Benchmarking the Ising Universality Class in $3 \leq d < 4$ dimensionss

Speaker:

Claudio Bonanno

INFN Firenze

✉ claudio.bonanno@fi.infn.it

Bootstrapping Nature:
Non-perturbative Approaches to Critical Phenomena

GGI – Nov. 2nd 2022

Talk based on:

CB, A. Cappelli, M. Kompaniets, S. Okuda, K. J. Wiese,
[arXiv:2205.06190]

Introduction

Critical phenomena have been extensively studied in non-integer space-time dimensions $d \in \mathbb{R}$:

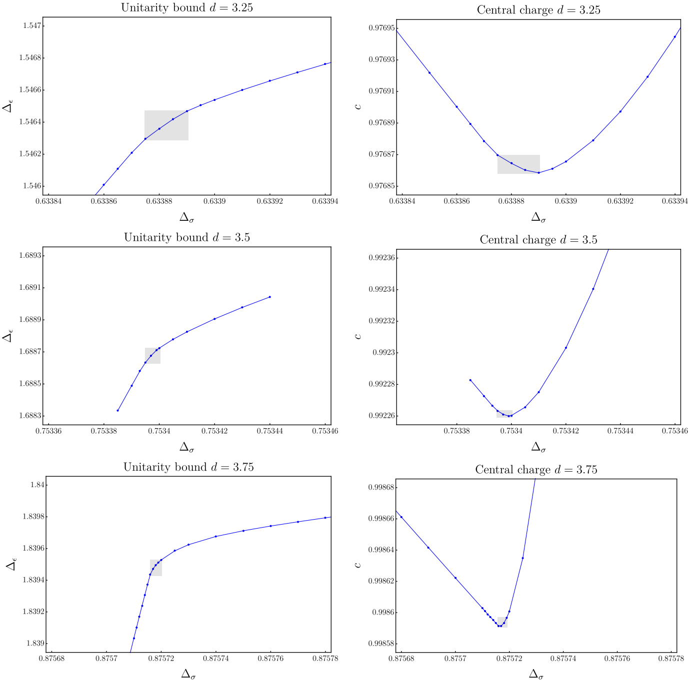
- Perturbative expansion around $d = 4$ gives results for non-integer d
- Universality Class may change in non-integer dimension (e.g., systems with long-range interactions or disorder)

In this talk we are mainly concerned with the investigation of the Ising Universality Class in fractional dimension $4 > d \geq 3$.

Our approach: numerical conformal bootstrap. We use 1-correlator $\langle \sigma\sigma\sigma\sigma \rangle$ setup **[El-Showk et al., 1403.4545]**, SDPB routine **[Simmons-Duffin, 1502.02033]** and the Extremal Functional Method (EFM) **[El-Showk and Paulos, 1211.2810]**.

Although more advanced methods have been recently introduced, see, e.g., the *navigator* **[Reehorst et al., arXiv:2104.09518]**, our setup turns out to be reliable enough to determine low-lying spectrum with up to per-mill precision.

We will also compare our results with those obtained by other perturbative and non-perturbative techniques: different boostrap approaches, perturbative expansion, resummed perturbative series, Monte Carlo, ...



Identification of **Ising Point**:

unitarity bound kink on $(\Delta_\sigma, \Delta_\epsilon)$ plane

$+$

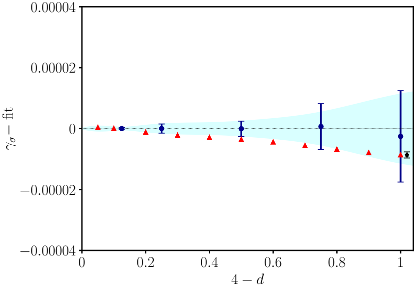
central charge c minimum

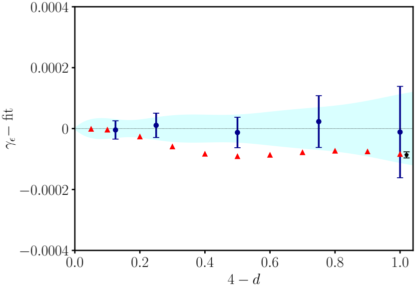
[El-Showk et al., 1403.4545].

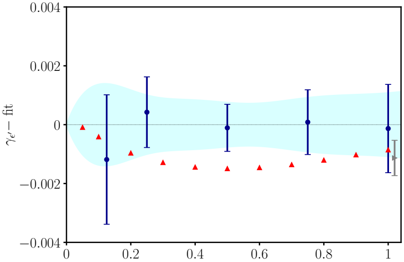
The position of the kink gets sharper as $d \to 4$.

Results for anomalous dimensions γ_σ, γ_ϵ, $\gamma_{\epsilon'}$

Lowest-lying $\ell = 0$ operators ($\gamma_\sigma, \gamma_\epsilon, \gamma_{\epsilon'} \to \eta$, ν, ω Ising critical exponents)







Very good agreement with recent 3-corr. boostrap results obtained by the navigator method for $4 > d \geq 3$ **[Henriksson et al., 2207.10118]**. Note that errorbars for 3-corr. results at $d = 3$ are *rigorous bounds* **[Kos et al., 1603.04436; Reehorst, 2111.12093]**.

Brief recap of perturbative eps. expansion

The perturbative β function takes the following form in the $\overline{\text{MS}}$ scheme:

$$\beta_{\overline{\text{MS}}}(g, y) = -yg + \sum_{k=2}^{n+1} \beta_k \, g^k, \qquad y \equiv 4 - d.$$

Using the fixed-point equation for the critical coupling $g = g^\star$

$$\beta_{\overline{\text{MS}}}(g^\star, y) = 0 \quad \longrightarrow \quad g^\star = g^\star(y)$$

we can convert a perturbative series in g into a perturbative series in y:

$$\gamma_{\mathcal{O}}(g) = \sum_{k=1}^{n} \gamma_{\mathcal{O},k} \, g^k \quad \longrightarrow \quad \gamma_{\mathcal{O}}(y) = \sum_{k=1}^{n} \overline{\gamma}_{\mathcal{O},k} \, y^k.$$

Expansion coefficients grow factorially:

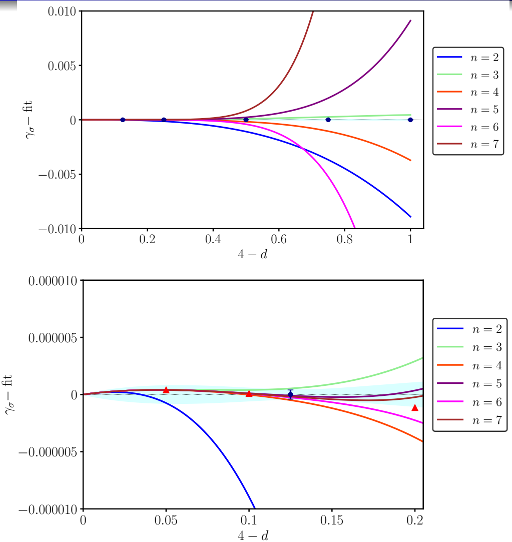
$$\overline{\gamma}_{\mathcal{O},k} \underset{k\to\infty}{\sim} (-1)^k \, C \, a^k \, k^b \, k! \, .$$

Thus, the more terms are added, the sooner the perturbative series diverges:

$$|\overline{\gamma}_{\mathcal{O},n}/\overline{\gamma}_{\mathcal{O},n-1}| \sim ayn \implies y \ll y_{\max} \sim \frac{1}{an}.$$

We expect perturbative predictions to be reliable when $y \ll y_{\max}$, i.e., only sufficiently close to $d = 4$. Let us check by comparing with bootstrap results.

Comparison with eps. expansion for γ_σ - 1



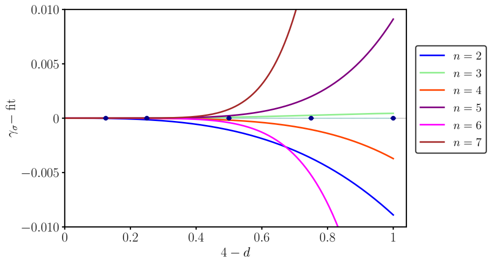
Close to $d = 3$, eps. expansion [Kompaniets et al., 1705.06483; Schnetz, 1606.08598] sensibly deviates from bootstrap results. Deviation increases with increasing order n.

In the range $4 > d \geq 3.8$, instead, perturbation theory prediction perfectly agrees with bootstrap results within errors. Triangles = navigator [Henriksson et al., 2207.10118].

Upon increasing n, perturbative series describes better and better $\gamma_\sigma^{(\mathrm{CB})}(y)$ up to some $y_{\mathrm{optimal}} \sim 1/n$ before starting to diverge.

Agreement between fully non-perturbative bootstrap results and perturbation theory close to $d = 4$ is a **non-trivial check** that non-perturbative effects become negligible for $y \to 0$ in the bootstrap equations.

Comparison with eps. expansion for γ_σ - 2

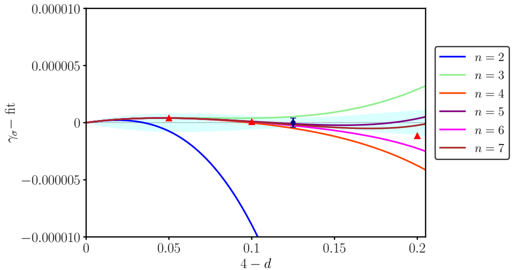


$$\gamma_\sigma^{(\text{eps}-\text{exp})}(y) =$$

$$\boxed{0.00925926y^2 + 0.00934499y^3}$$

$$-0.00416439y^4 + 0.0128282y^5$$

$$-0.0406363y^6 + 0.15738y^7$$



$$\gamma_\sigma^{(\text{CB})}(y) =$$

$$\boxed{0.00930647y^2 + 0.00889991y^3}$$

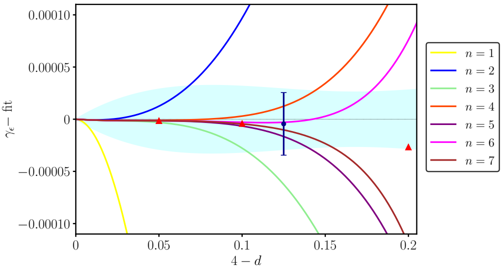
$$-0.00143511y^4 + 0.00178871y^5$$

$$-0.00053398y^6 + 0.00012867y^7$$

Agreement between fully non-perturbative bootstrap results and perturbation theory close to $d=4$ (up to $O(y^3)$ terms) is a **non-trivial check** that non-perturbative effects become negligible for $y \to 0$ in the bootstrap equations.

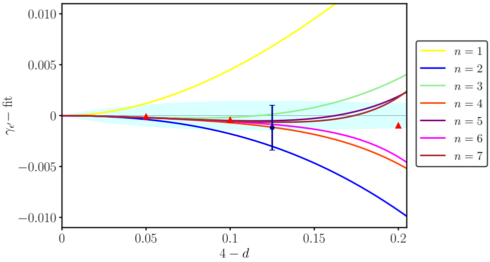
Comparison with eps. expansion for γ_ϵ and $\gamma_{\epsilon'}$

Similar considerations hold also for γ_ϵ and $\gamma_{\epsilon'}$. Also in this case agreement between bootstrap fit and perturbative series is very good for $d > 4 > 3.8$ (up to $O(y^3)$ terms).



$$\gamma_\epsilon^{(\mathrm{eps-exp})}(y) = \boxed{0.33333y + 0.11728y^2} - 0.124527y^3 + 0.30685y^4 - 0.95124y^5 + 3.57258y^6 - 15.2869y^7$$

$$\gamma_\epsilon^{(\mathrm{CB})}(y) = \boxed{0.33344y + 0.11410y^2} - 0.083458y^3 + 0.08138y^4 - 0.045297y^5 + 0.014290y^6 - 0.0017413y^7$$



$$\gamma_{\epsilon'}^{(\mathrm{eps-exp})}(y) = \boxed{2y - 0.62963y^2} + 1.61822y^3 - 5.23514y^4 + 20.7498y^5 - 93.1113y^6 + 458.7424y^7$$

$$\gamma_{\epsilon'}(y) = \boxed{2.00018y - 0.518007y^2} + 0.721997y^3 - 0.68447y^4 + 0.447649y^5 - 0.16290y^6 + 0.02616y^7$$

Brief recap of Borel resummation of perturbative series

The **Borel transform** is defined so as to remove the factorial growth of the coefficients of the perturbative series $\overline{\gamma}_{\mathcal{O},k} \underset{k\to\infty}{\sim} (-1)^k C a^k k^b k!$:

$$\mathcal{B}_{\gamma_{\mathcal{O}}}(t) \equiv \sum_{k=1}^{n} \frac{\overline{\gamma}_{\mathcal{O},k}}{k!} t^k \sim \frac{1}{(1+at)^{b+1}}$$

The Borel transform has finite radius of convergence $|t| < 1/a$.

Resummed perturbative series is obtained from the *inverse Borel transform*:

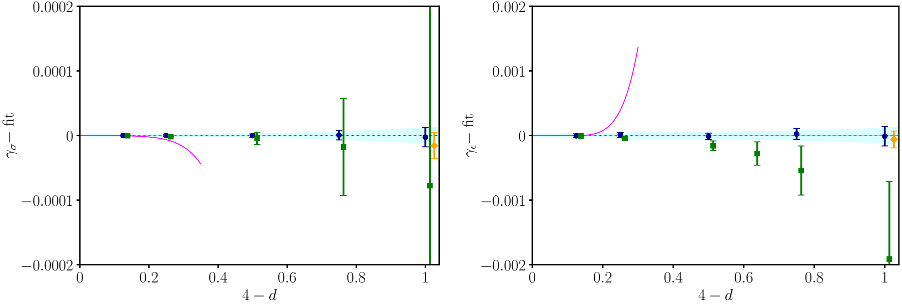
$$\widetilde{\gamma}_{\mathcal{O}}(y) = \int_0^{\infty} dt\, e^{-t}\, \mathcal{B}_{\gamma_{\mathcal{O}}}(yt),$$

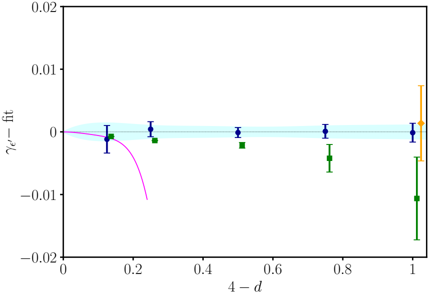
Series $\widetilde{\gamma}_{\mathcal{O}}$ has the same perturbative expansion of $\gamma_{\mathcal{O}}(y)$ by construction but is better behaved if $\mathcal{B}_{\gamma_{\mathcal{O}}}(t)$ is analytically continued outside $|t| < 1/a$.

Such analytic continuation can be done by several means, and introducing some parameters that can be varied to obtain a robust resumation scheme.

In this work we will mainly show resummed results obtained by the state-of-the-art techniques of **[Kompaniets and Panzer, 1705.06483]**.

Comparison with resummed perturbative series

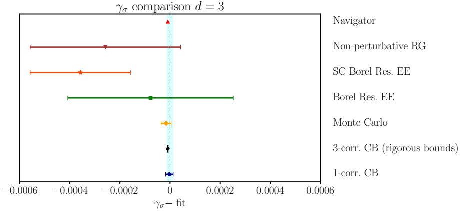


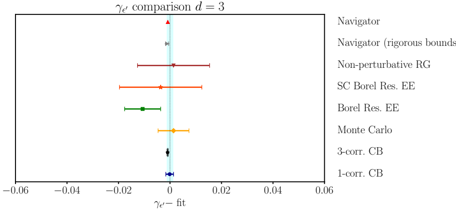


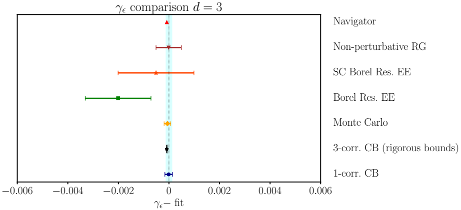
Resummation allows to improve agreement among bootstrap and perturbative results even close to $d = 3$.

For comparison purposes, we also report unresummed perturbative results (magenta solid curves) and Monte Carlo determinations for $d = 3$ (yellow diamond points) **[Hasenbusch, 1004.4486; 2105.09781]**.

For $d = 3$, the comparison can be further extended considering other results obtained by different methods.







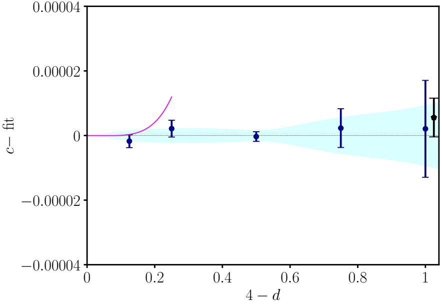
Different determinations for γ_σ, γ_ϵ, $\gamma_{\epsilon'}$ obtained by several perturbative and non-perturbative methods give overall very good agreeing results. It is highly non-trivial given the reached precision.

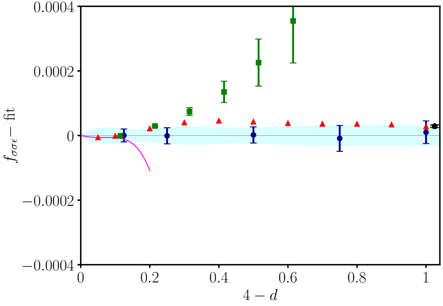
Our $d = 3$ CB 1-corr. result **[Cappelli et al., 1811.07751]**

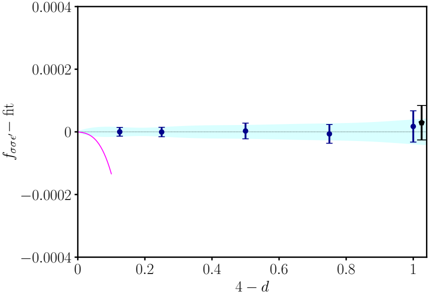
Non-perturbative RG **[Balog et al., 1907.01829; Depuis et al., 2006.04853]**

SC Borel Resummation **[Kompaniets and Wiese, 1908.07502]**

Results for OPE coefficients of low-lying $\ell = 0$ operators







Overall agreement among different determinations: 3-corr. CB (pentagons) **[Simmons-Duffin, 1612.08471]**, navigator (triangles) **[Henriksson et al., 2207.10118]**.

Lower precision of resummed results for $f_{\sigma\sigma\epsilon}$ due to lower eps. expansion known terms (up to $n = 4$).

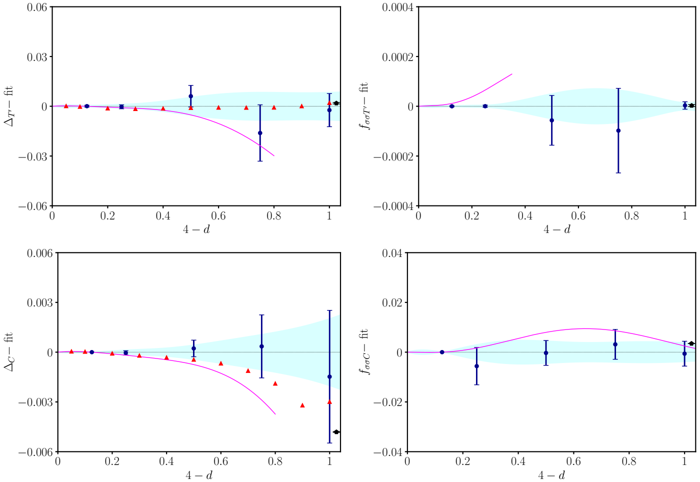
Results for low-lying $\ell > 0$ operators

Scaling dimensions and structure constants of low-lying operators in the $\ell = 2, 4$ channels can be determined with good precision too.

Very good agreement with recent navigator results **[Henriksson et al., 2207.10118]** in the whole range, and with perturbation theory for $4 > d > 3.8$.

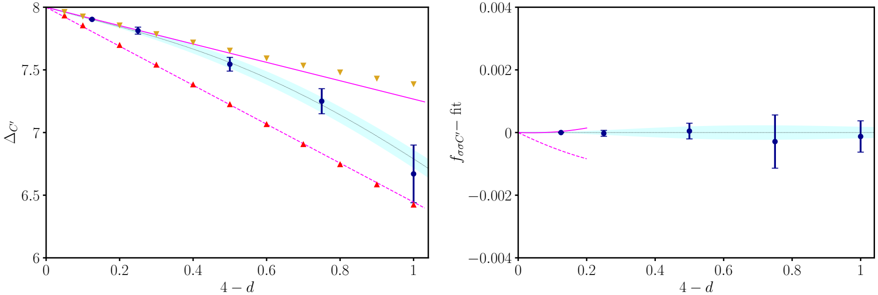
T'
($\ell = 2$)

C
($\ell = 4$)



Results for higher-dimensional operator: the case of C'

With our bootstrap setup we were not able to resolve the spectrum of higher-dimensional fields. As an example we show the case of C' ($\ell = 4$).



Comparing our results with those of the navigator [Henriksson et al., 2207.10118], we find a state that seems to approach correctly C' for $d \to 3$ but that gets closer and closer to C'' as $d \to 4$, which in this limit is degenerate with C'.

Taking a look at the OPE coefficient of our *would-be-*C' state we observe that, close to $d = 4$, it approaches the perturbative prediction for $f_{\sigma\sigma C''}$, which is larger than $f_{\sigma\sigma C'}$ $\longrightarrow$ it dominates in our mixted state close to $y \sim 0$.

Concluding remarks

- Our simple 1-correlator bootstrap is capable of precisingly determine the low-lying spectrum of the Ising CFT for $4 > d \geq 3$.

- Our results provide a benchmark for future more refined studies of Ising critical exponents in fractional dimensions (e.g., new resummation results).

- Very good agreement between bootstrap best fit and perturbative series up to $O(y^3)$ terms close to $d = 4$, more precisely perturbative series agrees with bootstrap results in the range $4 > d > 3.8$.

- Very good agreement among different numerical bootstrap determinations in the whole $4 > d \geq 3$ range.

- Very good overall agreement among several different perturbative (resummed eps. expansion) and non-perturbative (Monte Carlo, non-perturbative RG, 1-corr. and 3-corr. bootstrap) methods to compute Ising critical exponents for $d = 3$.

Future outlooks

- It would be interesting to perform analogous studies in the range $3 > d \geq 2$ to better understand how the strongly-interacting model at $d = 3$ reduces to the minimal Virasoro theory for $d = 2$.

- To this end it is of the utmost importance to precisely compute scaling dimensions and OPE coefficients of higher-dimensional operators, by using, e.g., the recent navigator method.