Physical and Mathematical Aspects of the Seiberg-Witten Solution

(P. Valtancoli, L. Vergnano, A.C.; Florence
Nucl. Phys. B (98))

Motivations

- Understand the physics of the S-W solution:
  - non-invariant electric & magnetic quantum #s
  - super conformal points
- Understand the mathematics:
  - analyticity & integrability
  - ansatz → general framework ?
  - relation with conformal field theory ?

Results

- Analyticity → Isomonodromy → $\boxed{c=-2 \text{ CFT}}$
- Math: Yang-Baxter & Pentagonal identities
  braiding fusing (Moore-Seiberg)
- Physics:
  - quark-monopole transmutation;
  - constraints on the BPS spectrum;
  - Higgs vs. Confinement phases (N=1)

Introduction #0: Higgs = Confinement

with scalar fields in the fundamental rep.

(Susskind ~'79)

$\langle W(C) \rangle \sim e^{-P(C)}$ in both phases

SU(2) gauge theory + lepton & Higgs doublets

$W^{\pm}, W^{3}$ $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

- **Higgs phase**

$\langle \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rangle = \begin{pmatrix} 0 \\ a \end{pmatrix}$ unitary gauge $\phi_1 = 0$, $\phi_2 = a + \delta\phi$ (real!)

Spectrum :
- ψ_1, ψ_2 two fermions
- $\delta\phi$ neutral Higgs
- $W^{\pm}, W^{3}$ three massive gauge bosons

"Confinement picture" : only gauge singlets

- $\phi_i^{+} \psi_i = a \psi_2 + \dots$; $\varepsilon_{ij} \phi_i^{+} \psi_j = a \psi_1 + \dots$
- $\phi_i^{+} \phi_i = a^2 + 2a\,\delta\phi + \dots$
- $\phi_i^{+} \overleftrightarrow{D}_\mu \phi_i = a^2 W_\mu^3 + \dots$; $\varepsilon_{ij} \phi_i \overleftrightarrow{D}_\mu \phi_j = a^2 W_\mu^{+} + \dots$

The same gauge-invariant composite fields can describe both phases:

Higgs ($a \gg \Lambda$, $g \ll 1$):

$\phi_i^\dagger \cdot \psi_i \sim a\,\psi_2$ point-like elementary fermion

Confinement ($a \sim \Lambda$, $g \sim 1$):

$\phi_i^\dagger \cdot \psi_i$ scalar-quark "meson" with resonances

→ Same phase : only quantitative changes, two regimes (Fradkin, Shenker)

→ spectrum should change smoothly

BUT: • Higgs phase is easy

• Confinement ???????

The Seiberg-Witten solution of $N=2$ SuSy SU(2) theory with matter can tell us something on the confinement regime.

The Weinberg-Salam theory is in the Higgs regime, $g_w \ll 1$, $a \gg \Lambda_w$; but Yukawa coupling of Higgs $\lambda \lesssim 1$: another story

Introduction #1 : SL(2,$\mathbb{Z}$) duality transformations

Effective Abelian gauge theory :

$$S[A] = \operatorname{Im} \frac{1}{32\pi} \int \tau \left(F_{\mu\nu} + i \tilde{F}_{\mu\nu} \right)^2 \qquad \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{e^2}$$

$$= \frac{1}{4e^2} \int F_{\mu\nu}^2 + \frac{\theta}{32\pi^2} \int F_{\mu\nu} \tilde{F}_{\mu\nu} \qquad \leftarrow \theta N_I$$

- $\theta \to \theta + 2\pi$ quantum symmetry (transf. T)
- $\tau \to -\frac{1}{\tau}$ change of field variables (transf. S)

Path-integral heuristic argument :

$$Z = \int \mathcal{D} A_\mu \, e^{iS[A,\tau]} = \int \mathcal{D} F_{\mu\nu} \mathcal{D} A^D_\mu \, e^{iS[A,\tau] + \frac{i}{4\pi} \int A^D_\mu \partial_\nu \tilde{F}^{\mu\nu}}$$

$$\frac{1}{16\pi} \operatorname{Im} \int (F^D + i \tilde{F}^D)(F + i \tilde{F})$$

$$Z = \int \mathcal{D} A^D_\mu \exp i \operatorname{Im} \frac{1}{32\pi} \int -\frac{1}{\tau} \left(F^D + i \tilde{F}^D \right)^2$$

Some facts :

- S & T generate $SL(2,\mathbb{Z}) : \tau \to \frac{a\tau + b}{c\tau + d}$
- $\theta = 0$; $S : e \to \frac{4\pi}{e} = g$ magnetic charge
- $\theta \neq 0$; monopole acquires electric charge $e \frac{\theta}{2\pi}$ $\to$ dyon (Witten, 79)

- generalized Dirac condition for two dyons

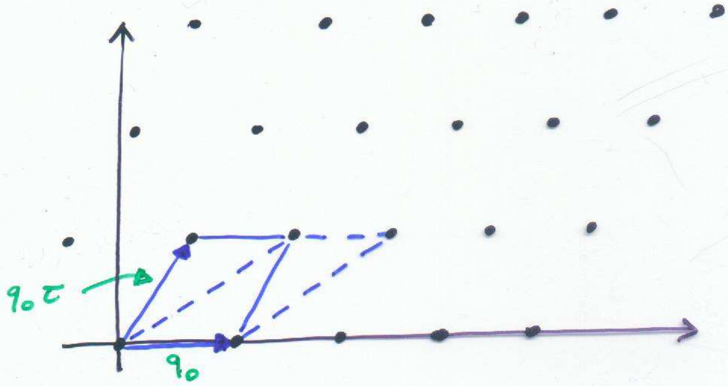
$\vec{v}_1 = (q_1, g_1)$, $\vec{v}_2 = (q_2, g_2)$

$|\vec{v}_1 \wedge \vec{v}_2| = q_1 g_2 - q_2 g_1 = 2\pi n \hbar \qquad n \in \mathbb{Z}$

- a $SL(2,\mathbb{Z})$-invariant spectrum $\{(q,g)\}$ should span a lattice

$q + ig = q_0 (n + \tau m)$

$n, m \in \mathbb{Z}$



- $SL(2,\mathbb{Z})$ transformation of τ = change of basic cell of the lattice

- basic cell $\leftrightarrow$ basic field variable

$(q_0, 0) \sim W^+$ boson

$(0, q_0\tau) \sim M$ elementary monopole

$\rightarrow$ Montonen-Olive conjecture:

" $\exists$ dual theory in which M is elementary and W^+ solitonic "

partially true in Seiberg-Witten $N=2$ Susy theories

Introduction #2: SU(2) S-W solution

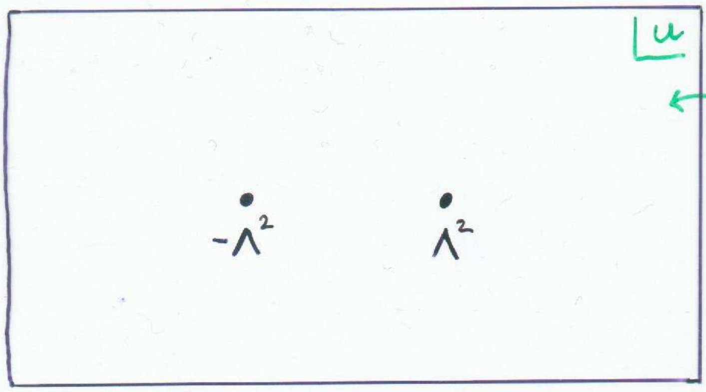
$N=2$ gauge multiplet $\begin{pmatrix} & & A^a_\mu & \\ & \lambda^a & & \chi^a \\ W_\alpha^{''} & & \phi^a & \end{pmatrix}$ Φ'' $\qquad a=1,2,3$

Higgs phase $\quad SU(2)\to U(1) \quad \approx$ low-energy effective $N=2$ QED

$V(\phi)\sim \mathrm{Tr}\left([\phi,\phi^+]\right)^2 \quad \to \quad \langle\phi\rangle = \frac{1}{2}a\sigma_3$

- a is arbitrary (massless Higgs, partner of photon)

→ moduli space : use coordinate $u=\langle \mathrm{tr}\,\phi^2\rangle \sim a^2$



$u\to\infty$ weak coupling

$u\sim\Lambda$ strong coupling

- holomorphic $SL(2,\mathbb{Z})$ section $\qquad y^2=(x-u)(x^2-\Lambda^4)$

$$\begin{pmatrix} a_D(u) \\ a(u)\end{pmatrix} \qquad \frac{da}{du}=\oint_{\gamma_1}\frac{dx}{y}\,, \quad \frac{da_D}{du}=\oint_{\gamma_2}\frac{dx}{y} \qquad \text{auxiliary torus}$$

- monodromy transformations $\quad (u-\Lambda^2)\to e^{i2\pi}(u-\Lambda^2)$

$$\begin{pmatrix} a_D \\ a\end{pmatrix} \to M_{\Lambda^2}\begin{pmatrix} a_D \\ a\end{pmatrix} \qquad M_{\pm\Lambda^2},\, M_\infty \in SL(2,\mathbb{Z})$$

- effective coupling $\quad \tau = \dfrac{\frac{da_D}{du}}{\frac{da}{du}}$

- BPS mass formula for the dyons (n_m, n_e) $\uparrow \mathbb{Z}$

$$m_{BPS} = \sqrt{2}\, | n_m a_D(u) + n_e a(u) |$$

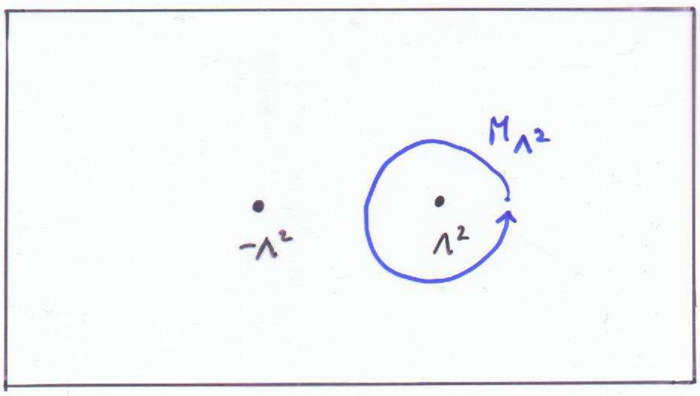
- singularity = new massless particle : a dyon

Ex: $\quad a_D(\Lambda^2) = 0 \;, \quad a(\Lambda) = \text{cost} \neq 0$

- monopole $(1,0)$ is massless at $u = \Lambda^2$
- $\tau \to \infty$; effective dual QED description : $\begin{cases} \tau_{QED} = -\frac{1}{\tau} \to 0 \\ \text{monopole} = \text{dual electron} \end{cases}$

$(n_m, n_e) \to (n_m, n_e)\, M_{u_i}^{-1}$ $\quad$ m_{BPS} is invariant

$(1,0)\, M_{\Lambda^2}^{-1} = (1,0) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (1,0)$ $\quad$ singularity is invariant under its monodromy



Talk begins: Isomonodromy

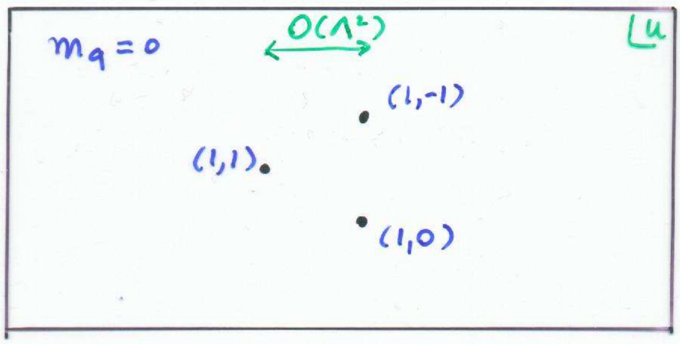
Isomonodromy : monodromies M_{u_i} are invariant under the displacement of the $\{u_j\}$

- rather obviously true here : $M_{u_0} \in \mathbb{Z}$
- first non-trivial case is 4 singularities
  → SU(2) theory with N_f quark hyper multiplets,
  $N_f = 1, 2, 3$: # sing. = $3 + N_f$ → $N_f \geq 1$

- very strong condition on $(a_D(u), a(u))^{(N_f)}$
- first instance of integrability : isomonodromic deformations

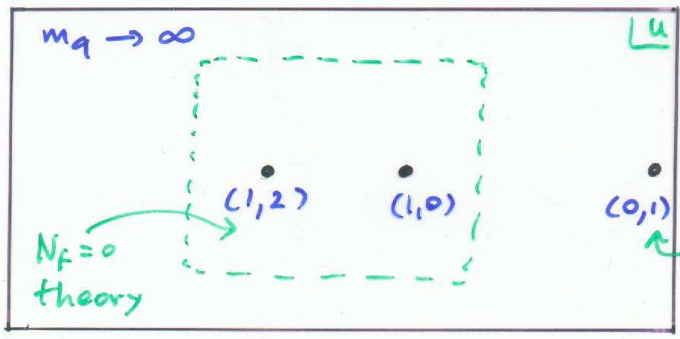
Ex: $N_f = 1$

STRONG COUPLING REGIME



three dyons

WEAK COUPLING REGIME

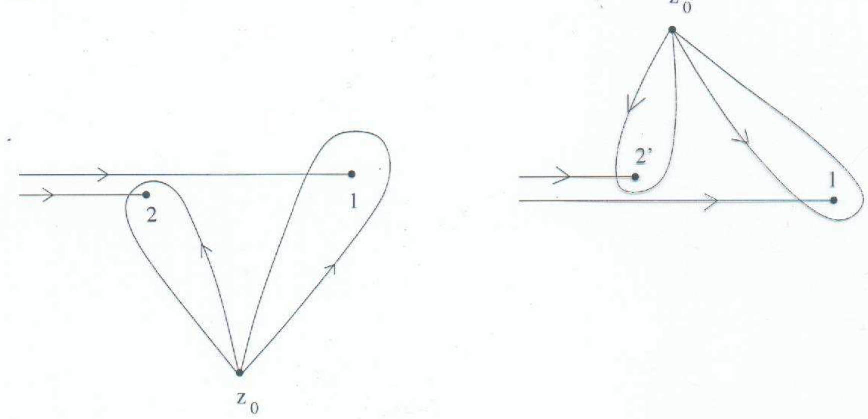


$\left\{ \begin{array}{l} u \sim m^2 \sim a^2 \\ \text{classical} \\ \text{massless quark} \\ \text{singularity} \end{array} \right.$

$\mathcal{L}_{eff} \sim (m-a) Q \tilde{Q}$

Monodromies : definitions & properties

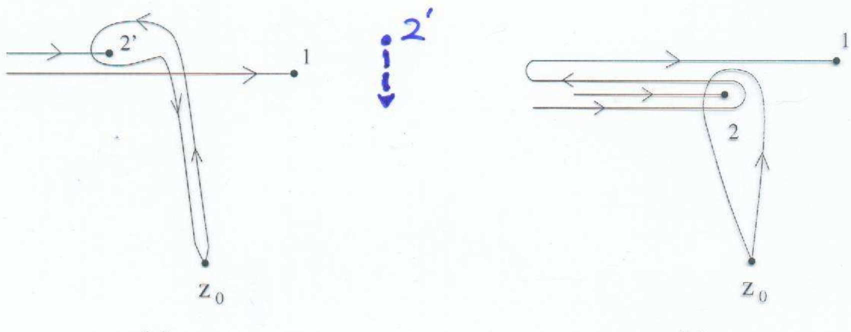
$N_f = 0$



(a) (b)

$$M_2 M_1 = M_\infty = M_1 M_{2'}$$

$M_2 = M_2(z_0)$
$M_{2'} = M_{2'}(z_0')$



(a) (b)

$M_2(z_0)$ independent of the motion of (2)

$$M_2 = M_1 M_{2'} M_1^{-1}$$

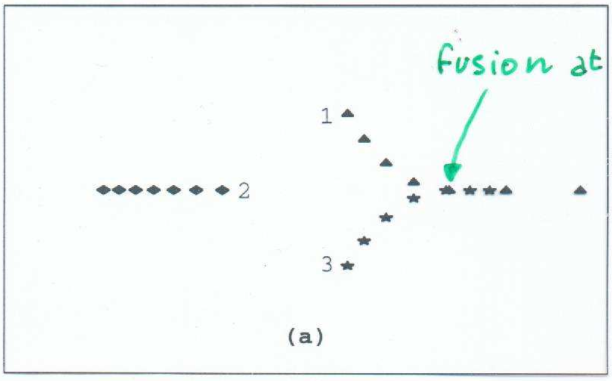
Formalism developed in (2+1)-gravity :
A. Cappelli, M. Ciafaloni, P. Valtancoli (91)
Bellini, Ciafaloni, Valtancoli (96)

Fusing & Braiding

$m_q \to \infty$

$\mathrm{Arg}(m_q) = 0$

fusion at $m_q = m_c = \frac{3}{4}\Lambda_1$

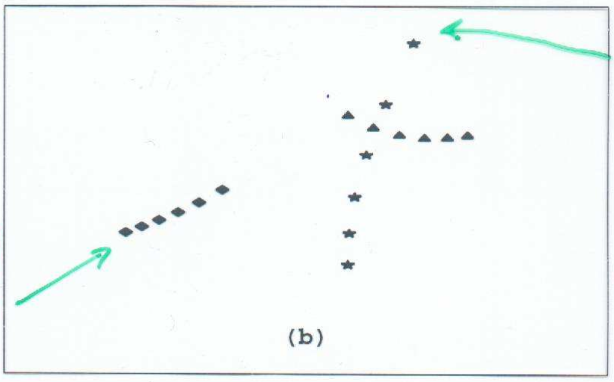


(a)

$\mathrm{Arg}(m_q) = \frac{\pi}{6}$

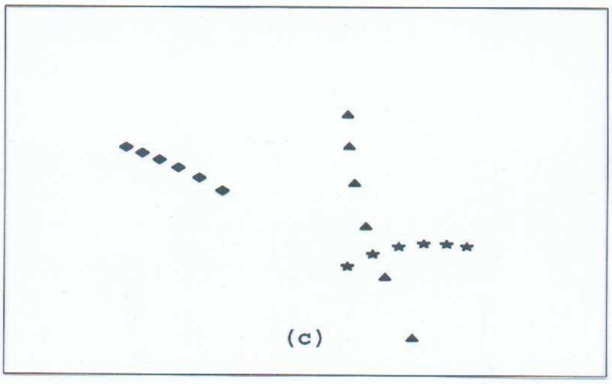
$\left\{ \begin{array}{l} u \sim m_q^2 \\ \mathrm{Arg}(u) \cong \frac{\pi}{3} \\ \text{quark sing} \end{array} \right.$

$\left. \begin{array}{l} u = \pm \Lambda_0^2 = \pm\sqrt{m_q \Lambda_1^3} \\ \mathrm{Arg}(u) = \frac{\pi}{12} \\ N_f = 0 \text{ sing.} \end{array} \right\}$

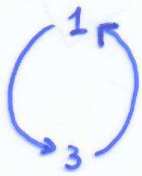


(b)

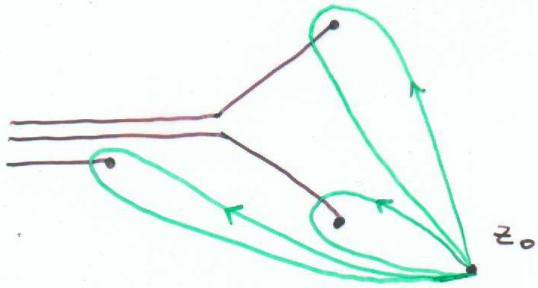
$\mathrm{Arg}(m_q) = -\frac{\pi}{6}$

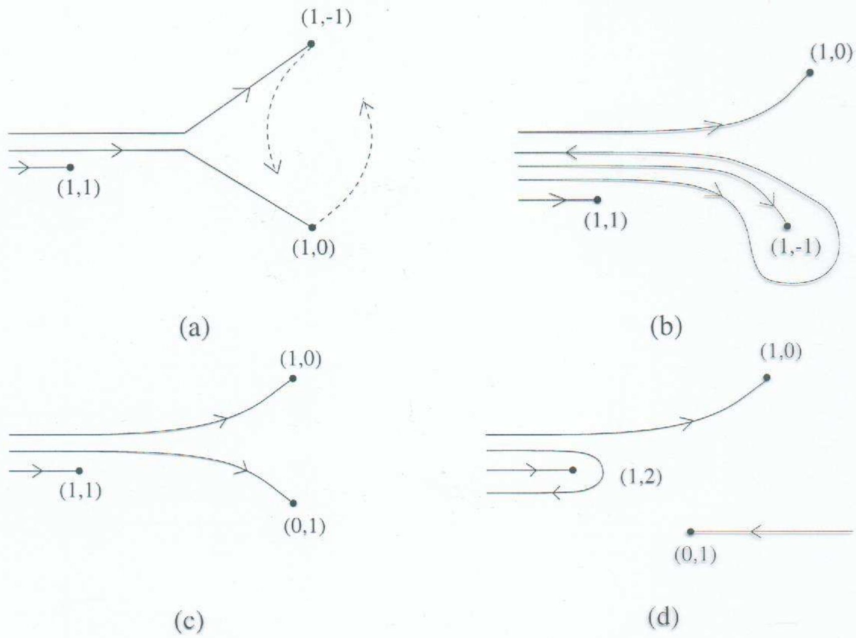


(c)



Braiding of monodromies





$$\begin{cases} M_{(m,n)} \rightarrow M^{\mp 1}_{(m',n')} \, M_{(m,n)} \, M^{\pm 1}_{(m',n')} \\ (m,n) \rightarrow (m,n) \pm (m',n') \, [\, mn' - nm' \,] \end{cases}$$

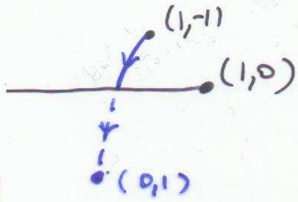
$(1,-1) \rightarrow (0,1)$ i.e. dyon $\longrightarrow$ quark

$(1,1) \rightarrow (1,2)$

• Braiding ~ quark-monopole transmutation

• Fusing ~ new type of singularity (power-law)
superconformal point (Argyres, Douglas)

• Braiding ~ analytic continuation
~ change of patch



• weak-coupling & strong-coupling regimes necessarely on two different patches

• same massless state in the spectrum, no discontinuity

• change of field coordinates, i.e. $A_\mu^D \to A_\mu$

• $m_q \in [0,\infty]$ needs two Abelian fields to be described

Fusing ~ true discontinuity

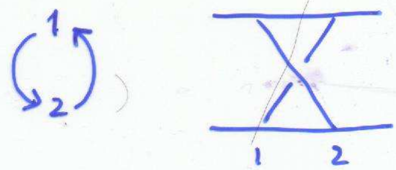
$$M_c = \underset{\substack{\uparrow\\ \text{strong-coupling patch}}}{M_{(1,0)} M_{(1,-1)}} = \underset{\substack{\uparrow\\ \text{weak-coupling patch}}}{M_{(0,1)} M_{(1,0)}}$$

• Fusing is required by consistency of the opposite braidings for $\mathrm{Arg}(m_q) > 0$ and $\mathrm{Arg}(m_q) > 0$

Isomonodromy → fusing → superconf. point
"unexpected"

Braiding & Fusing Identities

$$\sigma_{12}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} M_2 M_1 M_2^{-1} \\ M_2 \\ M_3 \end{pmatrix}$$



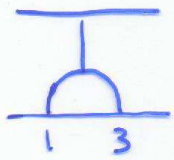
σ_{ij} , $i,j=1,2,3$ braid group $\mathcal{B}_3$ representation on $V_1 \otimes V_2 \otimes V_3$

Yang-Baxter identity

$$\sigma_{23}\,\sigma_{13}\,\sigma_{12} = \sigma_{12}\,\sigma_{13}\,\sigma_{23}$$

— • —

$$F_{31}\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} M_3 M_1 \\ M_2 \end{pmatrix}$$



Pentagonal identity (Moore - Seiberg)

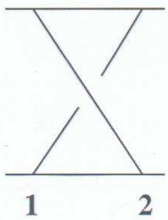
$$\sigma_{23}\,F_{12} = F_{12}\,\sigma_{13}\,\sigma_{23}$$

- These are all consequences of isomonodromy;
- σ_{ij} and F_{ij} operators exist which act on $(a_D(u), a(u))$, not only to its monodromies

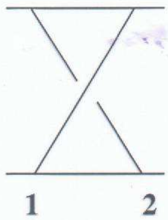
→ decomposition into "intermediate states" (Moore - Seiberg (88), "dual" amplitudes) see later

- differential equation for isomonodromy
  $N_f = 1$: Painlevé $\underline{VI}$ (see our paper)

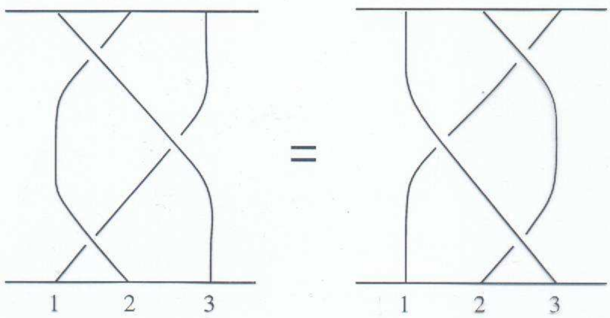
σ_{ij} :



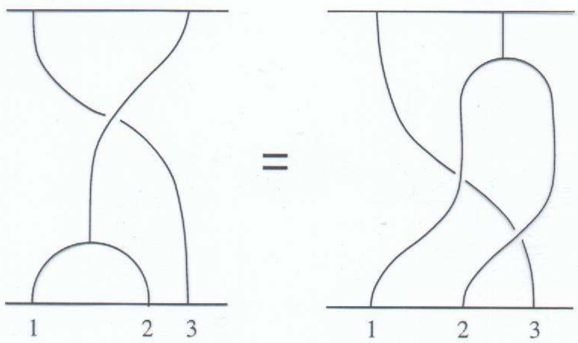
σ'_{ij} :



$=(\sigma_{ji})^{-1}$



$$\sigma_{23}\,\sigma_{13}\,\sigma_{12} = \sigma_{12}\,\sigma_{13}\,\sigma_{23}$$



$$\sigma_{23}\,f_{12} = f_{12}\,\sigma_{13}\,\sigma_{23}$$

Application of Braid relations

$N_f = 1, 2, 3$ BPS formula

$$m_{BPS} = \left| n_m a_D(u) + n_e a(u) + \sum_{i=1}^{f} s_i \frac{m_{qi}}{\sqrt{2}} \right|$$

$\{s_i\}$ = (pseudo) Baryonic quantum #s
not completely know before

$$\begin{pmatrix} a_D \\ a \\ \frac{m_i}{\sqrt{2}} \end{pmatrix} \to \left(\begin{array}{c|c} M & \begin{matrix} q_{Di} \\ q_i \end{matrix} \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} a_D \\ a \\ \frac{m_{qi}}{\sqrt{2}} \end{pmatrix} = \mathcal{M} \begin{pmatrix} a_D \\ a \\ \frac{m_{qi}}{\sqrt{2}} \end{pmatrix}$$

Affine monodromies : (q_i, q_{Di}) not know either

We have :
- used braid relations as equations
- matched $N_f \to (N_f - 1)$ theories at quark decoupling
- obtained complete affine monodro.
- obtained $\{s_i\}$ by enforcing the consistency under spectral flow

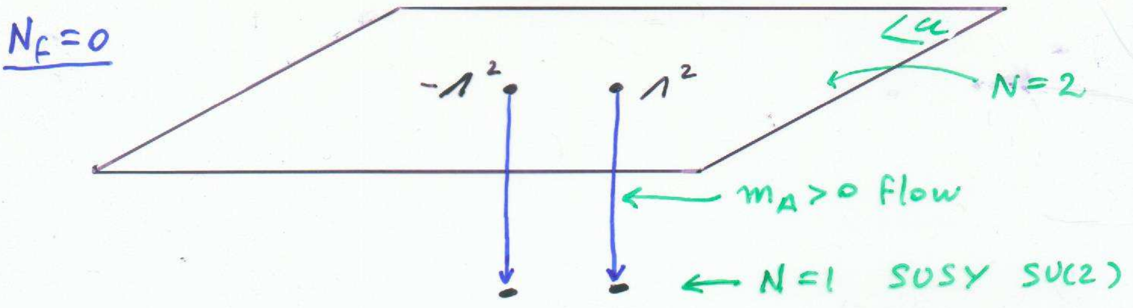
$$(n_m, n_e, s_i) \xrightarrow{m_\infty} (n_m', n_e', s_i')$$

both in the spectrum

Related works : ('97 - '98)
Bilal , Ferrari
Brandhuber , Stieberger
Konishi , Terao
Alvarez-Gaumé , Mariño , Zamora

Confinement as dual Higgs (Seiberg-Witten)

$N_f = 0$



At singular point $u = \Lambda^2$ add massless monopole field in the low-energy action

$$\begin{pmatrix} & A_\mu & \\ \lambda & & \chi \\ & \phi & \end{pmatrix}_D + \text{hypermultiplet} \quad M = \begin{pmatrix} \psi_M \\ M \end{pmatrix}, \; \tilde{M} = \begin{pmatrix} \psi_{\tilde{M}} \\ \tilde{M} \end{pmatrix}$$

Φ ↑ dual QED near $u \sim \Lambda^2$ — 1 Dirac — 2 Higgs

$N=1$ Superpotential — breaking $N=2 \to N=1$

$$W = \sqrt{2}\, \phi_D M \tilde{M} + m_A \, \mathrm{tr} \langle \Phi^2 \rangle$$

$$\begin{cases} \dfrac{\delta W}{\delta \phi_D} = 0 = \sqrt{2}\, M \tilde{M} + m_A \dfrac{\partial u}{\partial a_D} \\[2ex] \dfrac{\delta W}{\partial M} = 0 = a_D M = 0 \end{cases}$$

$+ \langle M \rangle = \langle \tilde{M} \rangle$ vanishing D term

- any point: $a_D \neq 0 \rightarrow \langle M \rangle = \langle \tilde{M} \rangle = 0 \rightarrow m_A = 0$ NO BREAKING
- $u = \Lambda^2$: $a_D = 0 \quad \langle M \rangle = \langle \tilde{M} \rangle = \left(- m_A \dfrac{\partial u}{\partial a_D} \right) \sim (m_A \Lambda)^{1/2}$

monopole condensation $\rightarrow$ confinement of original SU(2) gluons $A_\mu^\pm$, A_μ^3

('t Hooft; Polyakov)

two steps & two scales $\quad a = \Lambda \, , \; m_A$

$$SU(2) \longrightarrow U(1) \longrightarrow \text{full breaking}$$

$$\langle \phi^a \rangle = a = \Lambda \qquad\qquad \langle M \rangle = \sqrt{m_A \Lambda}$$

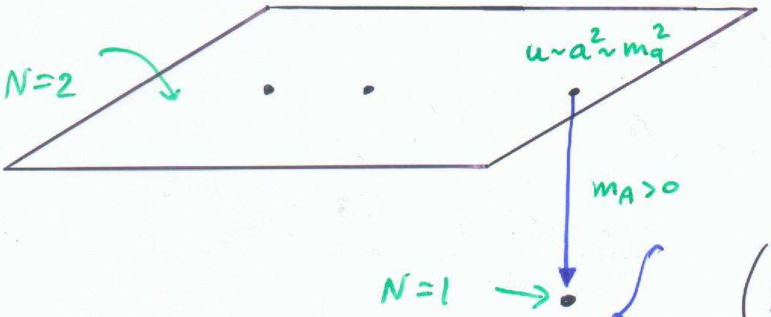
$$A_\mu^\pm \quad m \sim |g|\, a = |g|\,\Lambda \qquad\qquad A_\mu^3 \longrightarrow A_\mu^D \quad m \sim \frac{\sqrt{m_A \Lambda}}{|g|}$$

- $N=1$ SuSy patners have same mass m (Konishi)
- $N=2$ patners of gluons $\Phi^a = (\phi^a, \chi^a)$ do not have indipendent scale to be sent $\to \infty$
- further breaking of $N=1$ to real pure QCD.....
  .... phase transition (Alvarez-Gaumé et al.)
- Susy QCD has vacuum dominated by matter field condensates, not gluon condensates.

Confinement = Higgs in the N=1 Susy SU(2) theory with "lepton" doublets

Generalization of above to $N_f = 1$

Higgs phase ~ weak coupling $a \sim m_q \gg \Lambda$



$$\begin{pmatrix} & A_\mu & \\ \lambda & & \chi \\ & \phi & \end{pmatrix}$$ third component of SU(2)

"lepton" hypermultiplet

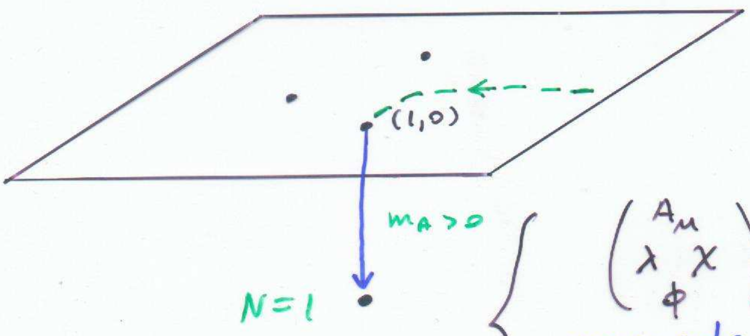
2 Dirac

$$Q_i = \begin{pmatrix} \psi_i \\ \varphi_i \end{pmatrix} \quad \tilde{Q}_i = \begin{pmatrix} \tilde{\psi}_i \\ \tilde{\varphi}_i \end{pmatrix} \qquad i = 1,2$$

4 Higgs

- similar to original ex. of SU(2) theory;
- double of Higgses $\langle \varphi_2 \rangle = \langle \tilde{\varphi}_2 \rangle$
- scales $a = m_q$, $\langle \varphi_2 \rangle = \sqrt{m_A m_q}$

Confinement phase ~ strong coupling $a \sim m_q \sim \Lambda$



$$\begin{pmatrix} & A_\mu & \\ \lambda & & \chi \\ & \phi & \end{pmatrix}_D$$ dual effective QED

monopole hyperm. as for $N_f = 0$

$$M = \begin{pmatrix} \psi_M \\ M \end{pmatrix} \quad \tilde{M} = \begin{pmatrix} \psi_{\tilde{M}} \\ \tilde{M} \end{pmatrix}$$

- scales: $a \sim \Lambda$, $\langle M \rangle = \langle \tilde{M} \rangle = \sqrt{m_A \Lambda}$

As expected (previous argument), there is a smooth transition in the spectrum, as $m_q \in [0, \infty]$

BUT : non-trivial mechanism!

- simple, "perturbative" description of the confinement regime by dual fields

- change of field variables dictated by the braid relations, i.e. quark-monopole transmutation
  dual fields & dual Higgs description is required by analytic continuation of (a_D, a) + isomonodromy

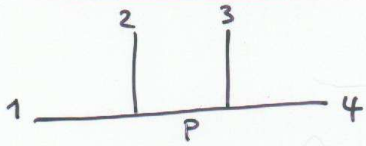
$\Rightarrow$ $\boxed{\text{Nice physical application of braids}}$

furthermore: (Bilal, Ferrari)

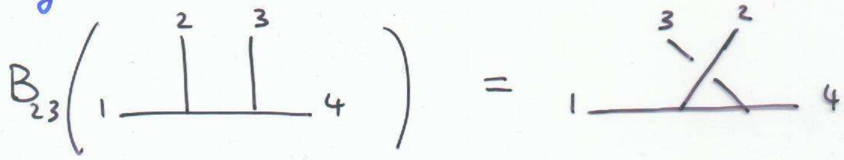
- almost all dyon states decay in the strong-coupling regime

Isomonodromy & Conformal Field Theory

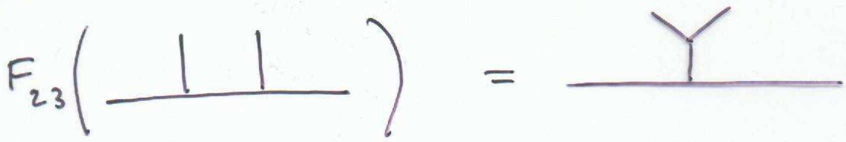
Moore & Seiberg (88): Yang-Baxter & Pentagonal identities for conformal blocks

$\langle \phi_1(z_1)\, \phi_2(z_2)\, \phi_3(z_3)\, \phi_4(z_4) \rangle_p \approx$ 

Braiding:



Fusing:



- These B & F act on fields, not on singularities

- Isomonodromy of conformal blocks (Dotsenko, Fateev) (84) of Rational CFT due to:
  a) local OPE $\quad \phi_1(z_1)\,\phi_2(z_2) = \sum_p (z_1 - z_2)^{h_p - h_1 - h_2}\, \phi_p(z_2)$

  b) Fuchsian diff. equations for conformal blocks (B.P.Z.; Christe, Ravanini)

$$\text{Conformal Symmetry} \longrightarrow \text{Isomonodromy}$$

$$\xleftarrow{\quad ? \quad}$$

Heuristic argument

2D Conformal symmetry $\rightarrow$ $\frac{\partial}{\partial \bar z} T(z) = 0$ $\left(\begin{array}{c} \partial_\mu T^{\mu\nu} = 0 \\ + T_\mu^{\ \mu} = 0 \end{array} \right)$

This is stable under analytic reparametrizations

$$\frac{\partial}{\partial \bar z} T(z) = 0 \iff \frac{\partial}{\partial \bar w} T(w) = 0 , \qquad z = \sum_{n \in \mathbb{Z}} \epsilon_n w^{n+1}$$

- Analyticity in Susy : chiral decomposition of certain representations

Ex: $N=1$ chiral rep. $\Phi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}$ χ_α Weyl fermion

Superpotential $W(\Phi)$ is also chiral (Seiberg)

$$\bar D_\alpha \Phi = 0 \ \ \& \ \ \bar D_\alpha W = 0 \ \rightarrow \ \frac{\delta}{\delta \bar \Phi} W(\Phi) = 0$$

This is also stable under analytic field redefinitions

$$\frac{\delta}{\delta \bar \Phi} W(\Phi) = 0 \iff \frac{\delta}{\delta \bar \Psi} W(\Psi) = 0 , \qquad \Psi = \sum_n \epsilon_n \Phi^{n+1}$$

Actually: $\langle \Phi \rangle = a \rightarrow$ reparametrization of moduli space

Isomonodromic deformations $\approx$ covariance above

But :

CFT : true invariances $\epsilon_{-1}, \epsilon_0, \epsilon_1$ $SL(2, \mathbb{C})$

Susy theories : no obvious $SL(2, \mathbb{C})$ invariance

Relation (a_D, a) with Conformal Blocks

- $(a_D(u), a(u))^{(N_f)}$ satisfy Fuchsian diff. equations with $(3+N_f)$ regular singularities $(N_f = 0,1,2,3)$

- 3 singularities $\longrightarrow$ Hypergeometric function
  $\longrightarrow$ $SL(2,\mathbb{C})$ invariance for free

$\Rightarrow$ $\frac{da_D}{du}$, $\frac{da}{du}$ can be written as 4-point conformal blocks

$N_f=0$: $\left(\frac{da_D}{du}, \frac{da}{du}\right) \sim \langle \phi_1(u)\, \phi_2(-1)\, \phi_3(1)\, \phi_4(\infty) \rangle_p$ $\quad (\Lambda^2 \equiv 1)$

$p = 1, 2$

Conditions:

- try minimal models $\quad c(p,p') = 1 - 6\frac{(p-p')^2}{pp'}$

- two-dimensional OPE $\rightarrow$ $\phi_1 = \phi_{1,2}$, $(r,s) = (1,2)$

- match logarithmic singularities
  $\rightarrow$ $c = c(1,p')$ logarithmic CFTs (Gurarie)

Result:

$N_f = 0$: $\quad \frac{da_D}{du} \sim \frac{\langle \phi_{12}(u)\, \phi_{12}(-1)\, \phi_{12}(1)\, \phi_{12}(\infty) \rangle_1}{((u+1)(u-1))^{1/4}} \sim F(\tfrac{1}{2}, \tfrac{1}{2}, 1; \tfrac{1-u}{2})$

$\frac{da}{du} \sim (1+u)^{1/2}\, F(\tfrac{1}{2}, \tfrac{1}{2}, 1; \tfrac{2}{1+u})$ $\sim$ other block

$c = c(1,2) = -2$

Other 3-sing. cases : $N_f = 1, 2, 3$ at the critical m_q values of fusion of $(1+N_f)$ sing.

Find similar operator representation of $\left(\frac{da_D}{du}, \frac{da}{du}\right)$ with

$N_f = 1$, $c(1,3) = -7$; $N_f = 2$, $c(1,4) = -\frac{25}{2}$;

$N_f = 3$, $c(1,6) = -24$.

Conclusion :

In these simple cases, Moore & Seiberg analysis of isomonodromy applies word by word.

General case $(3+N_f)$ sing.

Explicit expressions of $\frac{da_D}{du} = \oint_{\gamma_1} \frac{dx}{y}$, $\frac{da}{du} = \oint_{\gamma_2} \frac{dx}{y}$

have no $SL(2,\mathbb{C})$ covariance

→ ISOMONODROMY WITHOUT CONFORMAL SYMMETRY

- other covariance ?
- other integrable system ?

: - 2D topological field theories (Morozov et al; (K. Ito & S-K Yang)

- (quantum) Liouville theory (Matone, ...)

Conclusions

Some physical aspects of the S–W solution have been clarified:

- quark - monopole transmutation & Higgs vs. Confinement phases
- full BPS spectrum (baryonic #s)
- origin of superconformal points

Mathematical properties:

- Isomonodromy : signals integrability
- Isomonodromy without conformal symmetry w.r.t. the moduli $\{u\}$
- General framework behind S-W ansatz ?
- Integrability ?

→ Martinec, Warner; Witten, Donagi; ...
Systems described by same hyper-elliptic curves

→ Flohr : $c=-2$ CFT description of all S-W abelian differentials : conformal symmetry w.r.t. auxiliary coordinate of hyper-elliptic Riemann surface

Open problems

- The beta-function of S–W theory : not the naive one $\Lambda \frac{d\tau}{d\Lambda} = -2u \frac{d\tau}{du}$ (Ritz ; Dolan ; ....)

  → CFT & Liouville theory modelling (Matone, ..)
  → corrections to instanton calculus (Fucito et al.)
- higher-order terms in the effective action