Non-Abelian Anyons in the Quantum Hall Effect

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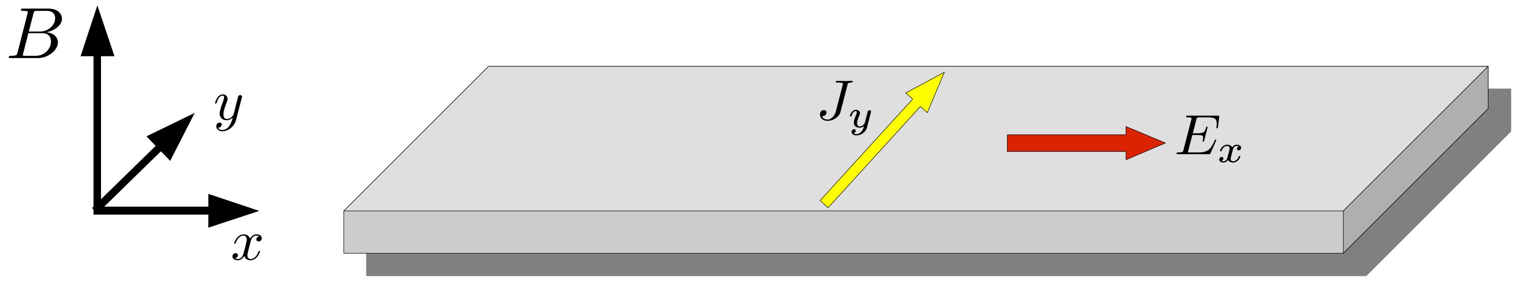
Outline

- Incompressible Hall fluids: bulk & edge excitations
- CFT description
- Non-Abelian statistics
- Partition function
- Signatures of non-Abelian statistics:
  - Coulomb blockade & thermopower

Quantum Hall Effect

- 2 dim electron gas at low temperature T ~ 10 mK

  and high magnetic field B ~ 10 Tesla



- Conductance tensor $J_i = \sigma_{ij} E_j, \quad \sigma_{ij} = R^{-1}_{ij}, \qquad i,j = x,y$
- Plateaux: $\sigma_{xx} = 0, \ R_{xx} = 0$ no Ohmic conduction → gap

$$\sigma_{xy} = R^{-1}_{xy} = \frac{e^2}{h}\nu, \quad \nu = 1(\pm 10^{-8}), 2, 3, \ldots \frac{1}{3}, \frac{2}{5}, \ldots, \frac{5}{2},$$

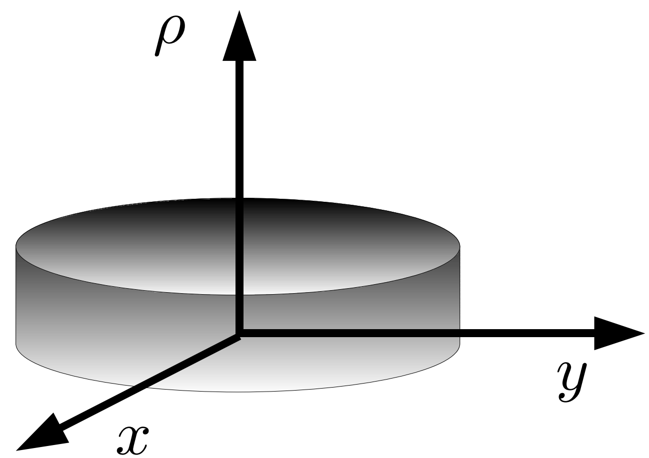
- High precision & universality
- Uniform density ground state: $\rho_o = \frac{eB}{hc}\nu$

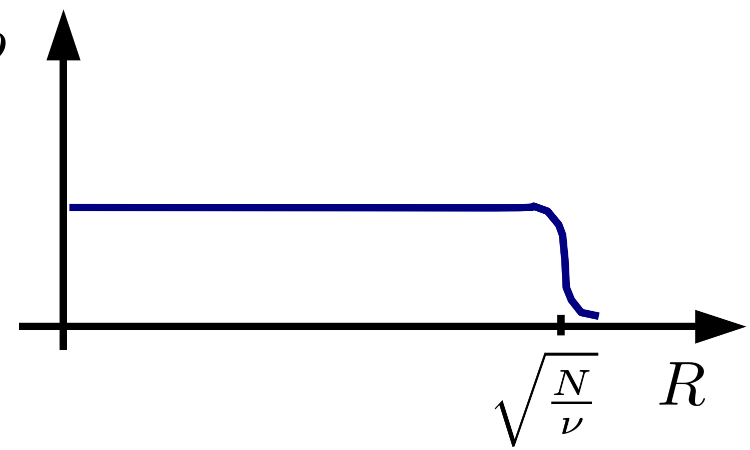
$$\boxed{\text{Incompressible fluid}}$$

Laughlin's quantum incompressible fluid

Electrons form a droplet of fluid:

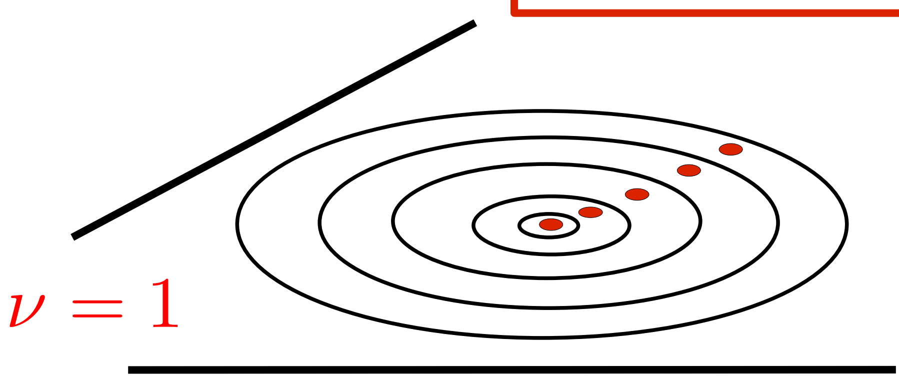
- incompressible = gap
- fluid = $\rho(x, y) = \rho_o = \text{const.}$

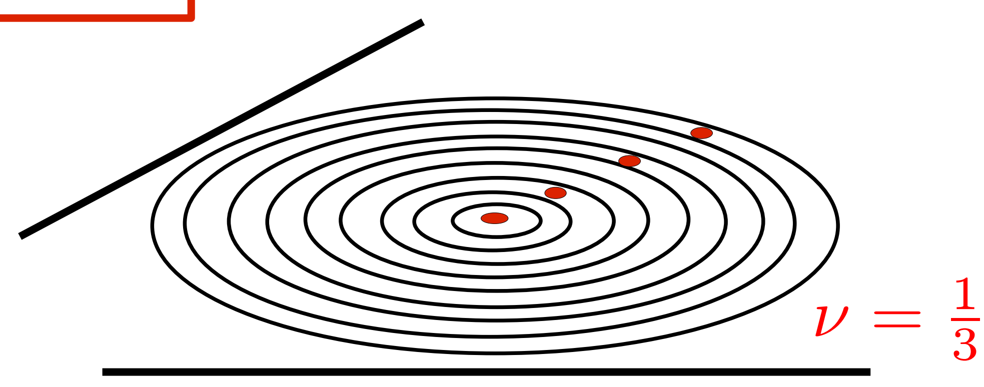




$\mathcal{D}_A = BA/\Phi_o,$ # degenerate orbitals = # quantum fluxes, $\Phi_o = \frac{hc}{e}$

filling fraction: $\boxed{\nu = \dfrac{N}{\mathcal{D}_A} = 1, 2, \ldots \dfrac{1}{3}, \dfrac{1}{5}, \ldots}$ density for quantum mech.





Laughlin's wave function

$$\boxed{\Psi_{gs}(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^{2k+1} e^{-\sum |z_i|^2/2}} \qquad \nu = \frac{1}{2k+1} = 1, \frac{1}{3}, \frac{1}{5}, \ldots$$

- $\nu = 1$ filled Landau level: obvious gap $\omega = \frac{eB}{mc} \gg kT$
- $\nu = \frac{1}{3}$ non-perturbative gap due to Coulomb interaction

→ effective theories

- quasi-hole = elementary vortex $\Psi_\eta = \prod_i (\eta - z_i)\, \Psi_{gs}$

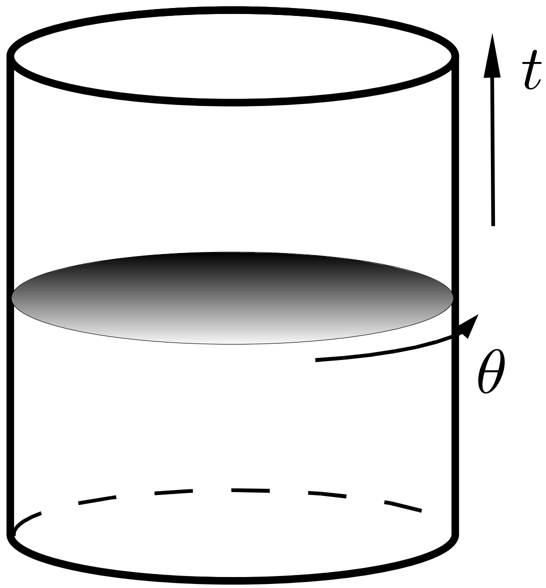
→ fractional charge $Q = \frac{e}{2k+1}$ & statistics $\frac{\theta}{\pi} = \frac{1}{2k+1}$

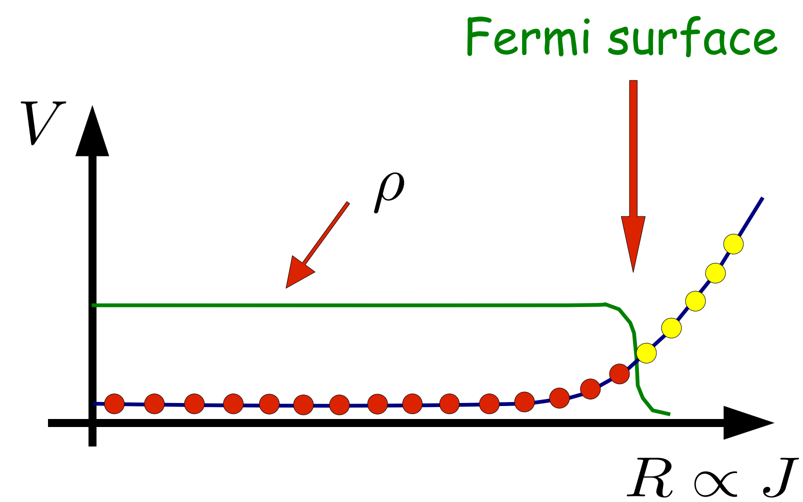
$$\Psi_{\eta_1,\eta_2} = (\eta_1 - \eta_2)^{\frac{1}{2k+1}} \prod_i (\eta_1 - z_i)(\eta_2 - z_i)\, \Psi_{gs}$$

$\boxed{\text{Anyons}}$ vortices with long-range topological correlations

Conformal field theory of edge excitations

The edge of the droplet can fluctuate: edge waves are massless

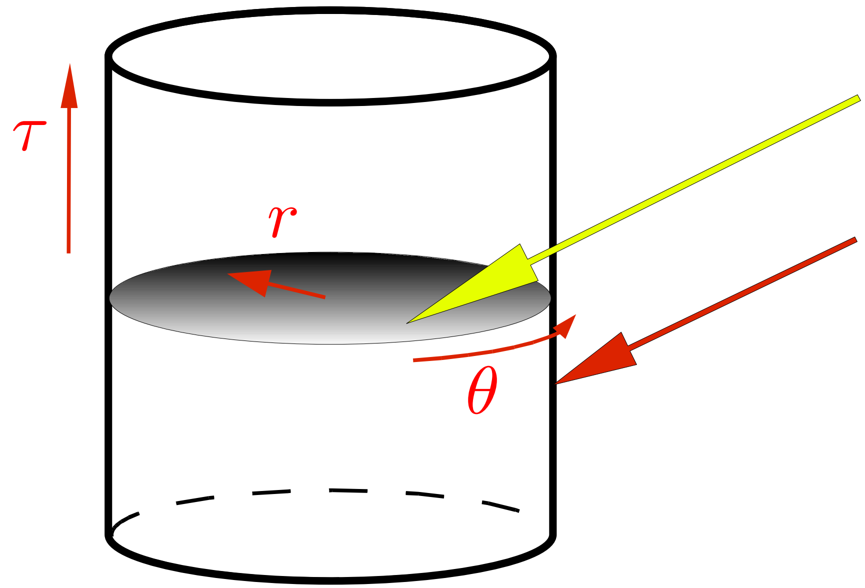




- edge ~ Fermi surface: linearize energy $\varepsilon(k) = \frac{v}{R}(k - k_F), \quad k = 0, 1, \ldots$
- relativistic field theory in 1+1 dimensions, chiral (X.G.Wen '89)

→ chiral compactified c=1 CFT (chiral Luttinger liquid)

CFT descriptions of QHE



$z = re^{i\theta}$ plane (bulk excit)

$\zeta = e^{\tau} e^{i\theta}$ cylinder (edge excit)

$\langle\ \phi_1 \quad \phi_2\ \rangle_{CFT}$

$(z_1 - z_2)^{\alpha^2}$ anyon wavefunction

$(\zeta_1 - \zeta_2)^{\alpha^2}$ edge-excit. correlator

- same function by analytic continuation from the circle:
  - both equivalent to Chern-Simons theory in 2+1 dim (Witten '89)
- c=1 Luttinger CFT:
  - wavefunctions: spectrum of anyons and braiding
  - edge correlators: physics of conduction experiments

Non-Abelian fractional statistics

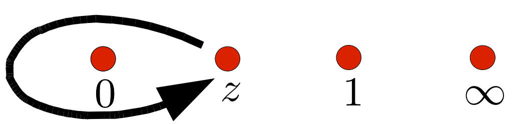
- $\nu = \frac{5}{2}$ described by Moore-Read "Pfaffian state" ~ Ising CFT x boson
- Ising fields: I identity, ψ Majorana = electron, σ spin = anyon
- fusion rules:
  - $\psi \cdot \psi = I$ 2 electrons fuse into a Bosonic bound state
  - $\boxed{\sigma \cdot \sigma = I + \psi}$ 2 channels of fusion = 2 conformal blocks

$$\langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle = a_1 F_1(z) + a_2 F_2(z)$$ Hypergeometric

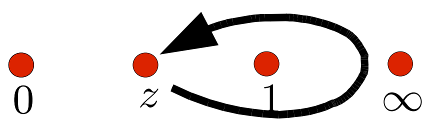
$\Longrightarrow$ state of 4 anyons is two-fold degenerate (Moore, Read '91)

- statistics of anyons ~ analytic continuation $\Longrightarrow$ 2x2 matrix

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (z e^{i2\pi}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (z)$$



$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} ((z-1) e^{i2\pi}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} (z)$$



(all CFT redone for Q. Computation: M. Freedman, Kitaev, Nayak, Slingerland,...., 00'-10')

Topological quantum computation

- qubit = two-state system $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$
- QC: perform $U(2^n)$ unitary transformations in n qubit Hilbert space
- Proposal: (Kitaev; M. Freedman; Nayak; Simon; Das Sarma '06)

> use non-Abelian anyons for qubits and operate by braiding

4-spin system $\alpha|F_1\rangle + \beta|F_2\rangle$ is 1 qubit (2n-spin has dim 2^{n-1})

- anyons topologically protected from decoherence (local perturbations)
- more stable but more difficult to create and manipulate

→ great opportunity

→ new experiments and model building

Models of non-Abelian statistics

- Study Rational CFTs with non-Abelian excitations:
  - best candidate: Pfaffian & its generalization, the Read-Rezayi states

$$\nu = 2 + \frac{k}{k+2}, \quad \begin{cases} k = 2, 3, \ldots \\ M = 1 \end{cases} \qquad U(1)_{k+2} \times \frac{SU(2)_k}{U(1)_{2k}}$$

  - alternatives: other (cosets of) non-Abelian affine groups $U(1) \times \frac{G}{H}$

- Identify their N sectors of fractional charge and statistics
  - Abelian (electron) & non-Abelian (quasi-particles)

- Compute physical quantities that could be signatures of non-Abelian statistics:
  - Coulomb blockade conductance peaks
  - thermopower & entropy

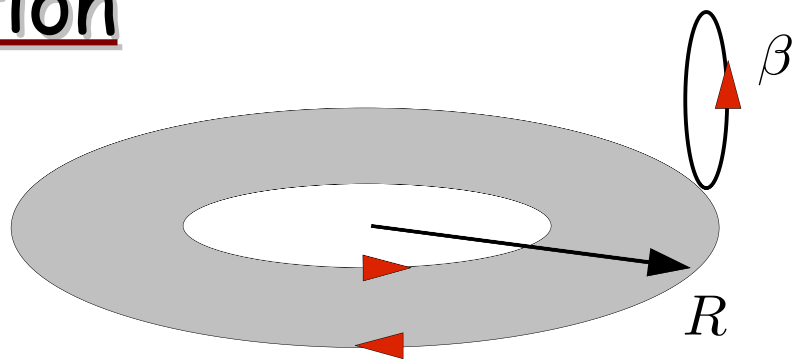
use partition function

- quantity defining Rational CFT (Cardy '86; many people)
- complete inventory of states (bulk & edge)
- modular invariance as building principle:
  - S matrix and fusion rules
  - further modular conditions for charge spectrum
  - straightforward solution for any non-Abelian state $U(1) \times \dfrac{G}{H}$
  - useful to compute physical quantities
- Inputs:
  - non-Abelian RCFT (i.e. $\dfrac{G}{H}$)
  - Abelian field representing the electron "simple current"
- Output is unique

Annulus partition function

$$i2\pi\ \tau = -\beta\frac{v}{R} + it, \quad \beta = \frac{1}{k_B T}$$

$$i2\pi\zeta = \beta(-V_o + i\mu)$$



$$Z_{\text{annulus}} = \sum_{\lambda=1}^{p} |\theta_\lambda(\tau,\zeta)|^2, \quad \theta_\lambda(\tau,\zeta) = \mathrm{Tr}_{\mathcal{H}^{(\lambda)}}\left[e^{i2\pi\tau(L_0 - c/24) + i2\pi\zeta Q}\right]$$

modular invariance conditions

geometrical properties & physical interpretation (A. C., Zemba, '97)

$T^2: \ Z(\tau+2,\zeta) = Z(\tau,\zeta),$ $\quad L_0 - \overline{L}_0 = \frac{n}{2}$ half-integer spin excitations globally

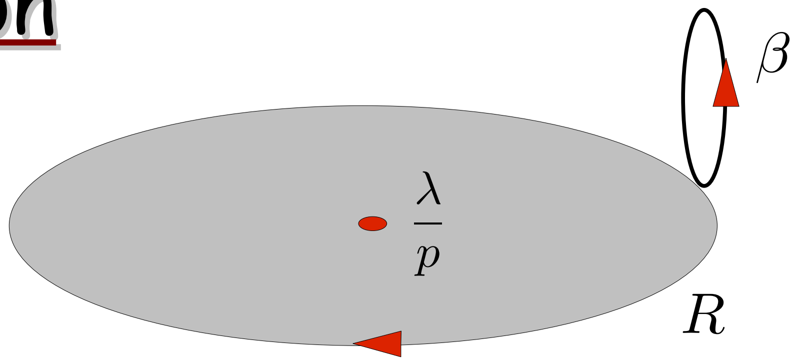
$S: \ Z\left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) = Z(\tau,\zeta),$ completeness $\quad \theta_\lambda\left(\frac{-1}{\tau}\right) = \sum_{\lambda'} S_{\lambda\lambda'}\theta_{\lambda'}(\tau)$ S matrix

$U: \quad Z(\tau,\zeta+1) = Z(\tau,\zeta),$ $\quad Q - \overline{Q} = n$ integer charge excitations globally

$V: \quad Z(\tau,\zeta+\tau) = Z(\tau,\zeta),$ $\quad \Delta Q = \nu$ add one flux: spectral flow

$$\theta_\lambda(\zeta+\tau) \sim \theta_{\lambda+1}(\tau)$$

Disk partition function



Annulus -> Disk (w. bulk q-hole $\bar{Q} = \frac{\lambda}{p}$)

$$Z_{\text{annulus}} \;\to\; Z_{\text{disk},\ \lambda} = \theta_\lambda(\tau, \zeta)$$

$$\theta_\lambda(\tau,\zeta) = K_\lambda(\tau,\zeta;p) = \frac{1}{\eta}\sum_n e^{i2\pi\left[\tau\frac{(np+\lambda)^2}{2p} + \zeta\frac{np+\lambda}{p}\right]}, \quad \nu = \frac{1}{p}, \quad c = 1$$

- $U: \; Q - \overline{Q} = n$ sectors with charge $Q = \frac{\lambda}{p} + n$

  basic quasiparticle + n electrons

- T^2 : electrons have half-integer dimension (=J),

  and integer relative statistics with all excitations

- # sectors $p = dim\,(S_{\lambda\lambda'})$ = Wen's topological order

→ <u>we recover phenomenological conditions on the spectrum</u>

Pfaffian & Read-Rezayi states

$$Z^{RR}_{annulus} = \sum_{\ell=0}^{k} \sum_{a=0}^{\hat{p}-1} \left| \, \theta^{\ell}_{a}(\tau,\zeta) \, \right|^2 \ , \qquad Z^{RR}_{disk} = \theta^{\ell}_{a} = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)} \; \chi^{\ell}_{a+2\beta}$$

Ex: Pfaffian (k=2)

ground state + electrons

$$Z^{\mathrm{Pfaffian}}_{\mathrm{annulus}} = |K_0 I + K_4 \psi|^2 \ + |K_0 \psi + K_4 I|^2 \ + |(K_1 + K_{-3})\,\sigma|^2$$
$$+ |K_2 I + K_{-2} \psi|^2 \ + |K_2 \psi + K_{-2} I|^2 \ + |(K_3 + K_{-1})\,\sigma|^2$$

non-Abelian quasiparticle

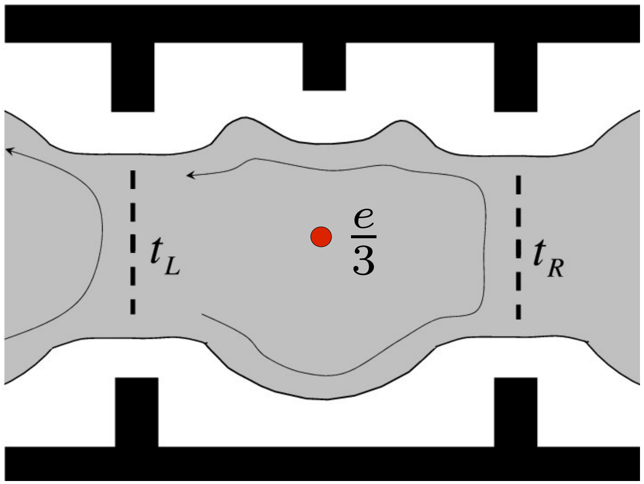
- K_λ charge parts $Q = \dfrac{\lambda}{4} + 2n$
- $I, \ \psi, \ \sigma$ Ising parts (Majorana fermion)
  - 6 sectors
  - also $Q = 0, \pm\dfrac{1}{2}$ Abelian excitations

- charge and neutral q. #'s are coupled by "parity rule"
- but S-matrix of θ_a^ℓ is factorized $S_{a\ell,a'\ell'} \sim e^{i2\pi aa'N/M} s_{\ell\ell'}$

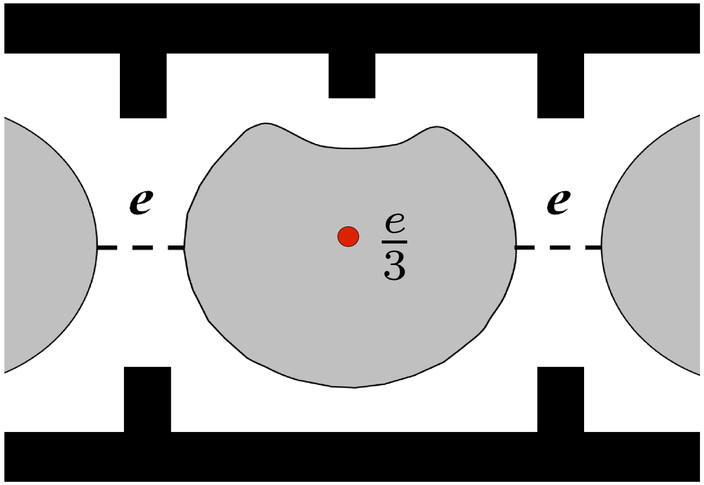
- generalization to other N-A models: (A.C, G. Viola, '10)
  - Non-Abelian Fluids $U(1) \times SU(2)_k$
  - Anti-Read-Rezayi $U(1) \times \overline{SU(2)_k}$
  - Bonderson-Slingerland $U(1) \times \mathrm{Ising} \times SU(n)_1$
  - N-A Spin Singlet state $U(1)_q \times U(1)_s \times \dfrac{SU(3)_k}{U(1)^2}$

→ unique result once RCFT and electron field have been chosen

Experiments on non-Abelian statistics



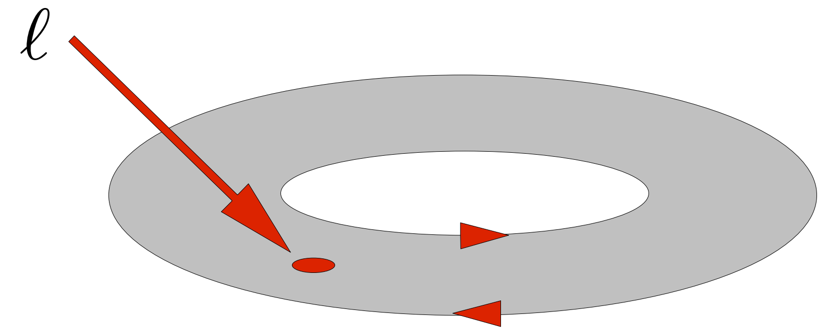
(a)



(b)

- (a) interference of edge waves (Chamon et al. '97; Kitaev et al 06)
  - ➡ Aharonov-Bohm phase, checks fractional statistics
  - – experiment is hard (Goldman et al. '05; Willett et al '09)
- (b) electron tunneling into the droplet (Stern, Halperin '06)
  - ➡ Coulomb blockade conductance peaks (Ilan, Grosfeld, Schoutens, Stern '08)
  - – check quasi-particle sectors (Stern et al.; A.C. et al. '09 - '10)
- ➡ Thermopower (Cooper, Stern; Yang, Halperin '09; Chickering et al. '10)

Thermopower



- fusion of n non-Abelian quasiparticles:
  - multiplicity $\sim (d_\ell)^n, \quad n \to \infty$
- Entropy: $\mathcal{S}(T=0) \sim n \log(d_\ell),$ $\quad d_\ell = \dfrac{s_{\ell 0}}{s_{00}} > 1$ (quantum dimension)

from Z: $\mathcal{S} = \left(1 - \tau \dfrac{d}{d\tau}\right) \log\left[\theta_a^\ell(\tau,\zeta)\overline{\theta_0^0(\tau,\zeta)}\right] \sim \log \dfrac{s_{\ell 0}}{s_{00}} + \mathcal{S}_{\text{Edge}}, \quad \tau \sim \dfrac{\beta}{R} \to 0$

- put temperature ∇T and potential ∇V_o gradients between the edges
- at equilibrium: $J = -\sigma\,\nabla V_o - \alpha\,\nabla T = 0$
- thermopower

$$\boxed{S_{\text{Seebeck}} = \frac{\alpha}{\sigma} = -\frac{\nabla V_o}{\nabla T} = \frac{\mathcal{S}}{eN_e}}$$

(Cooper,Stern; Yang, Halperin '09)

- it could be observable by varying B off the plateau center

$$S_{\text{Seebeck}} = \left|\frac{B - B_o}{e^* B_0}\right| \log(d_1)$$

(Chickering et al '10)

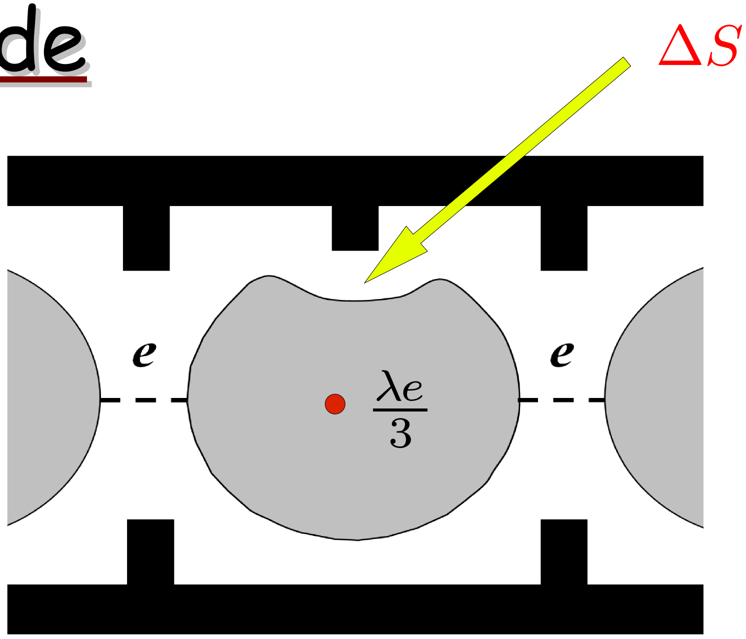
Coulomb blockade

- Droplet capacity stops the electron
- Bias & T ~ 0: needs energy matching

$$E(n+1, S) = E(n, S)$$

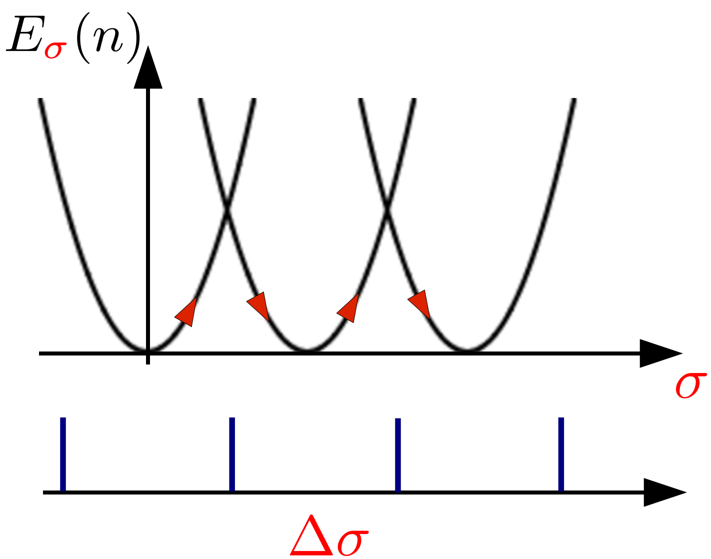
$\Rightarrow$ current peak

- energy deformation by $\Delta S \sim \Delta Q_{\mathrm{bkg}}$



$$E(n, S) = \frac{v}{R} \frac{(\lambda + pn - \sigma)^2}{2p} \propto (Q - Q_{\mathrm{bkg}})^2$$

$$\Delta\sigma = \frac{B \Delta S}{\Phi_o} = \frac{1}{\nu} \ , \qquad \Delta S = \frac{e}{n_o}$$



$U(1)$ $\Rightarrow$ equidistant peaks

$U(1) \times \frac{G}{H}$ $\Rightarrow$ modulated pattern

$$\boxed{\Delta\sigma_m^\ell = \frac{1}{\nu} + \frac{v_n}{v} \left(h_{m+2}^\ell - 2h_m^\ell + h_{m-2}^\ell \right)}$$

$$\frac{v_n}{v} \sim \frac{1}{10}$$

- compares states in the same sector
- spectroscopy of lowest CFT states
- $T = 0$: cannot distinguish NA state from "parent" Abelian state

$$\theta^{\ell}_{a} = \sum_{\beta=0}^{k-1} K_{a+\beta(k+2)}\ \chi^{\ell}_{a+2\beta}$$

(Bonderson et al. '10)

$T > 0$ corrections

$$\langle Q \rangle_T \sim \frac{\partial}{\partial V_o} \log \theta^{\ell}_{a}$$

- two scales: $0 < T_n < T_{ch}, \qquad T_n = \frac{v_n}{R}\ , \quad T_{ch} = \frac{v}{R} \sim 10\ T_n$

$$T < T_n : \quad \Delta\sigma^{\ell}_{m} = \cdots + \frac{T}{T_{ch}} \log\left(\frac{(d^{\ell}_{m})^2}{d^{\ell}_{m+2}\ d^{\ell}_{m-2}} \right),$$

d^{ℓ}_{m} multiplicity of neutral states

in (331) & Anti-Pfaff, not in Pfaff

$$T_n < T < T_{ch} : \qquad \cdots + \propto \frac{T}{T_{ch}}\ e^{-h^1_1 T/T_n}\ \frac{s_{\ell 1}}{s_{\ell 0}},$$

S matrix of non-Abelian part

→ test non-Abelian part of disk partition function

(Stern et al., Georgiev, AC et al. '09, '10)

$T > 0$ off-equilibrium

- energy offset $\Delta E_\sigma > 0$ and bias $\Delta V_o > 0$

  relevant regime: $T < \Delta V_o < \Delta E_\sigma$

→ thermal-activated Coulomb-Blockade conduction

$$\Gamma \sim d_a^\ell \left(\Delta E_\sigma^2 - \Delta V_o^2\right) e^{-\beta(\Delta E_\sigma - \Delta V_o)}$$

- real-time experiment of peak counting
- sensible to level multiplicity d_a^ℓ (but qualitative)

Conclusions

- non-Abelian anyons could be seen
- partition function:
  - it is simple enough
  - it defines the CFT, its sectors, fusion rules etc.
  - it is useful to compute observables
  - it can be the basis for further model building