Stability of 2D Topological Insulators

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Outline

- Chiral and non-chiral Topological States of Matter
- Stability of edge states of 2D Topological Insulators: $\mathbb{Z}_2$ anomaly
- Partition function: electromagnetic and gravitational (discrete) responses
- Stability of TI with interacting & non-Abelian edges

Topological States of Matter

- System with bulk gap but non-trivial at energies below the gap
- global effects and global degrees of freedom (edge states, g.s. degeneracy)
- described by topological field theory: Chern-Simons theory etc.
- quantum Hall effect is chiral (B field, chiral edge states)
- quantum spin Hall effect is non-chiral (edge states of both chiralities)
- other systems: Chern Insulators, Topological Insulators, Topological Superconductors, etc.
- Topological Band Insulators (free fermions) have been observed in 2 & 3 D

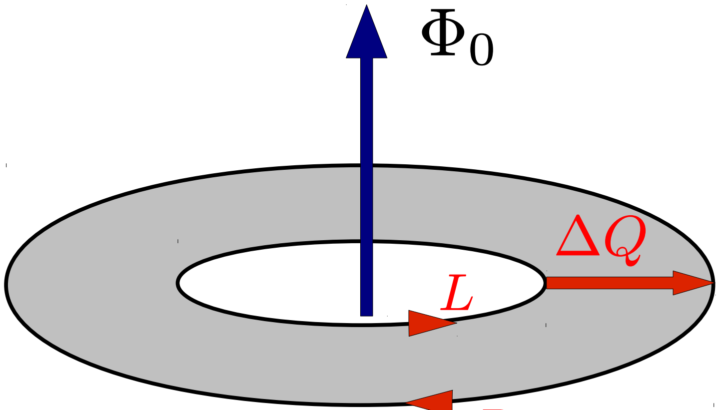
→ « fever »

Questions/Answers

- Q: systems of interacting fermions? (e.g. fractional insulators)
- A: use quantum Hall modelling and CFTs
- Q: but non-chiral edge states are stable?
- A: generically NO
- Q: stability with Time Reversal symmetry?
- A: YES/NO; there is a $\mathbb{Z}_2$ symmetry; if this is anomalous, they are stable

Chiral Topological States

Quantum Hall effect



- chiral edge states
- no Time-Reversal symmetry (TR)
- Laughlin's argument. $\nu = \frac{1}{3}$

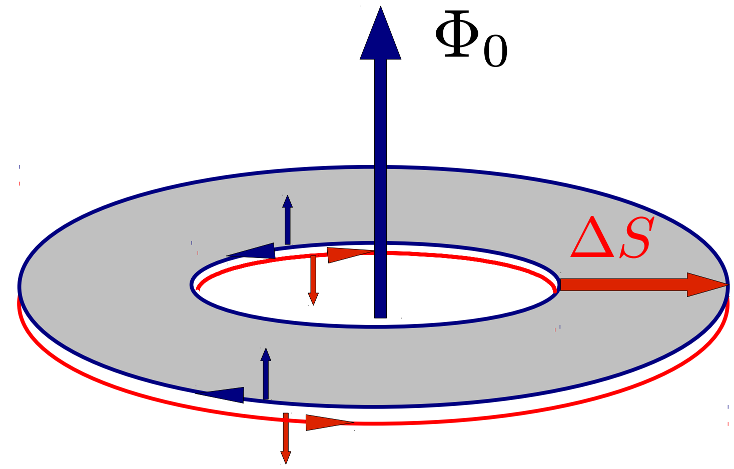
$$\Phi \to \Phi + \Phi_0, \quad H\left[\Phi + \Phi_0\right] = H\left[\Phi\right]$$

$$Q_R \to Q_R + \Delta Q = \nu, \qquad \Delta Q = \int dtdx\ \partial_t J^0_R = \nu \int F = \nu\, n \qquad \text{chiral anomaly}$$

- $\Phi \to \Phi + n\Phi_0$ spectral flow between charge sectors $\{0\} \to \{\frac{1}{3}\} \to \{\frac{2}{3}\} \to \{0\}$
- edge chiral anomaly = response of topological bulk to e.m. background
- chiral edge states cannot be gapped ⟷ topological phase is stable
- anomalous response extended to other systems and anomalies in any D=1,2,3,......

(S. Ryu, J. Moore, A. Ludwig '10)

Non-chiral topological states



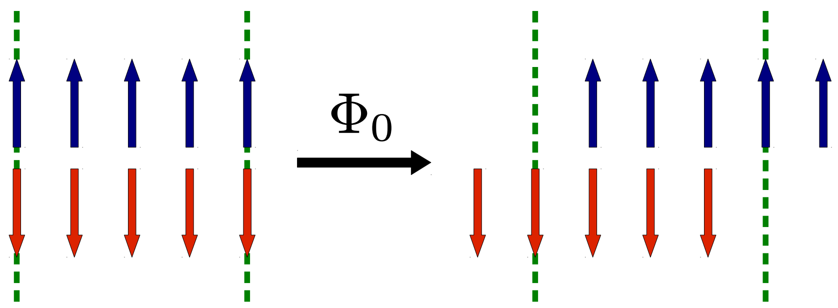
Quantum Spin Hall Effect

- take two $\nu = 1$ Hall states of spins $\uparrow\downarrow$
- system is Time Reversal invariant:

$$\mathcal{T}: \ \psi_{k\uparrow} \to \psi_{-k\downarrow} \ , \qquad \psi_{k\downarrow} \to -\psi_{-k\uparrow}$$

- non-chiral CFT with $U(1)_Q \times U(1)_S$ symmetry
- adding flux pumps spin → $U(1)_S$ anomaly

(X-L Qi, S-C Zhang '08)



$$\Delta Q = \Delta Q^{\uparrow} + \Delta Q^{\downarrow} = 1 - 1 = 0$$

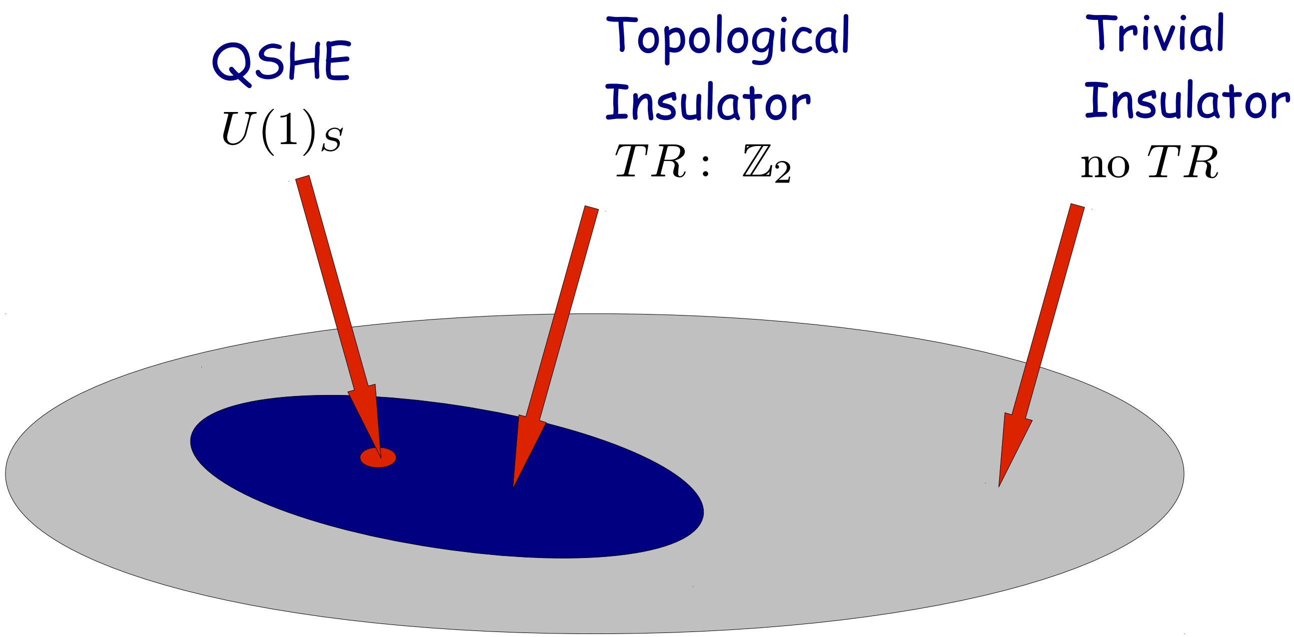
$$\Delta S = \tfrac{1}{2} - \left(-\tfrac{1}{2}\right) = 1$$

$$\Delta S = \Delta Q^{\uparrow} = \nu^{\uparrow} = 1$$

- in Top. Insulators $U(1)_S$ is explicitly broken by Spin-Orbit Coupling etc.
- no currents $\sigma_H = \sigma_{sH} = \kappa_H = 0$
- but TR symmetry keeps $\mathbb{Z}_2$ symmetry $(-1)^{2S}$

Kramers theorem

Symmetry Protected Topological Phases



- QSHE edge theory is used to describe Topological Insulator with Time-Reversal symmetry $\quad U(1)_S \;\to\; \mathbb{Z}_2$ of $(-1)^{2S}$

  Main issue: stability of TI $\longleftrightarrow$ stability of non-chiral edge states

- e.g. TR symmetry forbids mass term in CFT with odd number of free fermions

$$\mathcal{T} : H_{\text{int.}} = m \int \psi_\uparrow^\dagger \psi_\downarrow + h.c. \;\to\; -H_{\text{int.}}$$

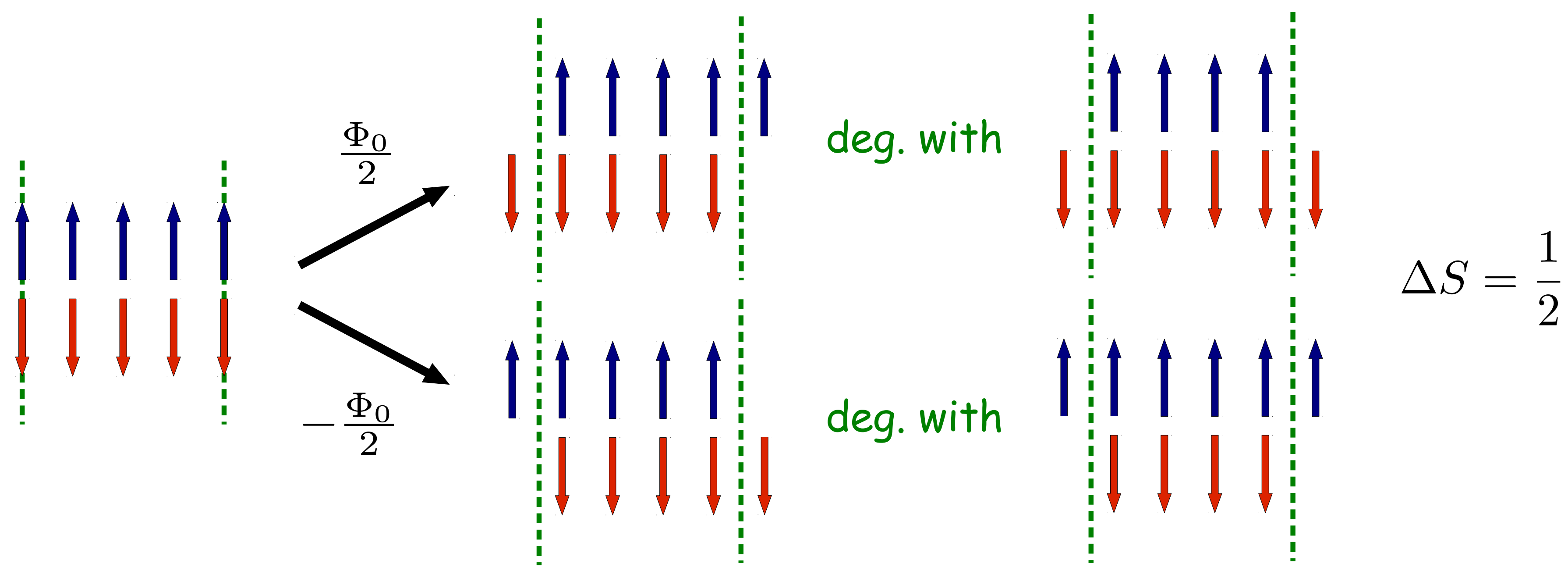
$\mathbb{Z}_2$ classification (free fermions)

Flux insertion argument

(Fu, Kane, Mele '05-06; Levin, Stern '10-13)

- TR symmetry: $\mathcal{T} H[\Phi] \mathcal{T}^{-1} = H[-\Phi]$ & $H[\Phi + \Phi_0] = H[\Phi]$
- TR invariant points: $\Phi = 0, \frac{\Phi_0}{2}, \Phi_0, \frac{3\Phi_0}{2}, \ldots$
- Kane et al. define a TR-invariant $\mathbb{Z}_2$ polarization (bulk quantity) that:

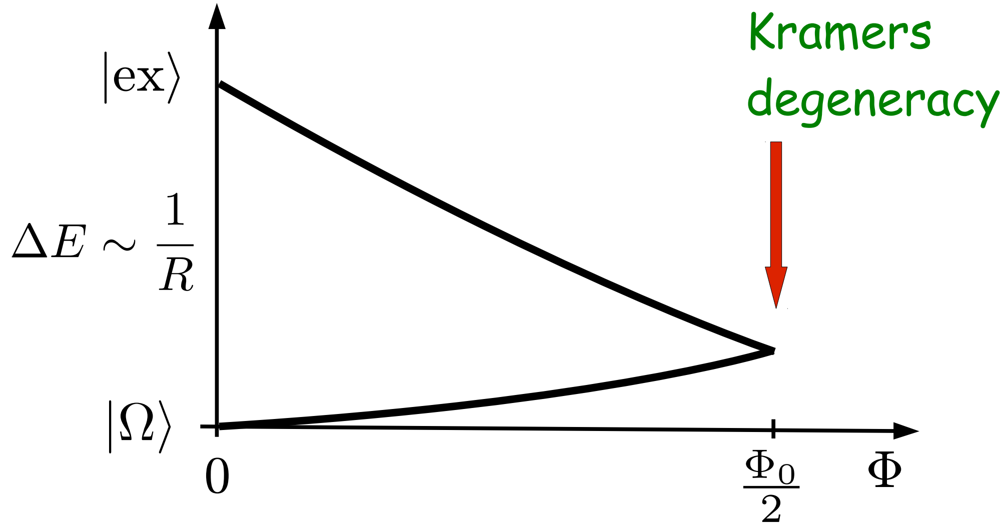
  - is topological and conserved by TR invariance
  - is equal to parity of edge spin $$(-1)^{2S} = (-1)^{N_\uparrow + N_\downarrow}$$
  - if $(-1)^{2S} = -1$ there exits a pair of edge states degenerate by Kramers theorem



$$\Phi = 0 : \quad (-1)^{2S}|\Omega\rangle = |\Omega\rangle$$

$$\Phi = \frac{\Phi_0}{2} : \; (-1)^{2S}|\Omega\rangle = -|\Omega\rangle$$

$|\mathrm{ex}\rangle$ gapless edge state



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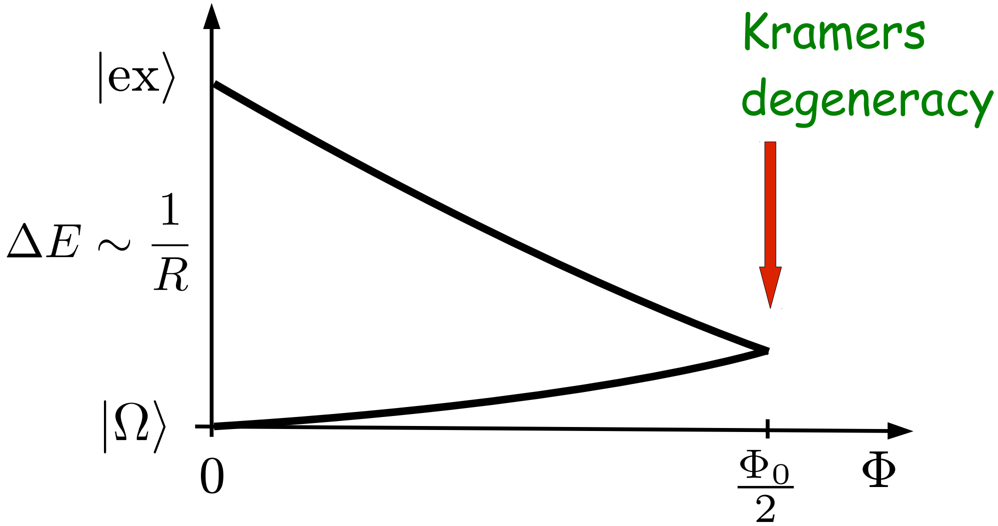

Conclusions

- topological phase is protected by TR symmetry if $\exists$ edge Kramers pair (N_f odd)
- spin parity is anomalous, discrete remnant of spin anomaly $U(1)_S \to \mathbb{Z}_2$

> Fu-Kane argument is Laughlin's argument for $\mathbb{Z}_2$ anomaly: $(-1)^{2\Delta S} = -1$

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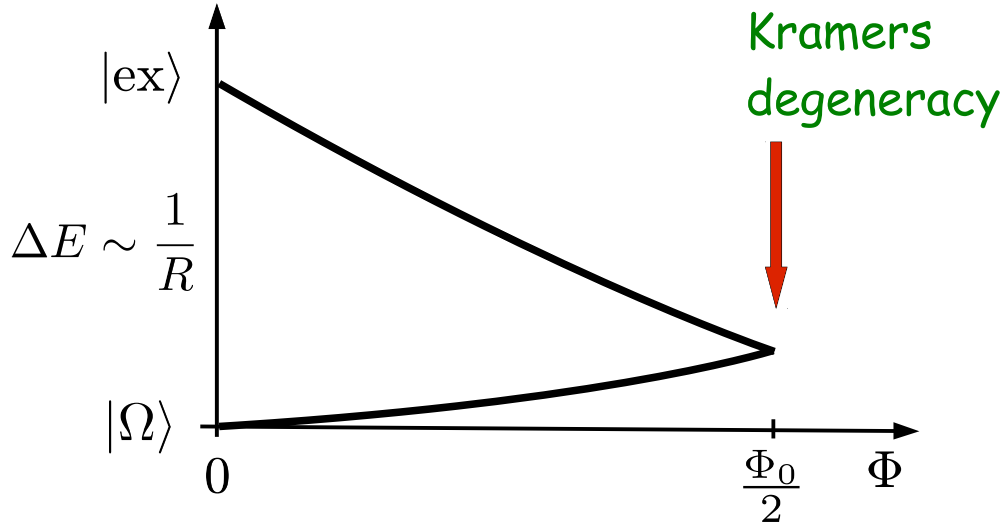

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Questions

- Can we extend this to Topological Insulators with interacting fermions?
- Can we use modular non-invariance, i.e. discrete gravitational anomaly of the partition function as another probe?

(S. Ryu, S.-C. Zhang '12)

Answers in this talk

- partition functions of edge states for QHE, QSHE and TI are completely understood (AC, Zemba '97; AC, Georgiev, Todorov '01; AC, Viola '11)
- we can study flux insertions and discuss stability
- modular transformations: e.m. & grav. responses

→ TI: $\mathbb{Z}_2$ classification extends to interacting & non-Abelian edges

$(-1)^{2\Delta S} = +1$ unstable, Z modular invariant

-1 stable, Z not modular invariant

$$2\Delta S = \frac{\sigma_{sH}}{e^*} = \frac{\nu^{\uparrow}}{e^*}$$

$$\frac{\text{spin-Hall conduct.} = \text{chiral Hall conduct.}}{\text{minimal fractional charge}}$$

(Levin, Stern)

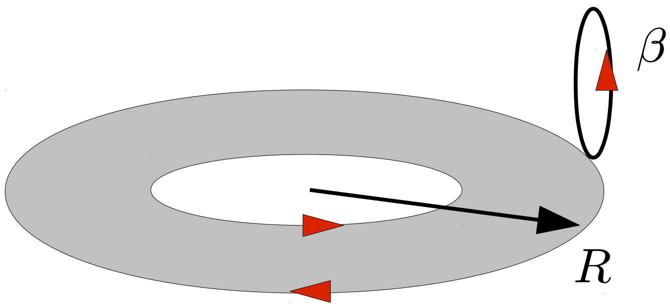
QHE partition function

Consider states of outer edge for $\nu = \frac{1}{p}, \ p \text{ odd}$ described by $c = 1$ CFT

$E \sim P \sim v\frac{L_0}{R}$ edge energy & momentum

$\tau = v\frac{i\beta}{2\pi R} + t,$ modular parameter

$\zeta = \frac{\beta}{2\pi}(iV_o + \mu)$ electric & chemical pot.



Partition function for one charge sector $Q = \frac{\lambda}{p} + n, \quad n \in \mathbb{Z}$

sum of characters for representations of $\widehat{U(1)}$ current algebra

$$K_\lambda(\tau, \zeta; p) = \mathrm{Tr}_{\mathcal{H}^{(\lambda)}}\left[\exp\left(i2\pi\tau L_0 + i2\pi\zeta Q\right)\right]$$
$$= \frac{1}{\eta(\tau)} \sum_n \exp\left(i2\pi\left(\tau\frac{(np+\lambda)^2}{2p} + \zeta\frac{np+\lambda}{p}\right)\right)$$

$\frac{\text{theta function}}{\text{Dedekind function}}$

$$K_{\lambda+p} = K_\lambda$$

Modular trasformations

$$K_\lambda(\tau,\zeta;p) = \frac{1}{\eta(\tau)} \sum_n \exp\left(i2\pi \left(\tau \frac{(np+\lambda)^2}{2p} + \zeta \frac{np+\lambda}{p} \right) \right), \qquad Q = \frac{\lambda}{p} + \mathbb{Z}$$

discrete coordinate changes respecting double periodicity

$$S : \tau \to -1/\tau$$

$$T : \tau \to \tau + 1$$

e.m. background changes:

$U : \zeta \to \zeta + 1$ adds the weight $e^{i2\pi Q}$

$V : \zeta \to \zeta + \tau$ adds a flux quantum $\Phi \to \Phi + \Phi_0,$

$S : \quad K_\lambda\left(\frac{-1}{\tau}, \frac{-\zeta}{\tau}\right) \sim \sum_{\mu=1}^{p} S_{\lambda\mu} K_\mu(\tau,\zeta)$ — unitary S matrix, completeness

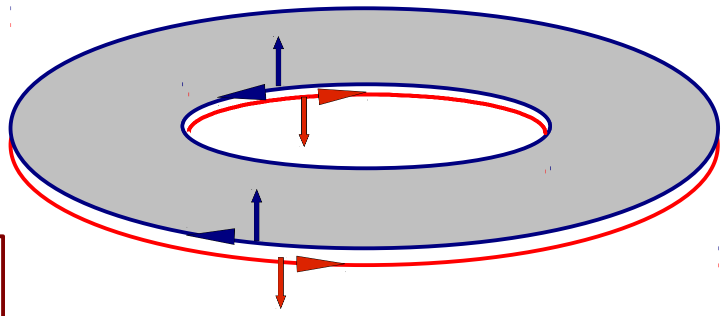
$T^2 : \quad K_\lambda(\tau+2,\zeta) \sim K_\lambda(\tau,z)$ — odd-integer electron statistics

$U : \quad K_\lambda(\tau,\zeta+1) \sim K_\lambda(\tau,z)$ — integer electron charge

$V : \quad K_\lambda(\tau,\zeta+\tau) \sim K_{\lambda+1}(\tau,z)$ — Φ_0 flux insertion: $Q \to Q + \nu$

Partition Function of Topological Insulators

Partition function for a single edge:



- combine two chiralities $K^{\uparrow}_{\lambda}\,\overline{K}^{\downarrow}_{\mu}$

$$Z^{NS}(\tau,\zeta) = \sum_{\lambda=1}^{p} K^{\uparrow}_{\lambda}\,\overline{K}^{\downarrow}_{-\lambda}, \quad S, T^2, U, V \text{ invariant}$$

In fermionic systems there are always four sectors of the spectrum:

$$NS,\ \widetilde{NS},\ R,\ \widetilde{R},\ \text{ risp. } (AA),\ (AP),\ (PA),\ (PP)$$

Ramond sector describes half-flux insertions: $\frac{\Phi_0}{2}, \frac{3\Phi_0}{2}, \ldots$

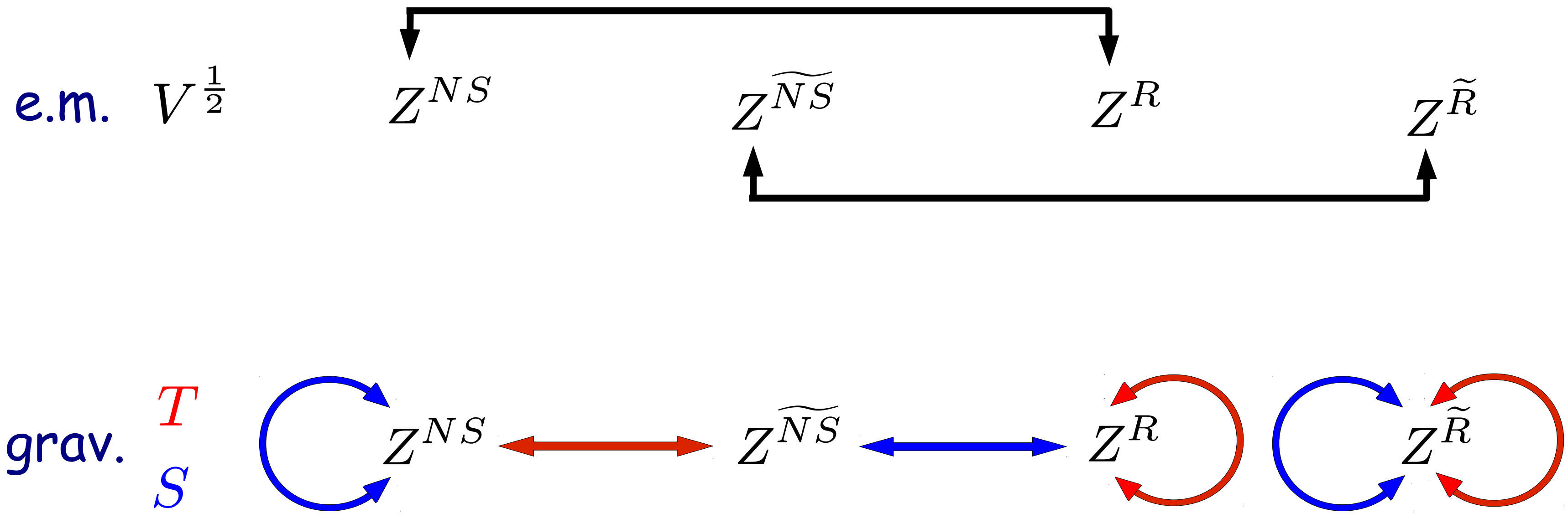
$$V^{\frac{1}{2}} : Z^{NS}(\tau,\zeta) \to Z^{NS}\left(\tau, \zeta + \frac{\tau}{2}\right) \sim Z^{R}(\zeta,\tau)$$

add half flux

Each sector has $\lambda = 1, \ldots, p$ vacua for the would-be fractional charges

$$Z^{R} = \sum_{\lambda=1}^{p} \widehat{K}^{\uparrow}_{\lambda}\overline{\widehat{K}}^{\downarrow}_{-\lambda}, \qquad \widehat{K}_{\lambda}(\tau,\zeta) \sim K_{\lambda}\left(\tau, \zeta + \frac{\tau}{2}\right)$$

E.m. & gravitational responses



Modular invariant partition function of a single TI edge can be found

$$Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}, \qquad S, T, U, V^{\frac{1}{2}} \quad \text{invariant}$$

But: is Z_{Ising} consistent with TR symmetry?

Stability and modular (non)invariance

- p fluxes are needed to create one electron in the same charge sector

$$V^p : K^{\uparrow}_{\lambda} \to K^{\uparrow}_{\lambda+p} = K^{\uparrow}_{\lambda}, \qquad \Delta Q^{\uparrow} = \frac{p}{p} = 1 \qquad\qquad \nu = \frac{1}{p}$$

- Flux argument: add $\frac{p}{2}$ fluxes and check if $(-1)^{2\Delta S} = -1$ i.e. Kramers pair

$$V^{\frac{p}{2}} : \ K_0 \to K_{p/2} \sim K^R_0, \qquad |\Omega\rangle_{NS} \to |\Omega\rangle_R , \qquad \Delta Q^{\uparrow} = \Delta S = \tfrac{1}{2}$$

$$(-1)^{2S}|\Omega\rangle_{NS} = |\Omega\rangle_{NS} \quad \longrightarrow \quad (-1)^{2S}|\Omega\rangle_R = -|\Omega\rangle_R \qquad \text{stable TI}$$

- Spin parities of Neveu-Schwarz and Ramond ground states are different

→ $\mathbb{Z}_2$ spin-parity anomaly recovered

→ $Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^R + Z^{\widetilde{R}}$ not consistent with TR symmetry

$(-1)^{2S} = 1 \qquad 1 \qquad -1 \qquad -1$

TR symmetry + anomaly → no modular invariance → stable insulator

TR symmetry + modular invariance → no anomaly → trivial insulator

General stability analysis

- Edge theory involves neutral excitations (possibly non-Abelian)
- electron is Abelian: "simple current" modular invariant (A.C., Viola '11)
- fractional charge sectors $\Theta^{\alpha}_{\lambda}(\tau,\zeta)$ parametrized by two integers (k,p)

$$\Theta^{\alpha}_{\lambda}(\tau,\zeta) = \sum_{a=1}^{k} K_{\lambda+ap}(\tau, k\zeta; kp)\ \chi^{\alpha}_{\lambda+ap}(\tau,0)$$ = {g.s.} + {1 el.} + {2 el.} +......

charged (K), neutral (χ)

- minimal charge: $Q = \frac{k\lambda}{kp}, \quad \lambda = 1, \quad e^{*} = \frac{1}{p}$
- Hall current: $V: \zeta \to \zeta + \tau, \quad \lambda \to \lambda + k, \quad \Delta Q = \nu^{\uparrow} = \frac{k}{p}$
- construct TI partition function as before $Z^{NS} = \sum_{\lambda\alpha} \Theta^{\alpha}_{\lambda} \overline{\Theta}^{\alpha}_{-\lambda}$
- Stability: add fluxes to create an excitation in the same charge sector

$$V^{\frac{p}{2}}: \quad \Delta S = \Delta Q^{\uparrow} = \frac{p}{2}\ \nu^{\uparrow} = \frac{k}{2}$$ Kramers pairs if k odd → stable TI

Levin-Stern index $2\Delta S = \frac{\nu^{\uparrow}}{e^*}, \quad (-1)^{2\Delta S} = (-1)^{k}$ fully general

Analysis of modular invariance vs. stability can be extended to any edge CFT

k even = unstable, TR-inv. modular invariant $Z_{\rm Ising} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}$

k odd = stable, modular vector $Z = \left(Z^{NS}, Z^{\widetilde{NS}}, Z^{R}, Z^{\widetilde{R}}\right)$

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Examples

- Jain-like TI $\nu^{\uparrow} = \frac{k}{2nk+1}$, $e^* = \frac{1}{2nk+1}$, $2\Delta S = \frac{\nu^{\uparrow}}{e^*} = k$ stable / unstable

- (331) & Pfaffian TI $\nu^{\uparrow} = \frac{1}{2}$, $e^* = \frac{1}{4}$, $2\Delta S = 2$ unstable

- Abelian TI $K = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}$ $\nu^{\uparrow} = \frac{3}{7}$, $e^* = \frac{1}{7}$, $2\Delta S = 3$ stable

- Read-Rezayi TI $\nu^{\uparrow} = \frac{k}{kM+2}$, $e^* = \frac{1}{kM+2}$, $2\Delta S = k$ stable / unstable

Remarks

- general expression of partition function allows to extend Levin-Stern stability criterium to any TI with interacting fermions

$\mathbb{Z}_2$ classification of TI protected by TR invariance

- neutral states are left invariant by flux insertions

- unprotected edge states do become fully gapped?

  - Abelian states: yes, by careful analysis of possible TR-invariant interactions

  (Levin, Stern; Neupert et al.; Y-M Lu, Vishwanath)

  - non-Abelian states: yes, use projection from "parent" Abelian states

  e.g. (331) -> Pfaffian because [projection, TR-symm.]=0 (A.C., to appear)

Conclusions

- $\mathbb{Z}_2$ spin parity anomaly characterizes Topological Insulators protected by Time Reversal symmetry (cf. Ringel, Stern; Koch-Janusz, Ringel)
- anomaly signalled by index $(-1)^{2\Delta S} = -1, \qquad 2\Delta S = \frac{\nu^{\uparrow}}{e^*} = \frac{k}{p}$
- Pfaffian TI is unstable
- To do:

  - explicit form of interactions gapping the edge of non-Abelian states (done)
  - stability of Topological Superconductors ⟷ Ryu-Zhang stability criterium
  - stability 3D systems and 2D-3D systems (cf. arxiv: 1306.3238, 3250, 3286)

Gapping interactions for Pfaffian TI

- Gapping interactions for Abelian states defined by K matrix

$$U_\alpha = \exp\left(i\Lambda_\alpha^T K \Phi_\uparrow - i\overline{\Lambda}_\alpha^T K \overline{\Phi}_\downarrow\right) \;+\; \text{h.c.} \qquad \alpha = 1, \dots, n = c$$

- For (331) state, they can be written in terms of Weyl fermions fields

$$U_1 = \Psi^\dagger_{\uparrow 1}\Psi_{\uparrow 2}\Psi^\dagger_{\downarrow 1}\Psi_{\downarrow 2} \;+\; \text{h.c.}$$

$$U_2 = \Psi^\dagger_{\uparrow 1}\Psi^\dagger_{\uparrow 2}\Psi_{\downarrow 1}\Psi_{\downarrow 2} \;+\; \text{h.c.}$$

- Projection (331) $\rightarrow$ Pfaffian, i.e. to identical layers $\Psi^\dagger_{\uparrow 1} \sim \Psi^\dagger_{\uparrow 2} \;\rightarrow\; V\chi$

$U_1 =: \chi\partial\chi : : \bar{\chi}\bar{\partial}\bar{\chi} :$ neutral

$U_2 =: V^2\, \bar{V}^2 :, \qquad V = e^{i\sqrt{2}\phi}$ charged field

- U_1 couples to fermion field, U_2 to charges modes, giving both mass
- Analysis extends to Read-Rezayi states and Ardonne-Schoutens NASS states

Topological Superconductors

- stability of TS ⟷ gapless non-chiral Majorana edge states
- N_f free Majorana, they are neutral → no flux insertion argument
- $\mathbb{Z}_2 \times \mathbb{Z}_2$ chiral spin parity $(-1)^{2S^\uparrow}(-1)^{2S^\downarrow} = (-1)^{N_\uparrow}(-1)^{N_\downarrow}$ no spin flip

  $\mathbb{Z}$ classification (free fermions)

- $N_f = 8$ unstable by non-trivial quartic interaction: $\mathbb{Z} \to \mathbb{Z}_8$ (Kitaev, many people)

  → proposal: study partition function and gravitational response (Ryu, S-C-Zhang)

- Standard invariant for any N_f

  $Z_{\text{Ising}} = Z^{NS} + Z^{\widetilde{NS}} + Z^{R} + Z^{\widetilde{R}}$ Is it consistent with $\mathbb{Z}_2 \times \mathbb{Z}_2$ parity?

- Yes, for $N_f = 8$: mod. trasf. $ST \sim V^{\frac{1}{2}}$ creates $\Delta S^\uparrow = \Delta S^\downarrow = 1$ in R sector OK
- Ryu-Zhang: test modular invariance of subsector $(-1)^{N_R} = (-1)^{N_L} = 1$

  → GSO projection $N_f = 8$

- general analysis of modular invariance + discrete symm. not understood yet
- many modular invariants are possible without charge matching $Z = \sum \mathcal{N}_{\lambda\mu} K^\uparrow_\lambda \overline{K}^\downarrow_\mu$