Anyon Physics

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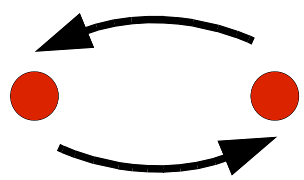
Outline

- Anyons & topology in 2+1 dimensions
- Chern-Simons gauge theory: Aharonov-Bohm phases
- Quantum Hall effect: bulk & edge excitations
- measure of fractional charge & statistics
- non-Abelian fractional statistics

  & topological quantum computation

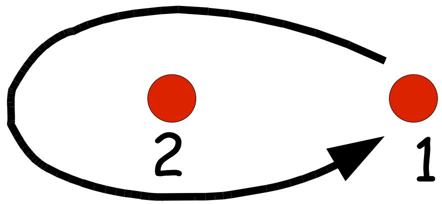
Fractional statistics in 2+1 dimensions

Exchange



$e^{i\theta}$

Monodromy



$$\Psi\left[(z_1 - z_2)e^{i2\pi}, z_2\right] = e^{i2\theta}\Psi\left[z_1, z_2\right]$$

- $\theta = \pi\nu,$ e.g. $\nu = 1/3$ fractional $\neq \pm 1$
- exchange of identical particles described by the braid group
- $e^{i\theta} \neq e^{-i\theta}$ violates P and T symmetries
- If excitation is described by multiplet of m states:

$$\Psi_a\left[z_1, z_2\right] \longrightarrow U_{ab}\ \Psi_b\left[z_1, z_2\right]$$

a,b = 1, . . . , m

m-dim unitary repres. of braid group = **Non-Abelian statistics**

Chern-Simons gauge theory

→ Special facts of 2+1 dimensions:

- matter current ⟷ gauge field: $J_\mu = (\rho, J_i), \quad \partial_\mu J_\mu = 0, \ \langle J_\mu \rangle = 0$

$$J_\mu = \varepsilon_{\mu\nu\rho}\partial_\nu \mathcal{A}_\rho$$

- low-energy effective action, $\not{P}$, $\not{T}$:

ext. source

$$S_{CS} = \frac{k}{4\pi}\int \varepsilon_{\mu\nu\rho}\mathcal{A}_\mu\partial_\nu\mathcal{A}_\rho + \mathcal{A}_\mu s^\mu + \frac{1}{M}\mathcal{F}^2_{\mu\nu}$$

eq. of motion → **no local degrees of freedom**

$$\mathcal{F}_{\mu\nu} = \frac{2\pi}{k}\,\varepsilon_{\mu\nu\rho}s^\rho, \qquad B = \frac{2\pi}{k}\,\delta^{(2)}(z - z_2)$$

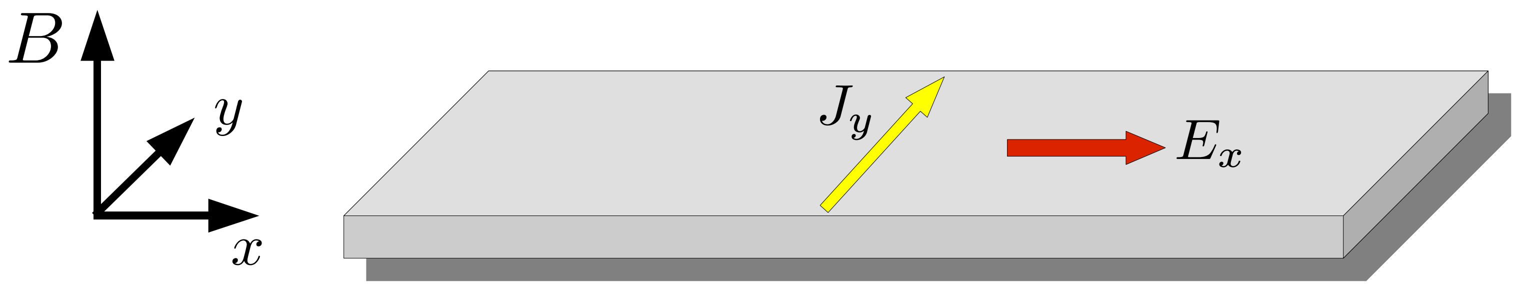
$$\exp\left(i\oint_{z_2}\mathcal{A}\right) = e^{i2\pi/k}$$

Aharonov-Bohm phase

Quantum Hall Effect

- 2 dim electron gas at low temperature T ~ 10 mK

  and high magnetic field B ~ 10 Tesla



- Conductance tensor $\qquad J_i = \sigma_{ij} E_j, \quad \sigma_{ij} = R^{-1}_{ij}, \qquad i,j = x,y$
- Plateaux: $\qquad \sigma_{xx} = 0, \;\; R_{xx} = 0$ $\qquad$ no Ohmic conduction → gap

$$\sigma_{xy} = R^{-1}_{xy} = \frac{e^2}{h}\nu, \qquad \nu = 1(\pm 10^{-8}), 2, 3, \ldots \frac{1}{3}, \frac{2}{5}(\pm 10^{-6})$$

- High precision & universality
- Uniform density ground state: $\qquad \rho_o = \frac{eB}{hc}\nu$

Incompressible fluid

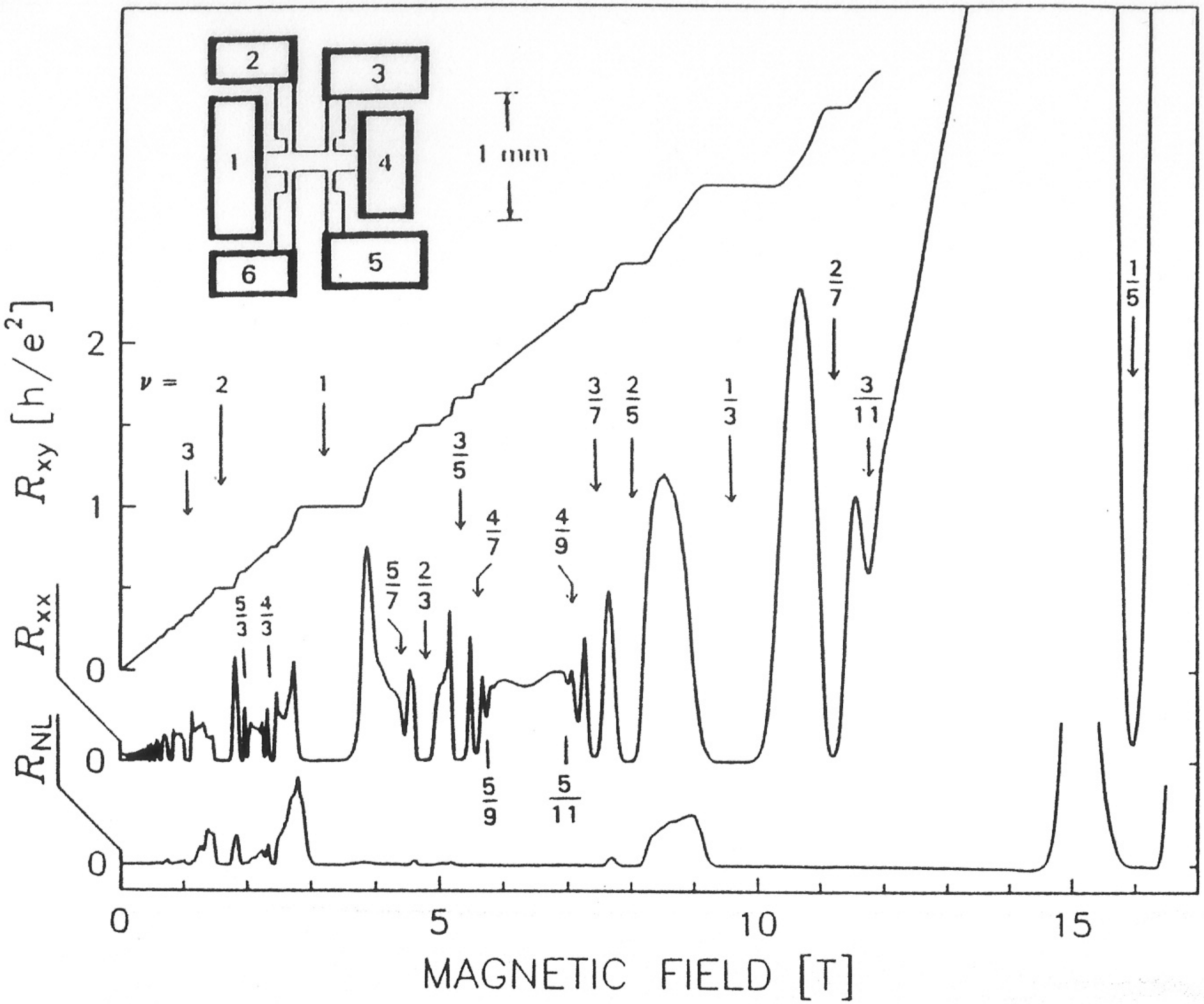
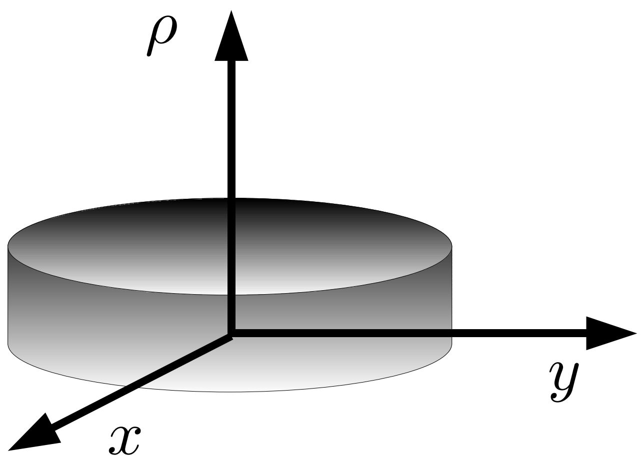


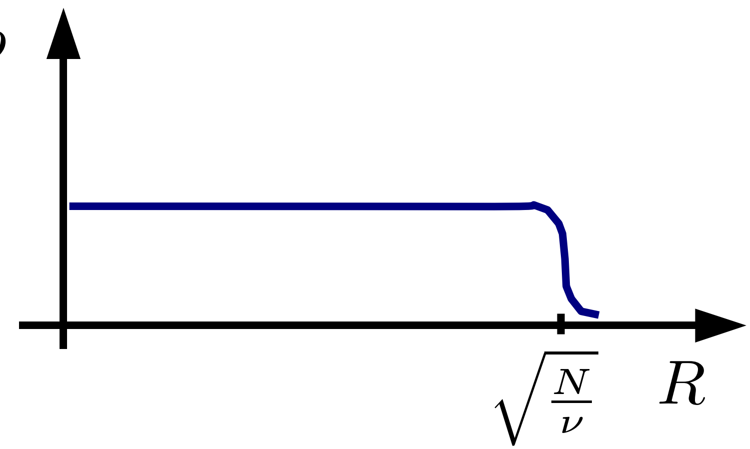
Figure 1. This figure shows a quantum Hall experiment trace. (Source: Goldman *et al.* [2].) The sample geometry is shown in the inset. The various resistances are defined as: Hall resistance $R_{xy} = V_{26}/I_{14}$; longitudinal resistance $R_{xx} = V_{23}/I_{14}$; and non-local resistance $R_{NL} = V_{26}/I_{35}$. Here, V_{jk} denotes the voltage difference between the leads j and k, and I_{jk} denotes the current from lead j to lead k. The experiment was performed at 40 mK.

Laughlin's quantum incompressible fluid

Electrons form a droplet of fluid:

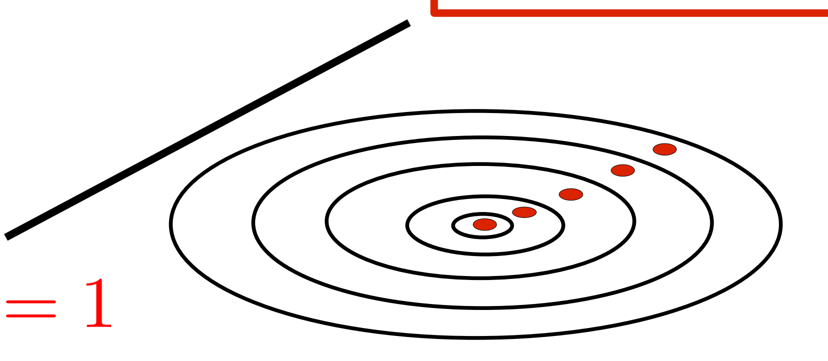
- incompressible = gap
- fluid = $\rho(x,y) = \rho_o = \text{const.}$



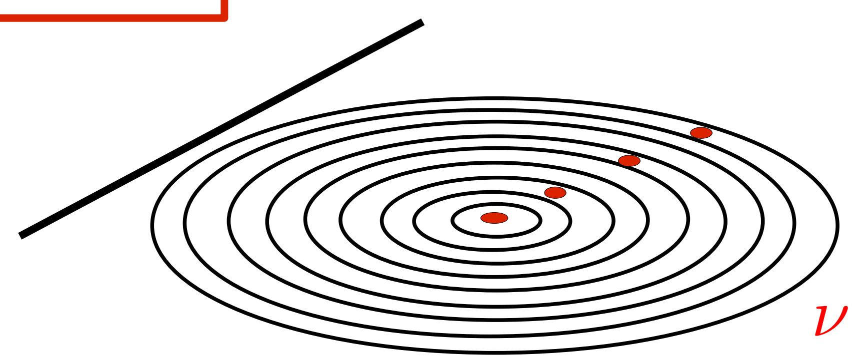


$\mathcal{D}_A = BA/\Phi_o$, # degenerate orbitals = # quantum fluxes, $\Phi_o = \frac{hc}{e}$

filling fraction: $$\nu = \frac{N}{\mathcal{D}_A} = 1, 2, \ldots \frac{1}{3}, \frac{1}{5}, \ldots$$ density for quantum mech.



$\nu = 1$



$\nu = \frac{1}{3}$

Laughlin's trial wave function

$$\Psi_\nu(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^{2k+1} e^{\sum |z_i|^2/2}$$

$$\nu = \frac{1}{2n+1} = 1, \frac{1}{3}, \frac{1}{5}, \ldots$$

- $\nu = 1$ filled Landau level: obvious gap $\omega = \frac{eB}{mc} \gg kT$
- $\nu = \frac{1}{3}$ non-perturbative gap due to Coulomb interaction
- ground state w. vortex condensation, like QCD but chiral
- quasi-hole excitation = elementary vortex $\Psi_\eta \propto \prod_i (\eta - z_i)$

→ fractional charge $Q = \frac{e}{2k+1}$ & statistics $\frac{\theta}{\pi} = \frac{1}{2k+1}$

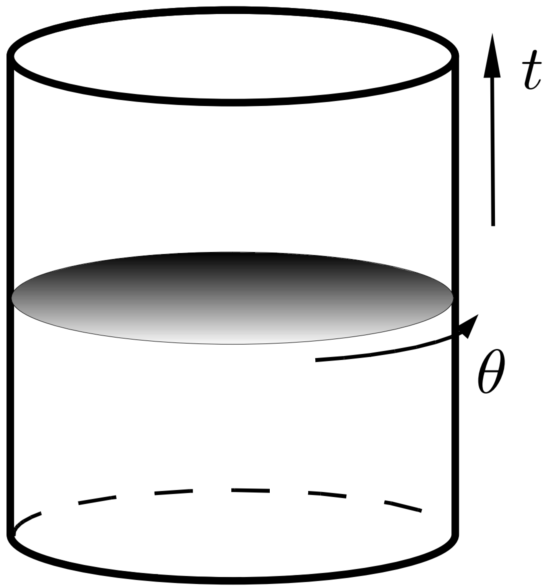
$$\Psi_{\eta_1,\eta_2} \propto (\eta_1 - \eta_2)^{\frac{1}{2k+1}} \prod_i (\eta_1 - z_i)(\eta_2 - z_i)$$

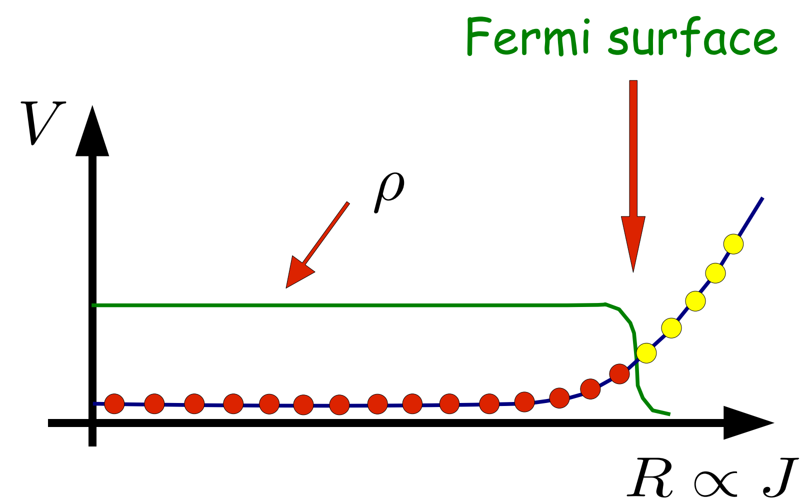
Anyons vortices w. long-range topological correlations

long-distance physics reproduced by effective field theory

Conformal field theory of edge excitations

The edge of the droplet can fluctuate: edge waves are massless



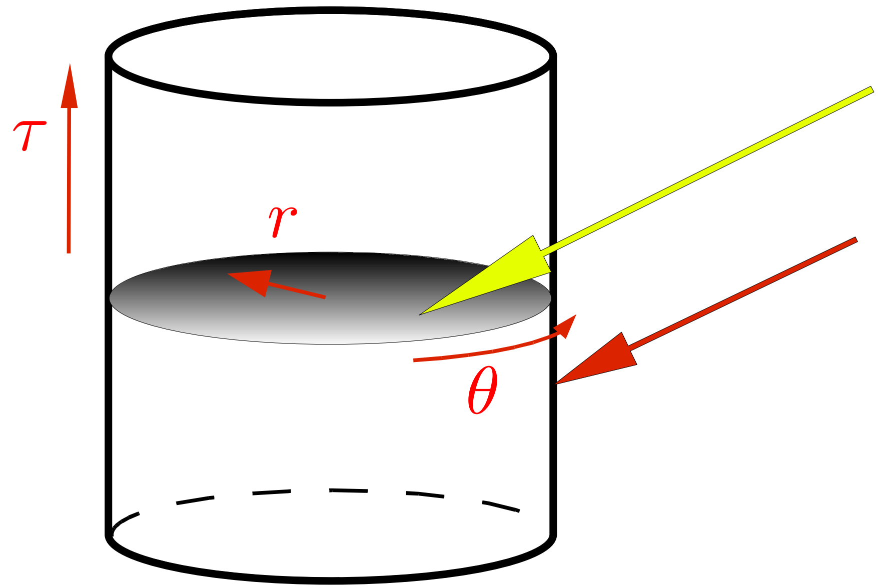


- edge ~ Fermi surface: linearize energy $\varepsilon(k) = \frac{v}{R}(k - k_F), \quad k \in Z^+$
- relativistic field theory in 1+1 dimensions, chiral (X.G.Wen)

  ➡ conformal field theory

- here compactified boson (c=1) = "chiral Luttinger liquid"
- vortex in the bulk ➡ charged excitation at the edge

CFT descriptions of QHE



$z = re^{i\theta}$ plane (bulk excit)

$\zeta = e^{\tau} e^{i\theta}$ cylinder (edge excit)

vertex operators $\langle e^{i\alpha\varphi_1} e^{i\alpha\varphi_2} \rangle_{CFT}$

$(z_1 - z_2)^{\alpha^2}$ anyon wave function

$(\zeta_1 - \zeta_2)^{\alpha^2}$ edge-excit. correlator

- same function by analytic continuation from the circle:

  both equivalent to Chern-Simons theory in 2+1 dim (Witten)

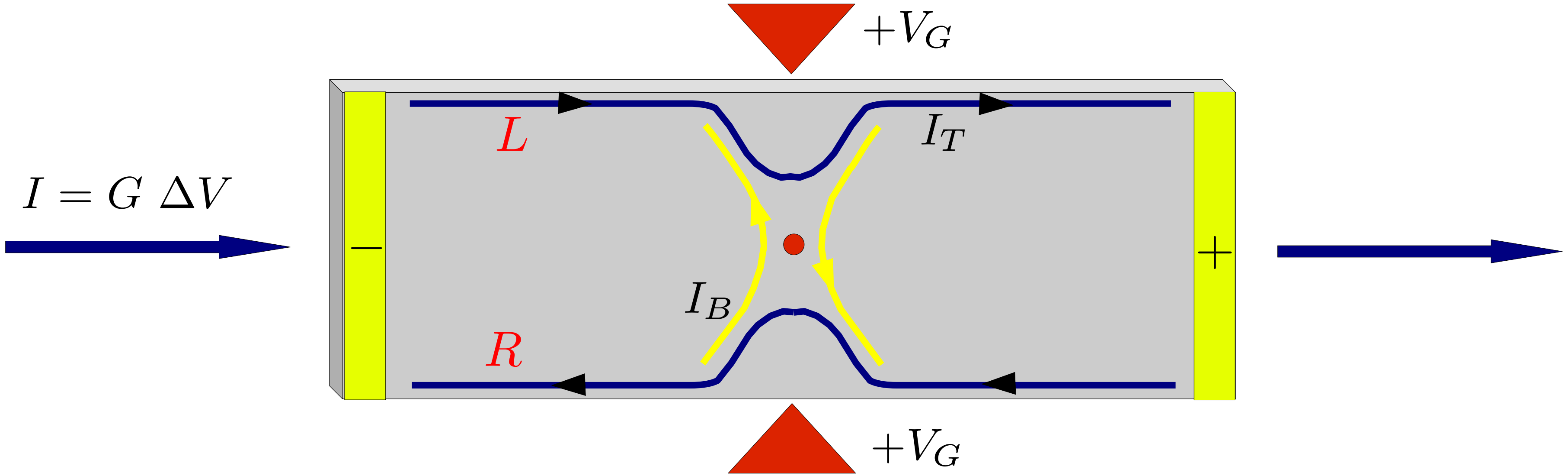
- spectrum of chiral boson CFT proofs Laughlin's fractional Q and $\frac{\theta}{\pi}$
  - wave functions: spectrum of anyons and braiding
  - edge correlators: conduction experiments (low V and small I)

CFT modelling of fractional QHE

- CFTs exactly describe nonperturbative quantum effects
- Big zoo of interacting theories (& integrable massive FT)
- experimental confirmations:
  - tunneling of edge excitations
- sophisticated technical tools all relevant:
  - repres. theory (affine and W_∞ algebras) (A.C.,Trugenberger,Zemba)
  - fusion rules (& modular invariance & boundaries)
  - n-point correlators (braid & fusion relations)

→ nice spin-off of string theory of '85-'95(-'05)

Measure of fractional charge



- electron fluid squeezed at one point: L & R edge excitations interact
- fluctuation of the scattered current: Shot Noise (T=0)
  - low current $I_B \ll I$ → tunnelling of weakly interacting carriers

$$S_I = \langle |\delta I(\omega)|^2 \rangle_{\omega \to 0} = \frac{e}{3}\ I_B$$

Poisson statistics

- CFT description & integrable massive interaction: (Fendley, Ludwig, Saleur)

$$G = \frac{e^2}{h}\frac{1}{3}\ F\left(\frac{V_G}{T^{2/3}}\right)$$

universality & "anomalous" scaling

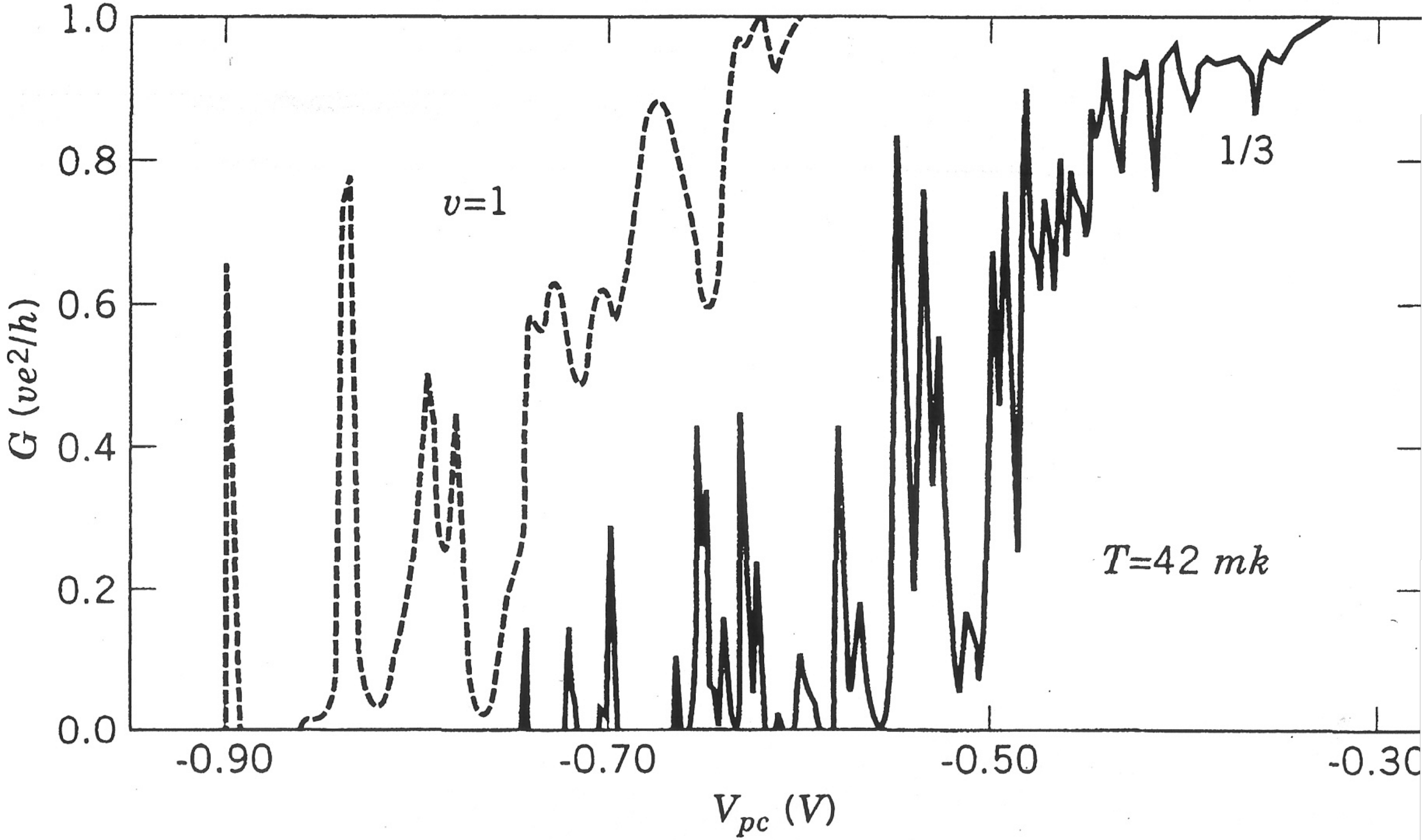
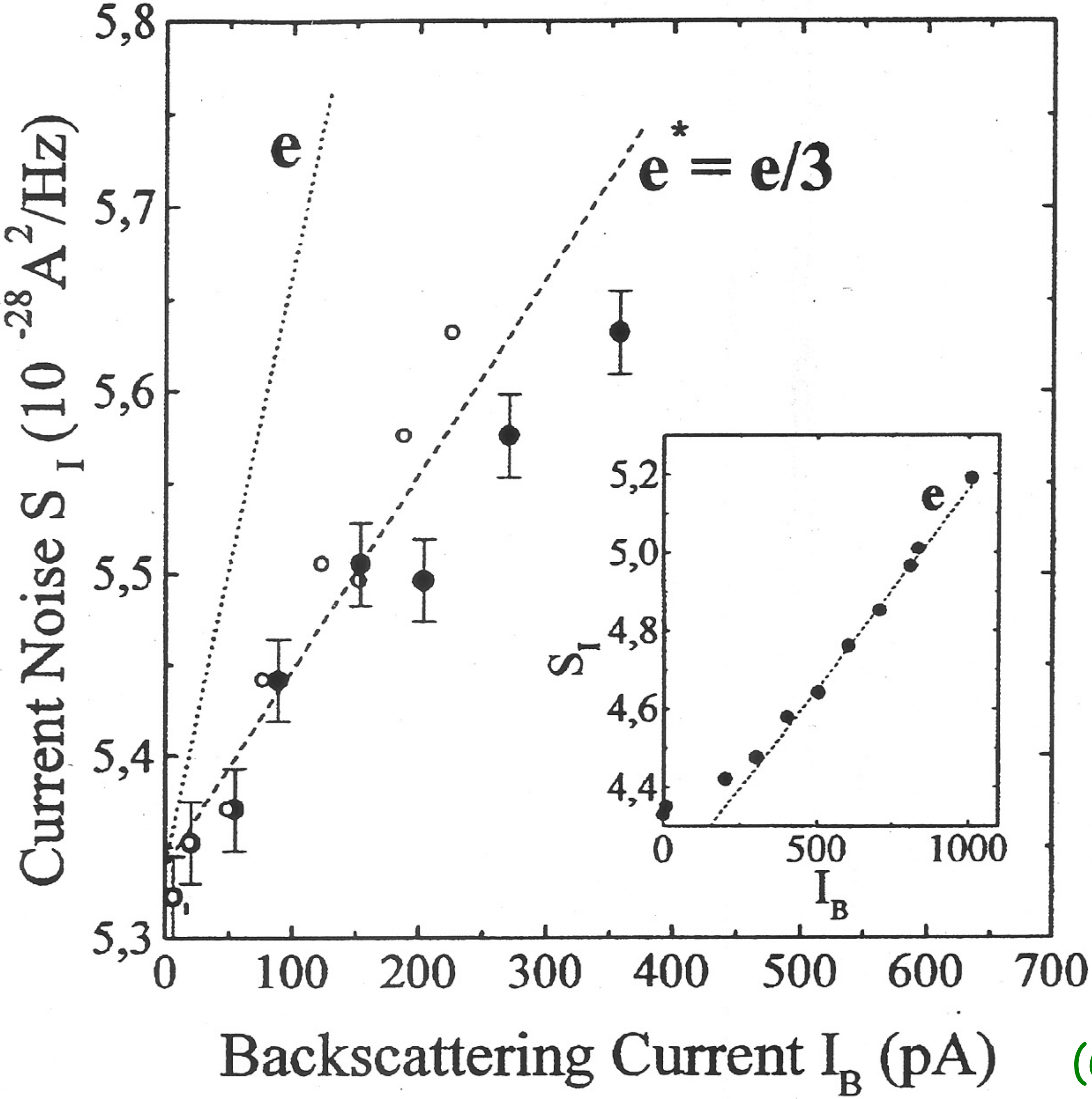


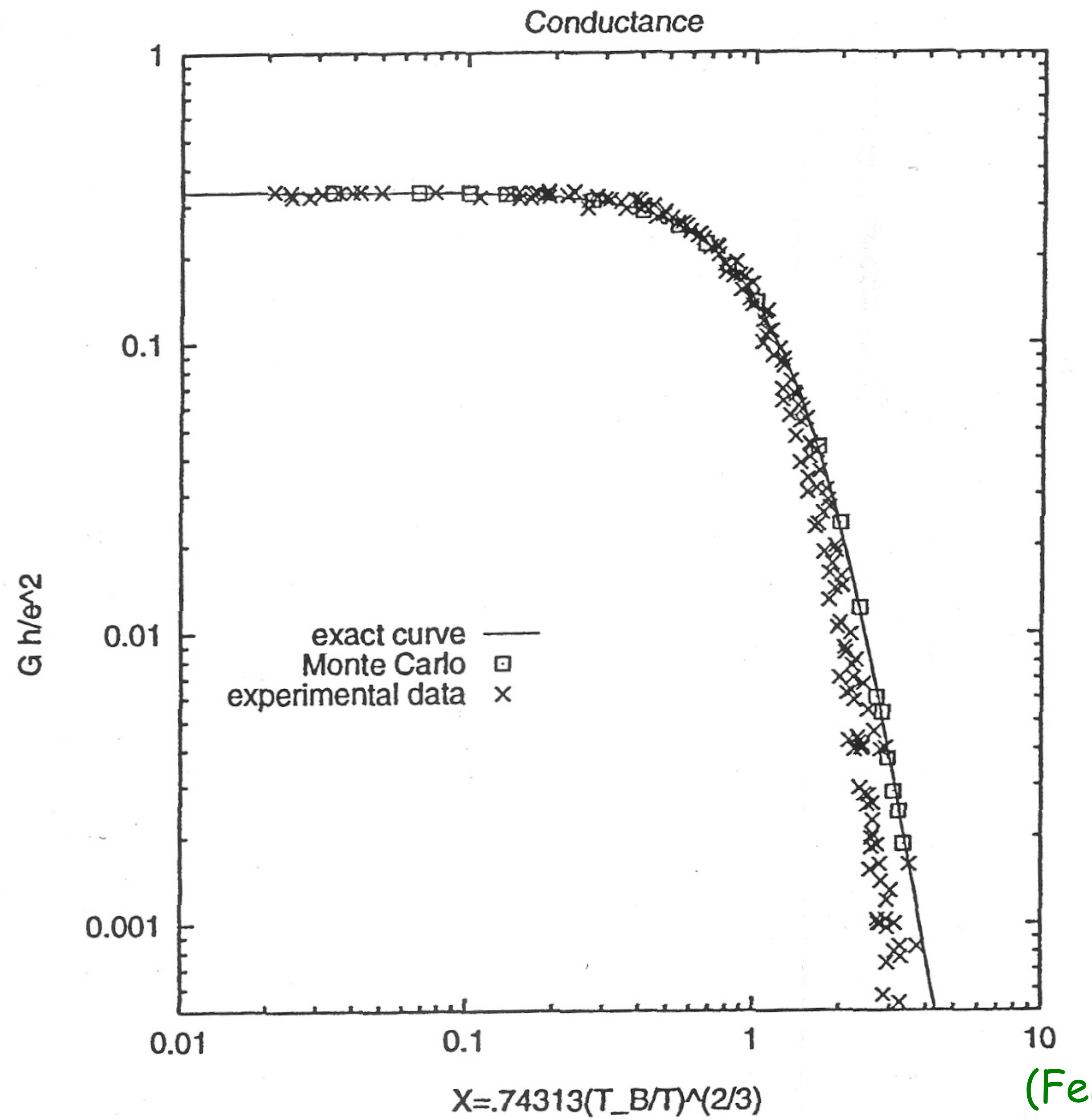
Figure 4.7. Two-terminal conductance as a function of gate voltage of a GaAs quantum Hall point contact taken at 42 mK. The two curves are taken at magnetic fields that correspond to $\nu = 1$ and $\nu = 1/3$ plateaus. (From Ref. [29].)

(Milliken et al '95)



(Glattli et al '97)

FIG. 2. Tunneling noise at $\nu = 1/3$ ($\nu_L = 2/3$) when following path A and plotted versus $I_B = (e^2/3h)V_{ds} - I$ (filled circles) and $I_B(1 - R)$ (open circles). The slopes for $e/3$ quasiparticles (dashed line) and electrons (dotted line) are shown. $\Theta = 25$ mK. Inset: data in same units showing electron tunneling for similar $G = 0.32e^2/h$ but in the IQHE



(Fendley et al '97)

Figure 4.14. Log-log scaling plot of the lineshape of resonances at different temperature: from Ref. [29]. The x axis is rescaled by $T^{2/3}$. The crosses represent experimental data a temperatures between 40 and 140 mK. The squares are the results of the Monte Carl

Remark 1. Can we prove the Laughlin state?

- Effective theories but no microscopic theory
- Exact eigenstate of model interactions → numerics (Haldane,.....)
- Gap is nonperturbative → need "Non Relativistic effective theory"
- NR fermions + extra Chern-Simons interaction (Fradkin et al.; Halperin et al.; Shankar et al.)
- Matrix gauge theory (Susskind '01; Polychronakos; A.C., I. Rodriguez)
  - electrons → D0 branes

    $\vec{x}_a(t),\ a = 1, \ldots, N$ → $\vec{X}_{ab}(t),$ NxN matrices
  - non commutative: $[X_1, X_2] = i\theta$ minimal area

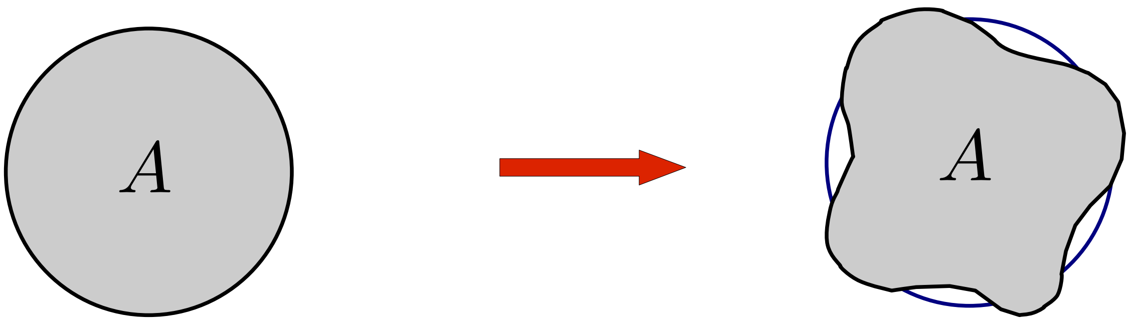
    $\rho_o = \frac{1}{2\pi\theta}, \quad \nu = \frac{1}{1+B\theta} = \frac{1}{1+2k}$
  - predicts Laughlin's states and more general Jain's states

    $$\frac{1}{\nu} = \frac{1}{n} + 2k$$ "composite fermion"

Remark 2. W-infinity symmetry

- one CFT for each plateau: which CFT?
- area-preserving diffeomorphisms of incompressible fluid:

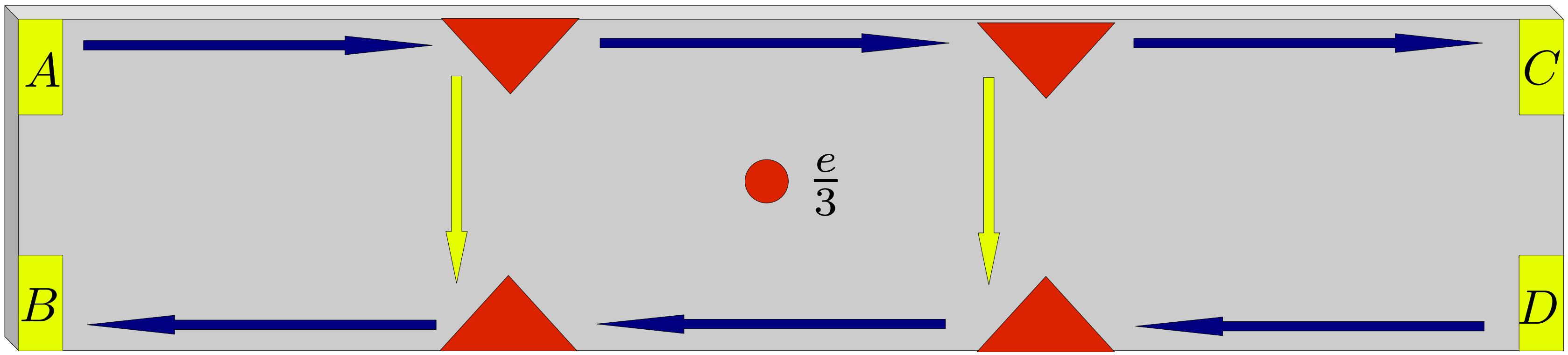
$$\int d^2x \ \rho(x) = N = \rho_o A \quad \longrightarrow \quad A = \text{constant}$$



- W-infinity symmetry can be implemented in CFT
- representations completely known (V.Kac, A. Radul)
- "minimal models" of W_∞ match Jain's states (A.C., Trugenberger, Zemba)

$$\frac{\widehat{U(1)}_{2k+1} \times \widehat{SU(n)}_1}{SU(n)} \ \text{CFT} \quad \longleftrightarrow \quad \nu = \frac{n}{2k\ n \pm 1} \ \text{plateaux}$$

Measure of fractional statistics



- need interference like double slit experiment
- 4-point function of edge states
- induce anyon(s) in the central cell → Aharonov-Bohm phase
- first experiment has side effects and instabilities (Goldman et al. '05)
- can manufacture better interference geometries (cf. Stern review '07)
- no doubts by low-energy effective theory

Non-Abelian fractional statistics

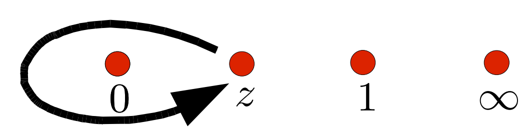
- $\nu = \frac{5}{2}$ described by Moore-Read "Pfaffian state" ~ Ising CFT x boson
- Ising fields: I identity, ψ Majorana = electron, σ spin = anyon
- fusion rules:
  - $\psi \cdot \psi = I$ 2 electrons fuse into bosonic bound state
  - $\sigma \cdot \sigma = I + \psi$ 2 channels of fusion = 2 "conformal blocks"

$$\langle \sigma(0)\sigma(z)\sigma(1)\sigma(\infty)\rangle = a_1 F_1(z) + a_2 F_2(z)$$ Hypergeometric functions

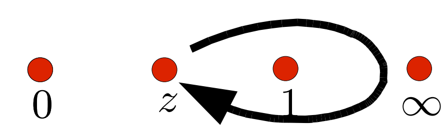
→ state of 4 anyons is two-fold degenerate (Moore, Read '91)

- statistics of anyons ~ analytic continuation → 2x2 matrix

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}(ze^{i2\pi}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}(z)$$



$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}((z-1)e^{i2\pi}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}(z)$$



(CFT tech: Verlinde; Moore, Seiberg; Alvarez-Gaume, Gomez, Sierra)

Quantum computation

- qubit = two-state quantum system, e.g. spin ½: $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$
- boolean gates ➡ unitary transformations on qubits

  ➡ discrete subgroup of $U(2^n)$ transformations in n qubit Hilbert space
- minimal set of generators:
  - 2x2 Pauli matrices + one specific 4x4 matrix

  ➡ "Universal Quantum Computation"
- many proposals of systems for QC: excitement & money
- quantum computer is unavoidable & useful (e.g. for war, electronic)
- big problem: decoherence by the environment

Topological quantum computation

- Proposal: use non-Abelian anyons for qubits and operate by braiding

  e.g. in Ising-like state $\nu = \frac{5}{2}$ (Kitaev; M. Freedman; Nayak; Das Sarma)

- anyons topologically protected from decoherence (local perturbations):
  - decay due to finite size $P \sim \exp(-L/\xi), \quad (\text{system size})/\ell = O(10^4)$
  - thermal pair creation $P \sim \exp(-\Delta/T), \quad \Delta/T = O(10^2)$
- use 4-spin system $\alpha|F_1\rangle + \beta|F_2\rangle$ as 1 qubit (2n spin has dim $= 2^{n-1}$)
- consider multi-gate bar geometry of before:

  → perform anyon exchanges by tuning the various gate voltages

- Ising is not universal QC; Z_3 parafermions $\nu = \frac{12}{5}$ are OK & others
- study other anyonic media, e.g. array of Josephson junctions

many ideas & open problems

Low-dimensional Quantum Field Theories and Applications

Workshop

Organizers:

Andrea Cappelli (INFN, Florence), Giuseppe Mussardo (SISSA,Trieste),
Hubert Saleur (CEA, Saclay), Paul Wiegmann (EFI, Chicago) and Jean-Bernard Zuber (Paris VI).

Period: from 01-09-2008 to 07-11-2008

Deadline: 31-03-2008

Topics

- Quantum dynamics in mesoscopic systems and cold atoms
- Conformal field theory and topological quantum computation
- Correlations and entanglement in lattice models and field theories
- Sigma models on noncompact groups and logarithmic conformal field theory
- Stochastic Loewner evolution and growth processes
- Integrability in the AdS/CFT correspondence
