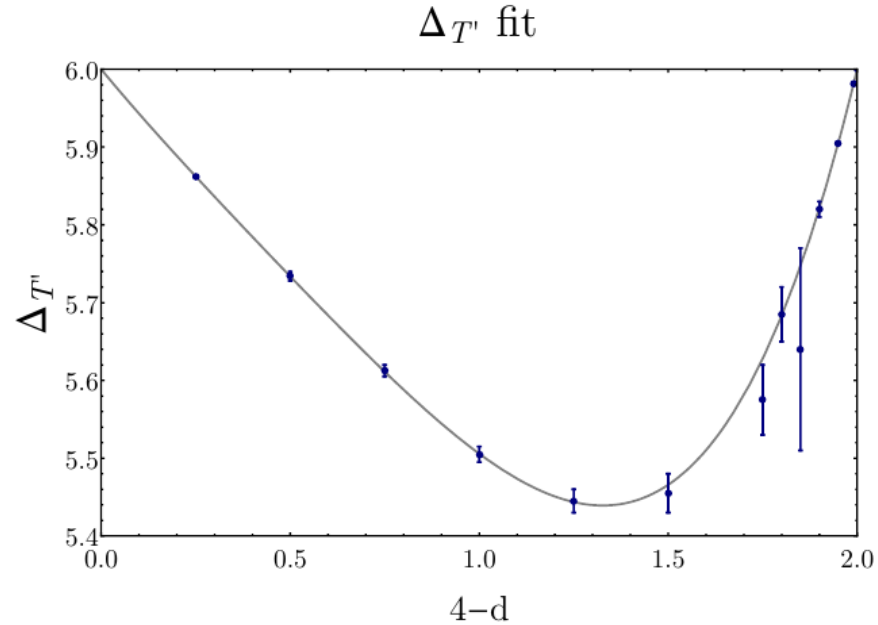
Subleading fields





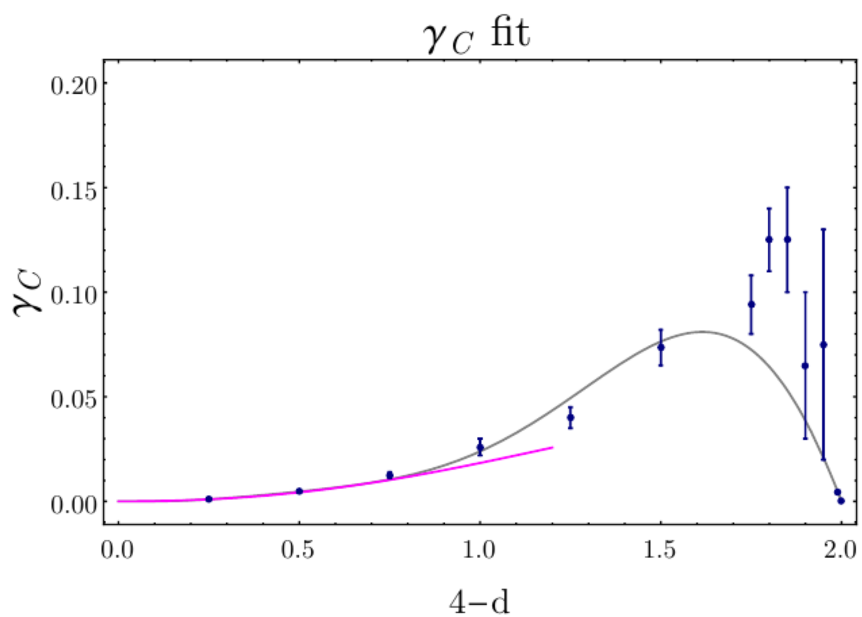
Δ_ε' fit
Δ_ε'
4.00
3.95
3.90
3.85
3.80
0.0
0.5
1.0
1.5
2.0
4−d

(a)



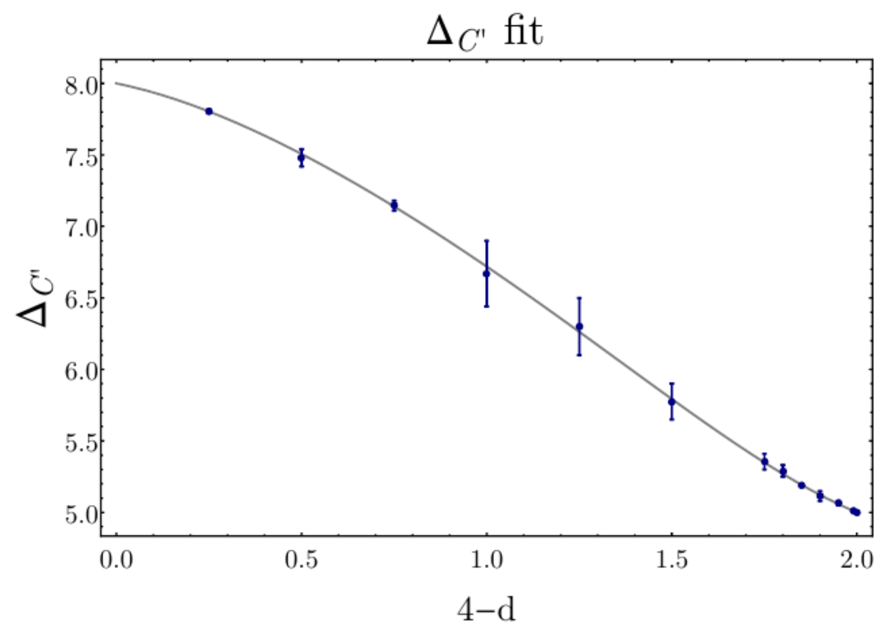
Δ_T' fit
Δ_T'
6.0
5.9
5.8
5.7
5.6
5.5
5.4
0.0
0.5
1.0
1.5
2.0
4−d

(b)



γ_C fit
γ_C
0.20
0.15
0.10
0.05
0.00
0.0
0.5
1.0
1.5
2.0
4−d

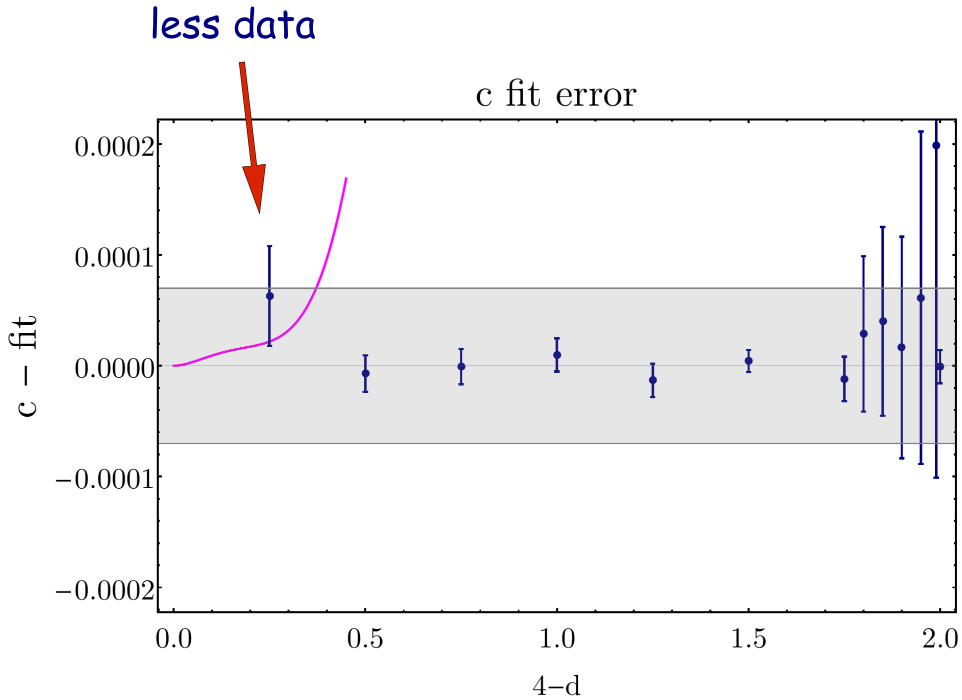
(c)

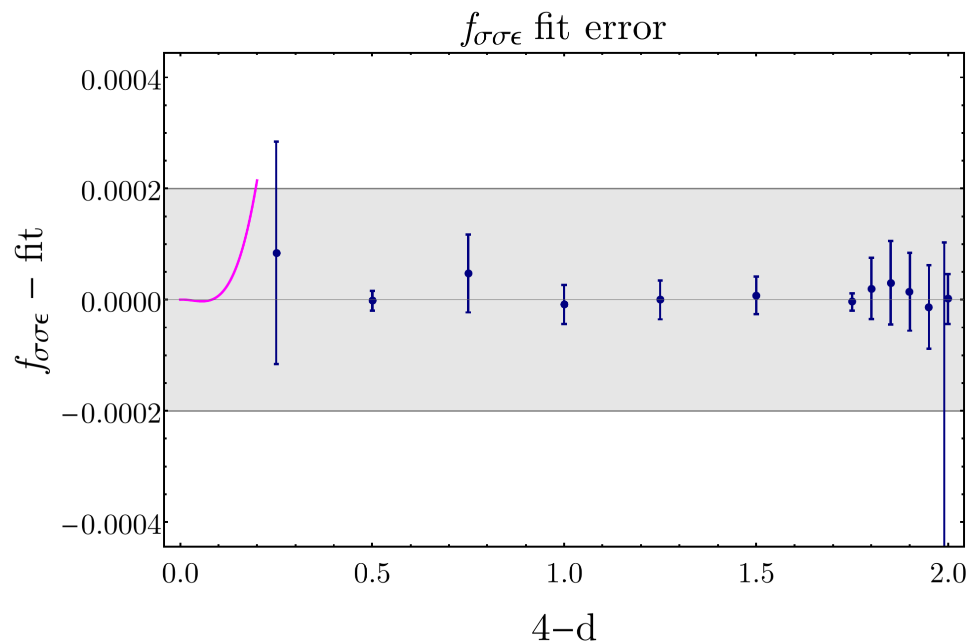


Δ_C' fit
Δ_C'
8.0
7.5
7.0
6.5
6.0
5.5
5.0
0.0
0.5
1.0
1.5
2.0
4−d

(d)

Structure constants



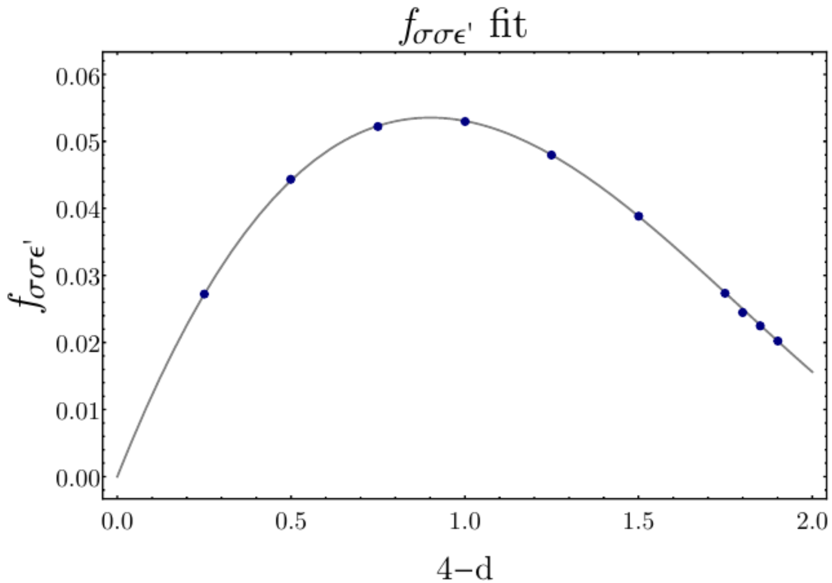


$$c(y) = 1 - 0.0173616y^2 - 0.0133068y^3 - 0.0385653y^4 + 0.0310843y^5 - 0.0196858y^6 + 0.00436051y^7,$$

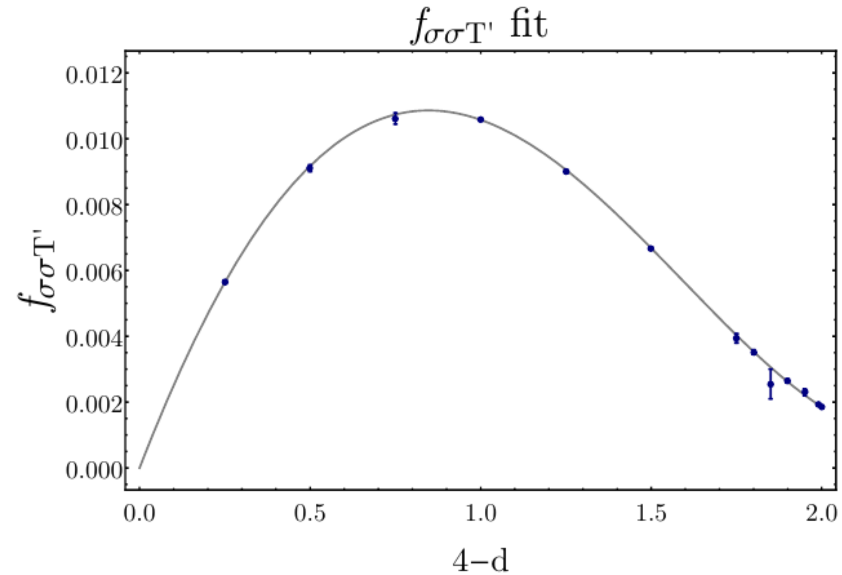
$$\mathrm{Err}(c) < 0.0007$$

$$f_{\sigma\sigma\varepsilon}(y) = 1.41421 - 0.235735y - 0.164305y^2 + 0.0631842y^3 - 0.0371191y^4 + 0.0137454y^5 - 0.00214024y^6,$$

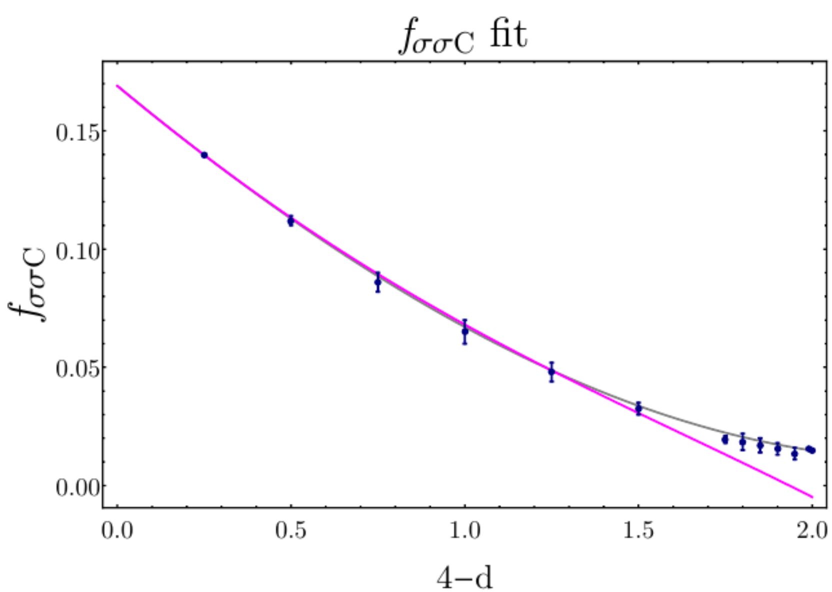
$$\mathrm{Err}(f_{\sigma\sigma\varepsilon}) < 0.0002$$


$f_{\sigma\sigma\epsilon'}$ fit
$f_{\sigma\sigma\epsilon'}$
0.00
0.01
0.02
0.03
0.04
0.05
0.06
0.0
0.5
1.0
1.5
2.0
4−d

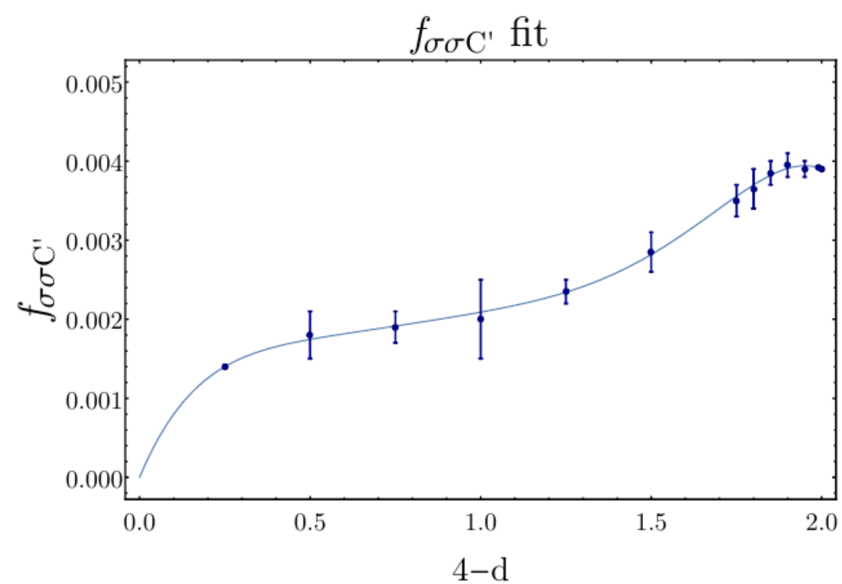
(a)


$f_{\sigma\sigma\mathrm{T}'}$ fit
$f_{\sigma\sigma\mathrm{T}'}$
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.0
0.5
1.0
1.5
2.0
4−d

(b)


$f_{\sigma\sigma\mathrm{C}}$ fit
$f_{\sigma\sigma\mathrm{C}}$
0.00
0.05
0.10
0.15
0.0
0.5
1.0
1.5
2.0
4−d

(c)


$f_{\sigma\sigma\mathrm{C}'}$ fit
$f_{\sigma\sigma\mathrm{C}'}$
0.000
0.001
0.002
0.003
0.004
0.005
0.0
0.5
1.0
1.5
2.0
4−d

(d)

Leading twists and d=2 limit

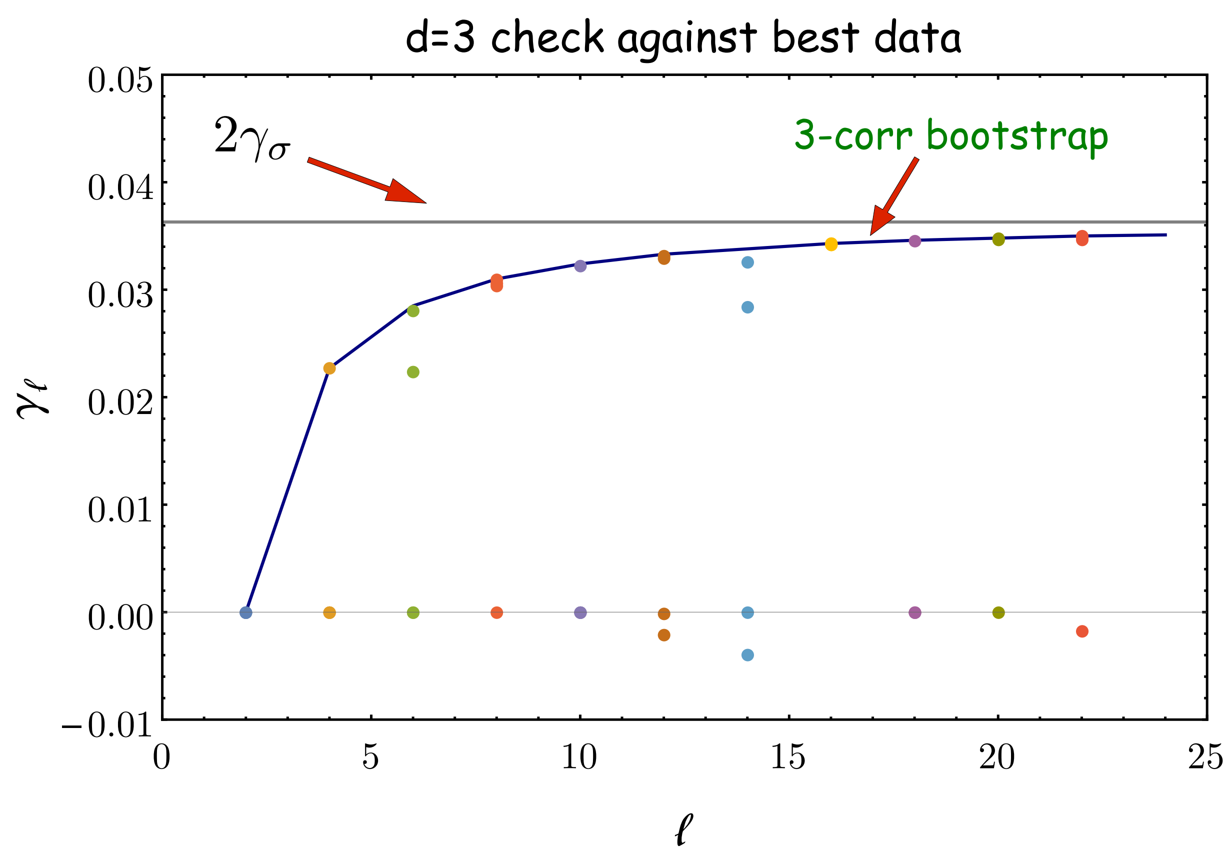
- d-dependence looks smooth but actually is not
- higher part of the spectrum changes completely for d>2; numerical fluctuations
  → qualitative results; d=3 check → rather good anyhow
- leading twists: $\mathcal{O}_\ell$ smallest Δ_ℓ for each $\ell = 4, 6, \ldots$

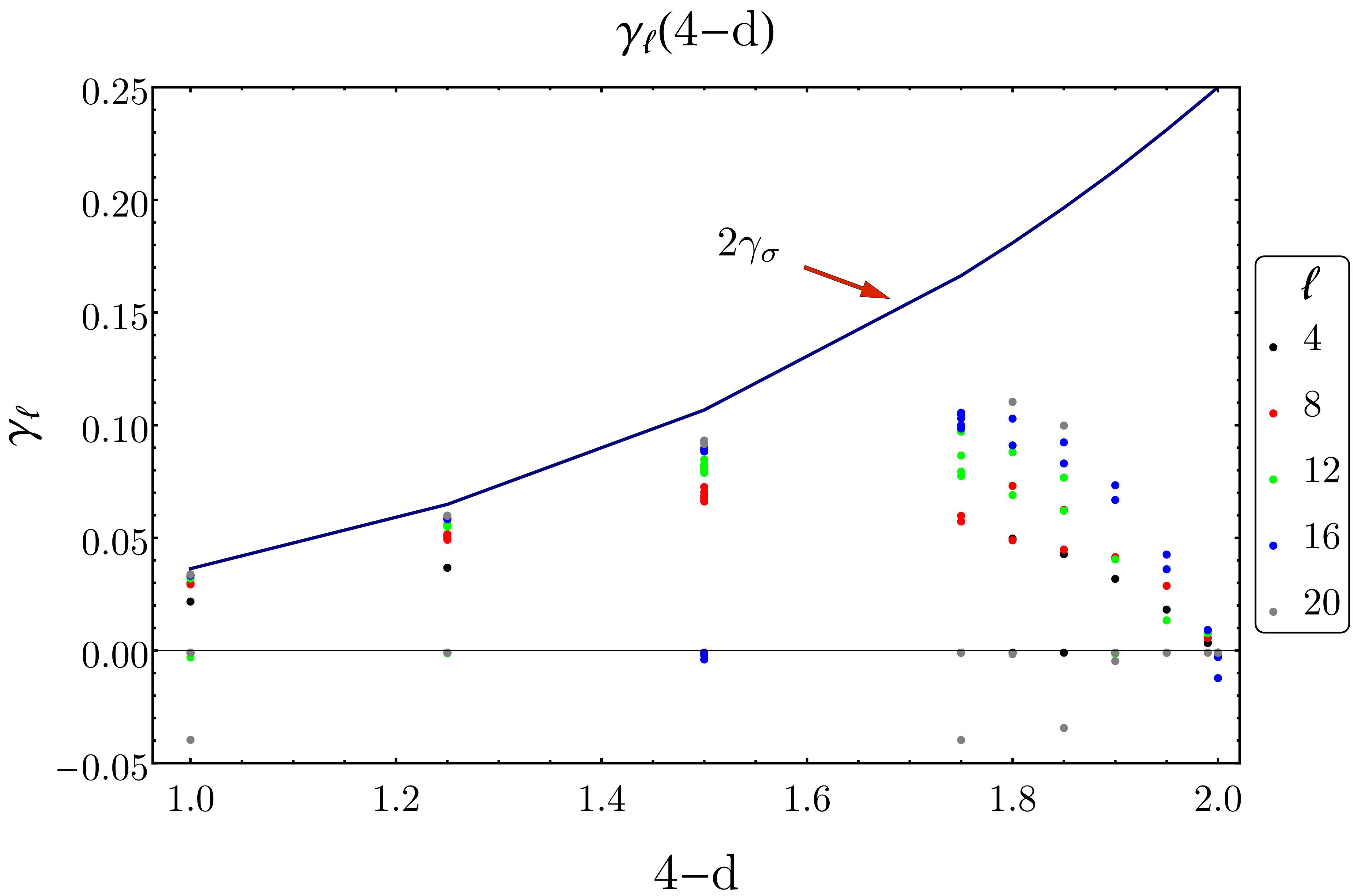
$$\Delta_\ell = \underbrace{d - 2 + \ell}_{\text{big}} + \underset{\text{small}}{\gamma_\ell}$$

$d = 3$: $\lim_{\ell \to \infty} \gamma_\ell = 2\gamma_\sigma$,

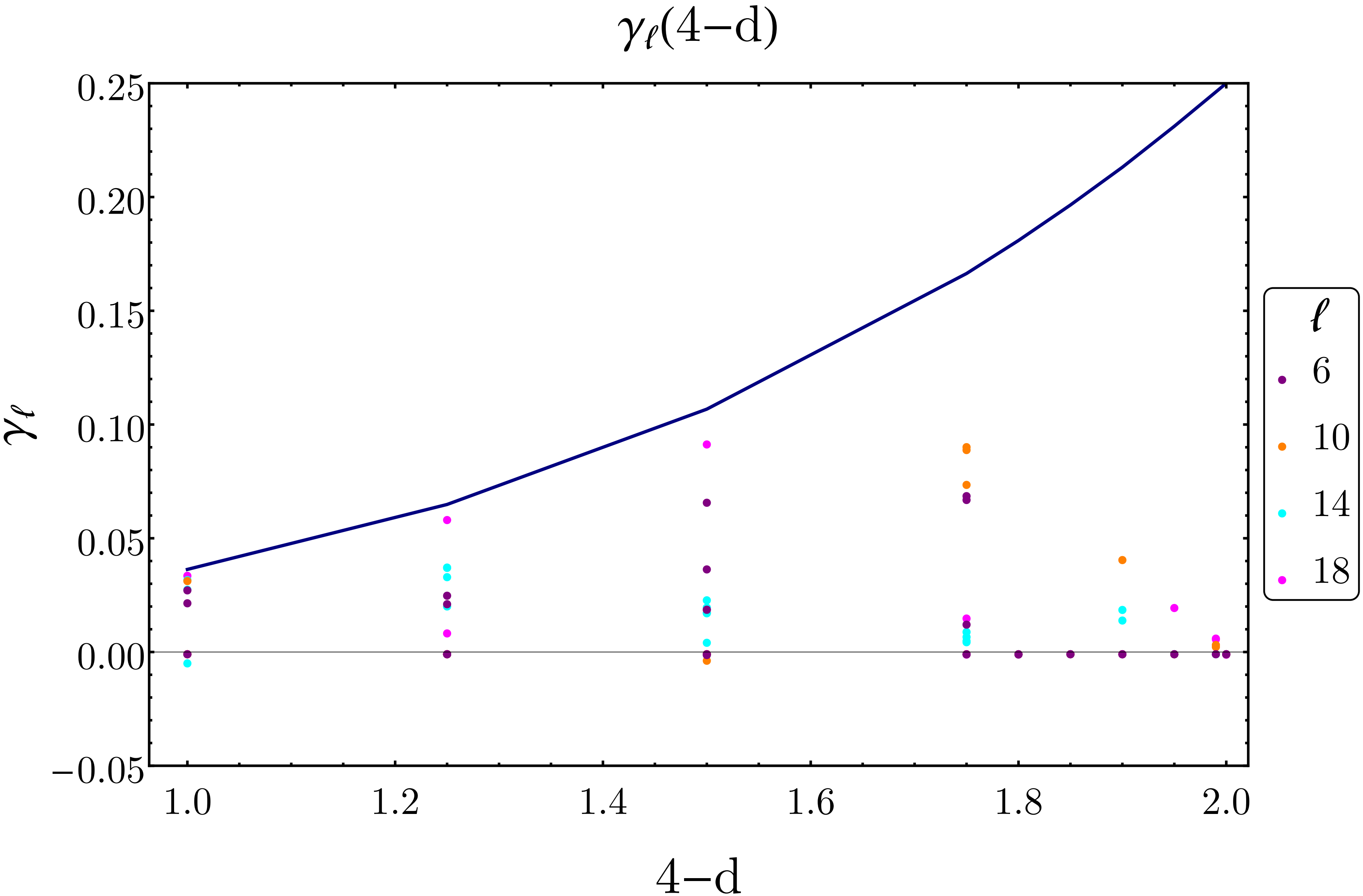
monotonic in ℓ,

$d = 2$: $\underset{\text{Virasoro Id rep.}}{\gamma_\ell} = 0 \quad \forall \ell$

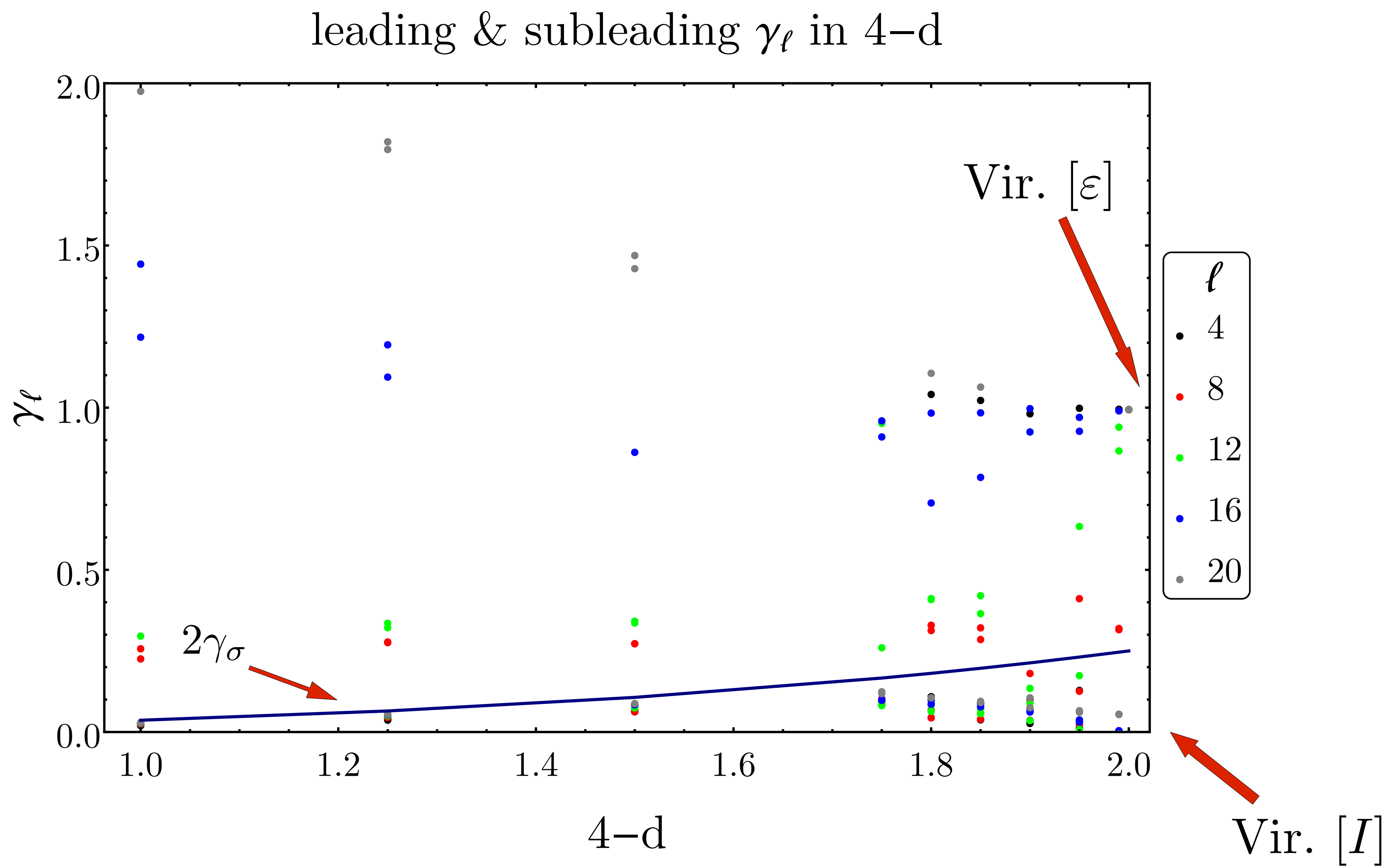




$\gamma_\ell(4\text{-d})$
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
γ_ℓ
$2\gamma_\sigma$
1.0
1.2
1.4
1.6
1.8
2.0
4−d
ℓ
4
8
12
16
20



γ_ℓ(4−d)
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
γ_ℓ
1.0
1.2
1.4
1.6
1.8
2.0
4−d
ℓ
6
10
14
18



Conclusion: d>2 behavior sets in at d ≥ 2.5 (i.e. 4 - d ≤ 1.5)

Decoupling of states: d=2 vs. d>2

- Numerically follow the path $\Delta_\sigma(c), \Delta_\varepsilon(c)$ from Tricritical Ising to Ising
- d=2: count quasiprimary states in both theories (A. Zamolodchikov '89)

no channel → necessary condition for decoupling

$$\sigma \cdot \sigma \equiv \phi_{12} \cdot \phi_{12} = \phi_{11} + \phi_{13}$$

$\boxed{1}$ decouples, well seen numerically

$\boxed{1}$ decouples, uncertain

$\boxed{2}$ decouples, unseen numerically

→ ONE clear state decoupling

at $\ell = 2$ owing to $\varepsilon \equiv \phi_{13} \sim \phi_{21}$

→ Follow it for d>2

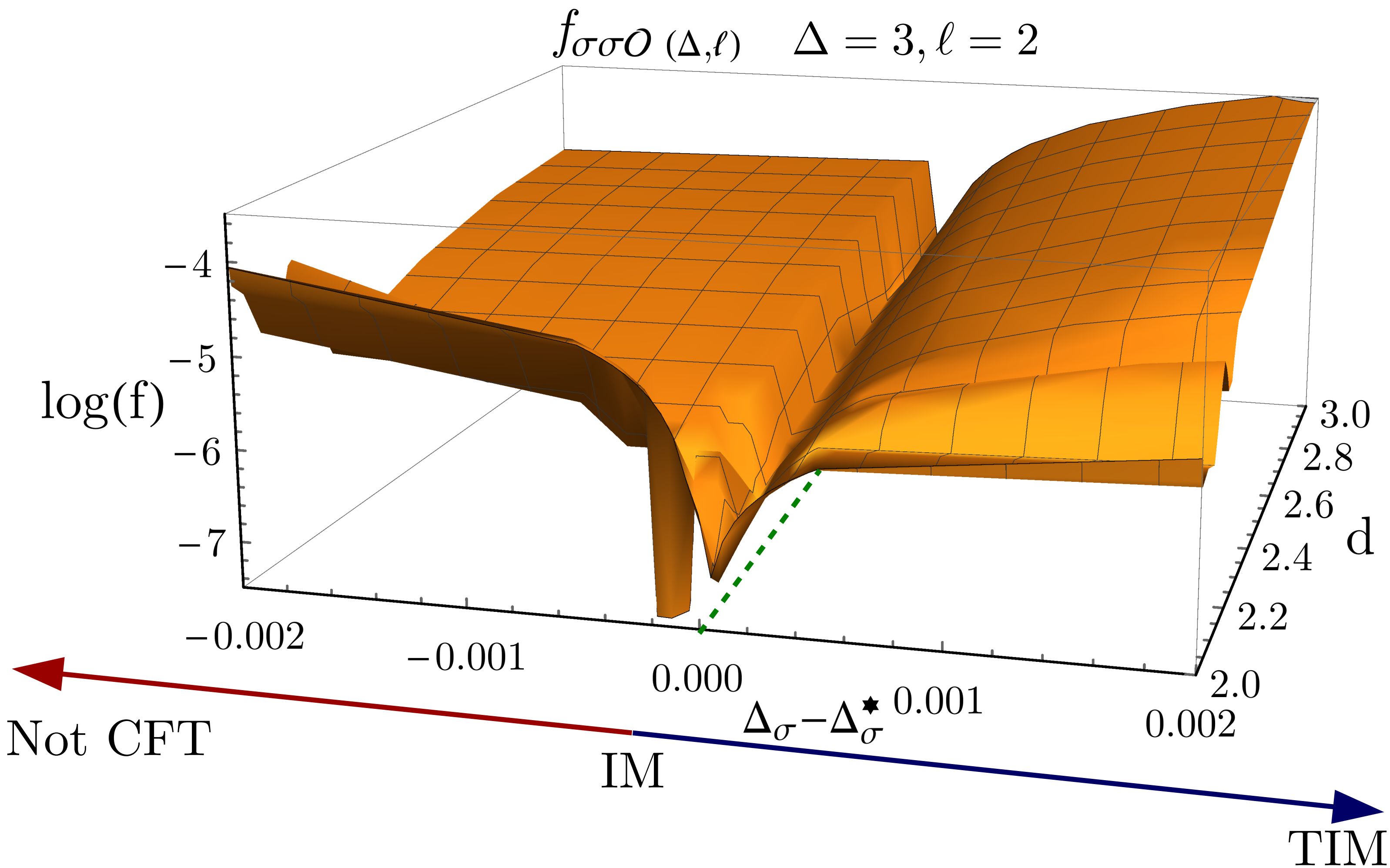
Tricritical Ising

ℓ \ Δ	1	2	3	4	5	6	7	8	9	10	11
10										4	6
8								3	4	0	0
6						2	2	0	0	3	$\boxed{4}$
4				1	2	0	0	2	$\boxed{2}$	0	0
2		1	$\boxed{1}$	0	0	1	$\boxed{2}$	0	0	2	4
0	1	0	0	1	$\boxed{1}$	0	0	1	4	0	1

Ising

ℓ \ Δ	1	2	3	4	5	6	7	8	9	10	11
10										2	2
8								2	1	0	0
6						1	1	0	0	2	0
4				1	1	0	0	1	0	0	0
2		1	0	0	0	1	0	0	0	1	1
0	1	0	0	1	0	0	0	1	1	0	0

It stays!



- It matches decoupling already seen at d=3
- How small? 10^{-2} w.r.t. far-off value $\forall\, 2 \leq d \leq 3$

$$f_{\sigma\sigma T} : f_{\sigma\sigma\mathcal{O}(3,2)} : f_{\sigma\sigma T'} = 10^2 : 10 : 1 \longrightarrow 10^2 : 10^{-1} : 1$$

Conclusions

- Precise exponents and structure constants

  - Ising universality class in $2 \leq d < 4$ can be useful for some problems

  - benchmarking for analytic methods (unresummed ε- exp is better than expected)

- Leading twists

  - a glimpse into an unknown world, the $d > 2.5$ conformal towers of states

- Decoupling of states at Ising point

  - a single evidence, "reduced bootstrap" is yet a mystery… or a red herring...

- Perspectives

  - analytic approximations, toy models, ???…………you tell me

Parameter	Value
findPrimalFeasible	true
findDualFeasible	true
detectPrimalFeasibleJump	false
detectDualFeasibleJump	false
precision	704
dualityGapThreshold	10^{-30}
primalErrorThreshold	10^{-30}
dualErrorThreshold	10^{-30}
initialMatrixScalePrimal	10^{20}
initialMatrixScaleDual	10^{20}
feasibleCenteringParameter	0.1
infeasibleCenteringParameter	0.3
stepLengthReduction	0.7
choleskyStabilizeThreshold	10^{-40}
maxComplementarity	10^{100}

Table 7: Parameters employed in the SDBP program.