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Boundary Conformal Field Theory and the Physics of D-branes

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PhD THESIS - XIV Ciclo

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December 2001

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Chapter 1

Introduction

String theory has been developing very fast in the recent years. The idea of *electric-magnetic duality* has been extended from supersymmetric gauge field theories [1] to string theory: the five supersymmetric ten-dimensional string theories [2], although markedly different in their perturbative excitations, have been related by non-perturbative dualities and shown to be different limits of a unique underlying theory. The duality groups, discrete versions of the continuous duality groups already known from the study of supergravity theories, mix fundamental string states with solitonic solutions of the low-energy equation of motions, that are $(p+1)$-dimensional space-time defects called p-branes. An important progress was the discovery [10] that these solitons can be described using open strings with Dirichlet boundary conditions and therefore they were called D-branes.

D-branes are intrinsic fundamental excitations of string theory and their tension and charge can be unambiguously calculated and expressed in terms of the string scale and coupling constant. Their properties were first studied in flat space-time where they can be described as hyper-planes on which the open strings end. However the general description of D-branes in non-trivial metric backgrounds should be given in the framework of boundary conformal field theory. Conformal field theory techniques have already played an essential role in the perturbative formulation of string theory [4]: in particular, the space of the conformal field theories with a suitable central charge coincides with the space of the classical string theory vacua. Conformal field theories with a boundary can similarly describe the D-branes configurations compatible with a given background.

D-branes are one of the most fruitful ideas introduced in string theory: first of all, they provide a link between gauge and gravitational theories. The origin of this link is in the open-closed string duality: a diagram describing the interaction between two D-branes through the exchange of closed string modes can be given a dual interpretation as a one-loop diagram of open strings stretched between the two D-branes. Since the exchange of closed strings corresponds to the gravitational interactions while the exchange of open strings corresponds to the gauge interactions, one can use the D-branes to relate gravitational and gauge theory problems. For instance D-brane configurations have been used to model black-holes and to obtain a statistical interpretation of their entropy [15]. Furthermore supersymmetric gauge theories have been described as effective theories on the brane worldvolume and their moduli space has been identified with the space of the possible deformations of the brane configuration [16].

Boundary conformal field theory methods [19, 24] have been first applied in string theory for the study of Type I models. These can be constructed in a systematic way [7] starting from a closed string theory and performing an orbifold-like construction using the world-sheet parity. One of the steps of the construction then requires the introduction of suitable open string sectors that can be interpreted as D-brane configurations. D-branes and in particular Type I models have also suggested new possibilities for string phenomenology, mainly related to the fact that the gauge and the gravitational interactions originate from two different sectors. As a consequence the gauge and the gravitational couplings are not directly related as in the closed string models [68]. Open string models are also of interest because the gauge fields are confined on the branes giving natural realizations in string theory of the recent proposal for large extra dimensions and *brane world* scenarios [69].

In this Thesis we report our work along two closely related line of research: boundary conformal field theory [**toi**] and Type I string compactifications [**shift**,**bsb**].

In boundary CFT, we have addressed the problem of determining the conformally invariant boundary conditions for a general bulk theory. In a seminal paper [24], Cardy gave a model-independent solution when the torus partition function is the charge-conjugation modular invariant and the boundary conditions preserve the full bulk symmetry. In our work, we have exploited orbifold [32] and simple current techniques [35] to obtain boundary states for more general modular invariants and boundary states that preserve only part of the bulk symmetries. These techniques have been also developed by other authors [42, 41, 54] and a general class of solutions has been found [43]. We have performed a detailed study of the boundary states for the conformal theories at $c = 1$ and the superconformal ones at $c = 3/2$ [**toi**]. These present both continuous families of theories as well as isolated points. The orbifold construction (often implemented by simple currents) maps one theory to another by modding out a discrete symmetry. Whenever orbifold relations between the various theories are present, it is easy to derive corresponding relations for the boundary states. A beautiful structure is displayed by the set of isolated conformal theories given by non-abelian orbifolds of the form $SU(2)/G$, where G is one of the symmetry groups of regular polyhedra in $\mathbb{R}^3$, the tetrahedron ($\mathbf{T}$), the octahedron ($\mathbf{O}$) and the icosahedron ($\mathbf{I}$) [55, 56]. The chiral algebras of these orbifolds present an interesting structure of twisted sectors and the boundary states provide nice examples of fractional branes. Furthermore the orbifold construction allows the determination of symmetry breaking boundaries starting from the Cardy states of another model.

In our study of Type I compactifications, we have addressed the problem of supersymmetry breaking in open string models. In closed string models the only known perturbative mechanism of supersymmetry breaking is the Scherk-Schwarz mechanism [63, 64]: supersymmetry is broken by the compactification, due to a different behaviour of the fermionic and bosonic fields in the internal space. The breaking scale is thus proportional to the compactification scale. The same mechanism displays a new feature in the presence of D-branes [66]: supersymmetry is broken at tree level only on the branes parallel to the direction used for the Scherk-Schwarz deformation, while the branes orthogonal to that direction have a supersymmetric massless spectrum. In this sector the breaking of supersymmetry is then induced via radiative corrections. This phenomenon was named *brane supersymmetry* [66, 81, 67]. The models studied in [**shift**] provide various examples of this phenomenon, as well as of brane configurations with henanced supersymmetry for the full spectrum, not only for the massless modes. Moreover they exhibit $N = 4 \to N = 1$ or $N = 2 \to N = 1$ partial supersymmetry breaking and involve interesting D-brane configurations. Indeed they contain four types of D-branes which combine in doublets and quadruplets of branes in order to be compatible with the action of the orbifold group. This class of models is characterized by the presence of supersymmetric branes embedded in a bulk space-time with less or none supersymmetries. In open string models one can consider also the opposite situation: non-supersymmetric branes floating in a supersymmetric universe. This mechanism is called *brane supersymmetry breaking* and was proposed in [72, 74]. Besides its interest for phenomenological applications, the mechanism of brane supersymmetry breaking makes possible the construction of chiral four-dimensional Type I models, whose tadpole conditions do not allow for supersymmetric brane configurations [**bsb**]. Since the type and the number of branes present in the vacuum follow form the parent closed string, in these models the breaking of supersymmetry is *required* by the consistency of the theory.

Models with brane supersymmetry breaking represent just a first step in the attempt of understanding string theory vacua without supersymmetry. Since supersymmetry is broken only in a localized region of space-time one can hope to have a better control of the system than in model where supersymmetry is broken from the start in the whole space-time.

This thesis is organized as follows. In the second Chapter, we briefly recall the main ideas at the basis of string theory and introduce the concept of a brane, first as a part of the perturbative vacuum solutions of Type I models, then as a solitonic solution of the low-energy equations of motion and finally as a probe for general string theory vacua. We explain that the space of conformal field theories with a suitable central charge represents the space of classical vacua of string theory and that the space of conformally invariant boundary conditions coincides with the possible brane configurations compatible with a given background. We also illustrate with a simple example the method of open descendants.

In the third Chapter we review some general features of conformal field theories and boundary conformal field theories. We introduce the boundary states and discuss the constraints they should satisfy in order to give consistent boundary conditions. We then address the problem of finding all the boundary conditions compatible with a given bulk modular invariant and discuss the completeness conditions for boundary states [27]. We then consider orbifold CFTs, paying attention to their chiral algebras and their characters. Finally we introduce the simple currents [35] as a useful tool for the construction of new modular invariants and show how orbifold and simple current techniques can be extended to the boundary states. The general analysis is illustrated by the examples of the compactified boson at $c = 1$ and of the $SU(2)_k$ Wess-Zumino-Witten models.

In the fourth Chapter we perform a careful analysis of the boundary states for rational conformal field theories at $c = 1$ and superconformal field theories at $c = 3/2$. We derive the chiral algebra for the **T**, **O** and **I** models, both in the conformal and in the superconformal case, and find complete sets of boundaries for various modular invariants.

In the fifth chapter we turn to Type I compactifications and study the open descendants of $T^6/Z_2 \times Z_2$ freely acting orbifolds which provide many instances of the phenomenon of brane supersymmetry. We then discuss vacua containing branes and anti-branes of the same type and finally illustrate the mechanism of brane supersymmetry breaking with two chiral four-dimensional models.

In Appendix A we collect the characters and the S matrices for the models discussed in Chapter 4 and in Appendix B we display the characters needed for the construction of the $T^6/Z_2 \times Z_2$ Type I compactifications discussed in Chapter 5.

Let us finally mention some work that we did in another line of research not described in this thesis [**ct1**,**ct2**]. The Zamolodchikov c-theorem implies the irreversibility of the renormalization-group flow in two dimensions, by proving that the conformal anomaly monotonically decreases along the flow. The proposals for extending this theorem above two dimensions have been tested by computing the Euler term of the trace anomaly for several free field theories in arbitrary dimension.

Chapter 2

String theory, branes and conformal invariance

Perturbative string theory [3] describes the propagation of a one-dimensional object in a given background space-time. The integration over all the possible string trajectories in the path integral formulation of quantum mechanics amounts to integrating over all possible two-dimensional surfaces. The embedding of the string world-sheet into the space-time is described by a two-dimensional field theory. The simplest example is the propagation of a string in flat Minkowski space-time. The world-sheet action is

$$S = -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu} + \frac{\gamma}{2\pi}\int d^2\xi\sqrt{-g}\mathcal{R}\ , \tag{2.1}$$

and it involves a two-dimensional metric $g_{\alpha\beta}$ defined on the world-sheet Σ and D scalar field $X : \Sigma \to \mathbb{R}^{1,D-1}$. The second term in (2.1) is a topological term, namely the Euler characteristic $\chi(\Sigma)$, with $\mathcal{R}$ the scalar curvature of the two-dimensional metric. Let us consider first a theory of closed oriented strings. In this case only closed oriented Riemann surfaces can appear as world-sheet. Their topology is completely specified by an integer h, called the genus, which counts the number of handles of the surface and is simply related to the Euler characteristic according to $\chi = 2(1-h)$. The integration over all the two-dimensional surfaces splits into a sum over topologies and an integration over two-dimensional metrics. Since each term in the sum is multiplied by a factor $e^{-\gamma\chi}$, $g_s = e^\gamma$ is identified as the coupling constant of the string perturbative expansion. The parameter γ can be interpreted as the vacuum expectation value of a scalar field, called the dilaton, generically present in the spectrum of any string theory model. We also note that at any order in the perturbative expansion there is only one contribution.

As a first step we want to identify the perturbative Hilbert space of the theory and therefore we need to quantize the action (2.1). This action is invariant under diffeomorphisms $\delta g_{\alpha\beta} = \partial_\alpha \epsilon_\beta + \partial_\beta \epsilon_\alpha$, Weyl re-scalings $\delta g_{\alpha\beta} = -2\omega g_{\alpha\beta}$ and Poincaré transformations in $\mathbb{R}^{1,D-1}$. We can remove part of the gauge freedom fixing a conformally flat metric $g_{\alpha\beta} = e^\Lambda \eta_{\alpha\beta}$. This gauge, called the conformal gauge, is preserved by suitable combinations of diffeomorphisms and Weyl re-scalings. These residual transformations are the conformal transformations, analytic and anti-analytic reparametrizations of the plane. The presence of this residual gauge symmetry is crucial for the consistency of the model, since it guarantees the decoupling of unphysical states of negative norm. Various methods of quantization have been applied to this system, the most efficient being BRST quantization [4]. One first introduces ghost and antighost fields, c_α and $b_{\alpha\beta}$, associated to the gauge fixing of reparametrization invariance and then a BRST operator Q. The physical Hilbert space is defined as the cohomology of Q, acting on the Hilbert space of the combined ghost (c, b) and matter $(X^\mu, \mu = 0, \ldots, D-1)$ systems.

Conformal invariance is anomalous at the quantum level. The anomaly can be explicitly calculated and the result is proportional to the *central charge* c of the conformal field theory. Since in string theory the conformal symmetry is a gauge symmetry, we should demand that $c_{tot} = c_m + c_g = 0$ *, where c_m is the central charge of the matter and c_g is the central charge of the ghosts. The latter contribution is universal and it is given by $c_g = -26$. Therefore the requirement of conformal invariance completely fixes the value of the central charge for the matter system: $c_m = 26$. From the point of view of BRST quantization, the requirement $c_{tot} = 0$ is equivalent to the nilpotency of the BRST operator $Q^2 = 0$. For the simple model in (2.1), $c_m = D$ and one obtains the well-known result that the critical dimension for the bosonic string is $D = 26$.

In order to determine the spectrum of the model, it is convenient to use the light-cone gauge. In this case one first fixes all the gauge symmetries of the system and then proceed to the quantization: the advantage is that of working with physical degrees of freedom with a positive definite inner product, the disadvantage is that Lorentz invariance is not manifest. We define $X^\pm = \frac{1}{2}(X^0 \pm X^{D-1})$ and fix $X^+ = x^+ + 2\alpha' p^+ \tau$. Using the constraint $T_{\alpha\beta} = 0$, coming from the equations of motion of the worldsheet metric, X^- can be expressed in terms of the transverse coordinates X^i that are the only physical degrees of freedom. The X^i satisfy a free wave equation in two dimensions and therefore can be expanded in two sets of normal modes, α^i_n and $\bar{\alpha}^i_n$, corresponding respectively to the left and to the right moving waves. The physical states are easily obtained acting with the creation operators α^i_{-n}, $\bar{\alpha}^i_{-n}$ with $n > 0$ on the ground state $|0\rangle \otimes |0\rangle$. From the mass-shell relation

$$m^2 = \frac{2}{\alpha'} \left(N + \bar{N} - \frac{D-2}{12} \right) , \tag{2.2}$$

*Allowing $c_{tot} \neq 0$ would require the quantization of the Liouville mode.

where N and $\bar{N}$ are the left and right number operators, one see that in the critical dimension the states $\alpha_{-1}^{i}\bar{\alpha}_{-1}^{j}|0\rangle \otimes |0\rangle$ are massless. This two index tensor can be decomposed in a traceless symmetric part, in an antisymmetric part and in a trace part which are exactly the right degrees of freedom for a graviton $g_{\mu\nu}$, an antisymmetric two-form field $B_{\mu\nu}$ and a scalar field ϕ, the dilaton.

Conformal invariance greatly simplifies the calculation of scattering amplitudes, since a worldsheet with n asymptotic string states can be mapped by a conformal transformation to a compact worldsheet with n punctures, corresponding to the insertions of local operators of the conformal field theory. The calculation of a perturbative scattering amplitude in string theory is then equivalent to the evaluation of CFT correlators on arbitrary Riemann surfaces. There is however a crucial difference between CFT correlators and string theory amplitudes since the latter, due to the prescription of integrating over all the two-dimensional metrics, are also to be integrated on the moduli space of Riemann surfaces whereas CFT correlators are usually evaluated at a given point in the moduli space. The first term in the perturbative series is the sphere, the second the torus. A torus can be described as the quotient of the complex plane by a two-dimensional lattice Γ. Using rotation and scale invariance, one can always identify one of the two basis vectors of the lattice with the unit vector along the real axis. Therefore the torus is completely specified by a single complex parameter $\tau = \tau_1 + i\tau_2$ belonging to the upper-half plane, *i.e.* $\tau_2 > 0$. If two modular parameters τ and τ' are related by an $SL(2,\mathbb{Z})$ transformation, they define different fundamental cells for the same lattice Γ and therefore the resulting tori are equivalent. The moduli space of compact Riemann surfaces at genus one is then a fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper-half plane, as for instance

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} \; |\tau_2 > 0, -\frac{1}{2} < \tau_1 \leq \frac{1}{2}, |\tau| \geq 1 \right\} \; . \tag{2.3}$$

The path integral on the torus, without any vertex operator insertion, is a particularly important amplitude. In an operator formalism it coincides with the trace of the evolution operator on the Hilbert space, that is with the partition function $Z(\tau)$ of the model. According to the previous discussion, $Z(\tau)$ must be a well-defined function on the fundamental domain $\mathcal{F}$. Since the modular group $SL(2,\mathbb{Z})$ is generated by the two transformations $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$ the requirement of *modular invariance* for the partition function can be stated as $Z(\tau) = Z(\tau + 1) = Z(-1/\tau)$. The requirement of modular invariance at genus one is of fundamental importance for the consistency of string theory. First of all integration over the parameter τ in a string amplitude is the analogous of the integration over a Schwinger parameter in the evaluation of a Feynman diagram; the restriction to a fundamental domain removes from the integration region all the potentially divergent contributions, leading to finite amplitudes. Moreover, as we will see in more detail in the sequel, modular invariance is a completeness condition for the spectrum of the closed string states. Finally modular invariant string spectra are free from gauge and gravitational anomalies. Modular invariance at higher genus is an important constraint as well. For instance modular invariance at genus two implies the spin-statistics relation.

Let us now turn to the superstring. In this case a consistent quantization of the theory requires the presence of $N = 1$ superconformal symmetry on the worldsheet. After gauge fixing and with a flat background, the action reduces to a model of free scalars and Majorana fermions

$$S = -\frac{1}{4\pi\alpha'} \int d^2\xi \left(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} + i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi^\nu \eta_{\mu\nu} \right) . \tag{2.4}$$

The total central charge of the system should vanish again. Since in this case the reparametrization and super-reparametrization ghosts have conformal anomalies $c_g + c_{sg} = -26 + 11$, the matter should contribute $c_m = 15$. The critical spacetime dimension is $D = 10$ since each super-coordinate (X^μ, ψ^μ) contributes $3/2$ to the total central charge.

Let us consider for the moment one chiral part of the spectrum, say the left moving modes. For the worldsheet supercurrent one can chose either periodic or anti-periodic boundary conditions. The sector where the supercurrent is anti-periodic is called the NS sector, while the sector where the supercurrent is periodic is called the R sector. In the NS sector the fermionic modes ψ^i_n carry a label $n \in \mathbb{Z} + 1/2$ and the mass-shell condition reads $\alpha' m^2 = N - (D-2)/16$. At the critical dimension, it contains a tachyonic vacuum state $|0\rangle$ and a massless vector $\psi^i_{-1/2}|0\rangle$. In the R sector we have modes ψ^i_n with $n \in \mathbb{Z}$ and the mass-shell condition reads $\alpha' m^2 = N$. The fermionic zero modes ψ^i_0 fulfill the anti-commutation relations of a Clifford algebra $\{\psi^i_0, \psi^j_0\} = 2\delta^{i,j}$ and therefore the massless states are spinors of $SO(8)$. It is convenient to introduce the worldsheet fermion number operator $(-1)^F$ and decompose $\mathcal{H}_{NS}$ and $\mathcal{H}_R$ into eigenspaces of $(-1)^F$. The resulting structure of the Hilbert space becomes quite clear if we note that a system of $D = 2n$ free fermions, like the one in (2.4), realizes an $\widehat{SO(2n)}_1$ affine algebra. For the moment all we need to know about these algebras are only the few following facts: the Hilbert space decomposes according to the four integrable representations of $\widehat{SO(2n)}_1$ which are respectively a scalar O_{2n}, a vector V_{2n} and two spinor representations S_{2n}, C_{2n}. To each representation we can associate a character, *i.e.* the trace on the sates in the Hilbert space belonging to such representation. In our case the characters are

$$\begin{aligned} O_{2n} &= \frac{\theta_3^n + \theta_4^n}{2\eta^n} , \qquad V_{2n} = \frac{\theta_3^n - \theta_4^n}{2\eta^n} , \\ S_{2n} &= \frac{\theta_2^n + i^{-n}\theta_1^n}{2\eta^n} , \qquad C_{2n} = \frac{\theta_2^n - i^{-n}\theta_1^n}{2\eta^n} . \end{aligned} \tag{2.5}$$

where the η function is defined as an infinite product

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ , \tag{2.6}$$

and under the modular group transforms according to $\eta(\tau+1) = \exp(i\pi/12)\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{-i\tau_2}\eta(\tau)$. For the θ functions we use the following definition

$$\theta\left({a \atop b}\right)(z|\tau) = \sum q^{\frac{1}{2}(n+a)^2} e^{2\pi i (n+a)(z+b)} \ , \tag{2.7}$$

and $\theta_3 = \theta\left({0 \atop 0}\right)$, $\theta_4 = \theta\left({0 \atop 1/2}\right)$, $\theta_2 = \theta\left({1/2 \atop 0}\right)$, $\theta_1 = \theta\left({1/2 \atop 1/2}\right)$. From their modular transformation properties

$$\begin{aligned} \theta\left({a \atop b}\right)(z|\tau+1) &= e^{-i\pi a(a-1)} \theta\left({a \atop b+a-1/2}\right)(z|\tau) \ , \\ \theta\left({a \atop b}\right)(z/\tau| -1/\tau) &= \sqrt{-i\tau} e^{2i\pi ab - i\pi z^2/\tau} \theta\left({b \atop -a}\right)(z|\tau) \ , \end{aligned} \tag{2.8}$$

we can calculate the T matrix

$$T_{2n} = e^{-in\pi/12} \mathrm{diagonal}(1, -1, e^{in\pi/4}, e^{in\pi/4}) \ , \tag{2.9}$$

and the S matrix

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-n} & -i^{-n} \\ 1 & -1 & -i^{-n} & i^{-n} \end{pmatrix} \ , \tag{2.10}$$

for the characters in (2.5). For later use we also report the matrix $P = T^{1/2} S T^2 S T^{1/2}$

$$P = \begin{pmatrix} c & s & 0 & 0 \\ s & -c & 0 & 0 \\ 0 & 0 & \zeta c & i\zeta s \\ 0 & 0 & i\zeta s & \zeta c \end{pmatrix} \ , \tag{2.11}$$

with $\zeta = \exp(-in\pi/4)$, $c = \cos(n\pi/4)$ and $s = \sin(n\pi/4)$. The O_{2n} and V_{2n} characters belong to the NS sector, their highest weight states being the vacuum $|0\rangle$ and the vector $\psi^i_{-1/2}|0\rangle$ respectively. The S_{2n} and C_{2n} are instead characters for representations of the Ramond algebra and their highest weight states are the two spinor representations of $SO(2n)$.

In order to obtain a consistent model of closed superstrings, we have to combine the four $\widehat{SO(8)}_1$ characters in a modular invariant way. Different modular invariants correspond to different prescription for summing over the various spin structures on the torus, *i.e.* to different GSO projections for the closed string spectrum [5]. Let us list the different possibilities, omitting for brevity the contribution of the scalar fields

$$\mathcal{T}_{IIB} = |V_8 - S_8|^2 \ , \tag{2.12}$$
$$\mathcal{T}_{IIA} = (V_8 - S_8)(\bar{V}_8 - \bar{C}_8) \ , \tag{2.13}$$
$$\mathcal{T}_{0B} = |O_8|^2 + |V_8|^2 + |S_8|^2 + |C_8|^2 \ , \tag{2.14}$$
$$\mathcal{T}_{0A} = |O_8|^2 + |V_8|^2 + S_8\bar{C}_8 + C_8\bar{S}_8 \ . \tag{2.15}$$

We note that closed superconformal field theories contain four sectors corresponding to all the possible choices for the periodicity of the supercurrent for the left and the right movers. The NS-NS and R-R sectors contain spacetime bosons while the NS-R and R-NS sectors contain spacetime fermions. The first model in (2.15) is the Type IIB string theory. Its massless spectrum has $N = 2$ supersymmetry and can be easily read from the $SO(8)_1$ characters, that is the main reason why we introduced them. The term $|V_8|^2$, being the product of two vectors, contains the graviton, the dilaton and a NS-NS two-form. The term $|S_8|^2$ is the product of two spinors and can be decomposed as a sum over antisymmetric forms: it gives a scalar, a two form and a self-dual four-form. The two terms $V_8\bar{S}_8$ and $S_8\bar{V}_8$ gives two left-handed gravitini and two right-handed spinors. The spectrum of the Type IIA string can be read similarly from the expressions in (2.15), as well as the spectrum of the Type 0B and Type 0A. We note that these last two models are examples of non-supersymmetric string theory vacua and that their spectrum contains a tachyonic field.

Since string theory is supposed to be a theory of quantum gravity, one would like to study its spectrum and its dynamics in more general backgrounds than flat Minkowski space, for instance on spaces of the form $\mathbb{R}^{(1,d)} \times \mathcal{M}_n$, where $\mathcal{M}_n$ is a compact manifold and $d + 1 + n = 10$. According to the previous discussion, string theory compactifications to $d + 1$ dimensions can be described combining the free scalars and fermions of the $d + 1$ non-compact directions with an arbitrary *internal* superconformal field theory of central charge $c = 3(9-d)/2$. The space of the classical vacua of string theory then corresponds to the space of superconformal field theories with fixed central charge. The internal CFT has a clear geometric interpretation whenever it can be formulated as a superconformal σ-model.

We turn now to the description of two other types of string theory: the heterotic string and the Type I string. The heterotic string [6] is a theory of closed oriented strings and hence the structure of the perturbative series described before is not modified. However the worldsheet theory is based on different chiral algebras for the left and for the right moving modes, more precisely on the superconformal algebra for the left modes and on the conformal algebra for the right modes. The resulting theory has $N = 1$ space-time supersymmetry and contains non-abelian gauge fields.

On the other hand, the Type I string is a theory of closed and open unoriented strings and therefore in the perturbative series one has to consider also the contribution of unoriented surfaces and of surfaces with boundaries. Non-orientable Riemann surfaces can be described introducing a new structure, the *crosscap*, which can be visualized as a disc with diametrically opposite points identified. The Euler characteristic for a surface of genus h with b boundaries and c crosscaps is given by $\chi = 2(1-h) - b - c$ and since three crosscaps are topologically equivalent to one handle and one crosscap, one can consider only surfaces with one or two crosscaps. The study of a surface $\mathcal{C}$ containing boundaries and crosscaps is greatly simplified if one use the so-called Schottky double of $\mathcal{C}$, a closed oriented surface $\hat{\mathcal{C}}$ such that $\mathcal{C} = \hat{\mathcal{C}}/\sigma$, where σ is an anti-conformal involution. For instance both the disc and the crosscap are double-covered by a sphere and the relevant involutions are the reflection with respect to the equator and the antipodal reflection, respectively.

The systematic relation between surfaces with boundaries and crosscaps and their double-covering surfaces suggests that a systematic method should exist for associating open string models to any left-right symmetric closed string model. The basic idea of this method [7], known has the method of open descendants, is to construct a theory of open and closed unoriented string starting from a theory of closed string and gauging the worldsheet parity Ω. As a simple example of this method we illustrate how Type I string theory can be obtained as an open descendant of Type IIB string theory. Other examples of this construction will be discussed in details in Chapter 5. We restrict our attention to the determination of the spectrum of the model, encoded in the four surfaces with $\chi = 0$: the torus, the Klein bottle, the annulus and the Möbius strip.

Let us consider the closed string Hilbert space, which can be decomposed in a sum over left and right moving modes $\mathcal{H} = \bigoplus_{i,j} \mathcal{H}_i \otimes \bar{\mathcal{H}}_j$. The torus partition function for the Type IIB reads

$$\mathcal{T} = \int_{\mathcal{F}} \frac{d\tau_2}{\tau_2^2} \frac{1}{(\sqrt{\tau_2}\eta\bar{\eta})^8} |V_8 - S_8|^2 \ , \tag{2.16}$$

where the first factor gives the contribution of the transverse scalar fields, while the second factor gives the contribution of the transverse fermionic fields. The contribution of the longitudinal oscillators is exactly canceled by the ghosts which also provide the correct integration measure over the moduli space.

The action of Ω on the closed string Hilbert space is to exchange the left and the right sector. In order to construct the open descendant we insert a projector $\frac{1}{2}(1+\Omega)$ in the trace over the closed string states. As a result the torus amplitude (2.16) is divided by two, signaling that the states are combined in left-right invariant pairs. However states already invariant under Ω must by properly symmetrized or antisymmetrized. This is performed by the Klein bottle amplitude, which is precisely the trace of Ω over the closed string Hilbert space

$$\mathcal{K} = \frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{(V_8 - S_8)}{\eta^8}(2it) \ . \tag{2.17}$$

From the combination $\mathcal{T}/2 + \mathcal{K}$ one can read the closed spectrum of Type I string. According to eqs.(2.16) and (2.17) the NS-NS sector is symmetrized and therefore the graviton and the dilaton are preserved while the two-form is projected out. The R-R sector is antisymmetrized and gives a two-form. The NS-R and the R-NS sectors are paired and contribute a left-handed gravitino and a right-handed spinor. The resulting low-energy theory is the Type I supergravity.

For the calculation of the annulus amplitude we have to specify the boundary conditions for the worldsheet fields. The natural choice, which is compatible with superconformal invariance and with ten-dimensional Poincaré invariance, is represented by Neumann boundary conditions, namely $\partial_\sigma X^\mu = 0$ at the boundary. The open string spectrum in the light-cone gauge can then be determined proceeding as for the closed string. The main difference is that now, due to the boundary conditions, only one set of oscillators α_n^i acts on the vacuum. Moreover the mass-shell conditions reads

$$m^2 = \frac{1}{\alpha'}\left(L_0 - \frac{c}{24}\right) \ . \tag{2.18}$$

The physical massless states form a vector field $\alpha_{-1}^i|0\rangle$ and the low-energy dynamics of a system of open strings is then described by the Yang-Mills action. For the moment we have only an abelian gauge theory but non-abelian gauge symmetries can be introduced in a simple way if we associate to each end of an open string a non-dynamical label i which can assume N distinct values. These labels are the so-called Chan-Paton factors and result in an interesting modification of the n-points scattering amplitudes $A(1, ..., n)$ of open strings. Indeed if we associate to any external open string state a matrix Λ, the scattering amplitudes are multiplied by the trace of these matrices $A(1, ..., n)Tr(\Lambda_1...\Lambda_n)$ and at low energy they correctly reproduce the interaction vertices of a non-abelian gauge theory. The requirement of consistent factorization of these amplitudes on intermediate states restricts the possible choices for the matrices Λ to the classical groups $U(N)$, $SO(N)$ and $USp(2N)$.

We now have all the necessary information for writing down the open string partition function

$$\mathcal{A} = \frac{N^2}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{V_8 - S_8}{\eta^8}(it/2) \ , \tag{2.19}$$

where the factor N is the Chan-Paton multiplicity. The Ω projection is implemented on the open string spectrum by the Möbius amplitude

$$\mathcal{M} = \frac{\epsilon N}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^6} \frac{\hat{V}_8 - \hat{S}_8}{\hat{\eta}^8}(it/2 + 1/2) \ . \tag{2.20}$$

Since the modular parameter of the Möbius strip has a non-vanishing real part, we have redefined the characters by a phase in order to work with a real basis. This new basis of real characters is given by

$$\hat{\chi}\left(\frac{it+1}{2}\right) = T^{-1/2}\chi\left(\frac{it+1}{2}\right) = q^{h_i-\frac{c}{24}}\sum_k (-1)^k d_k q^k \ , \tag{2.21}$$

where $q = \exp(-\pi t)$ and $T_{ij}^{1/2} \equiv \exp(-i\pi(h_i - c/24))\delta_{ij}$. The massless spectrum described by the amplitudes (2.19) and (2.20) consists of gauge fields in the adjoint of an orthogonal group for $\epsilon = 1$ and in the adjoint of a symplectic group for $\epsilon = -1$.

All these amplitudes can be given two different interpretations, corresponding to two inequivalent choices for the temporal direction on the worldsheet. With *vertical* time the Klein bottle describes a closed string propagating in a loop and coming back to itself with reversed orientation, while with *horizontal* time it describes a closed string propagating between two crosscaps. In a similar way the annulus describes a one-loop open string diagram or the propagation of a closed string between two boundary and finally the Möbius strip describes an open string in a loop coming back to itself with reversed orientation or a closed string propagating between a boundary and a crosscap. The two descriptions are related by an S transformation $t \mapsto 1/t$ for the Klein bottle and for the annulus, and by a P transformation $it/2+1/2 \mapsto i/2t+1/2$ for the Möbius strip. In the basis of real hatted characters introduced in (2.21), the P modular transformation relating the direct and the transverse channel is implemented by the matrix $P = T^{1/2}ST^2ST^{1/2}$.

In order to complete the construction of the open descendant, let us transform the three amplitudes (2.17), (2.19) and (2.20) to the transverse channel:

$$\tilde{\mathcal{K}} = \frac{2^5}{2}\int_0^\infty dl \frac{(V_8 - S_8)}{\eta^8}(il) \ , \tag{2.22}$$

$$\tilde{\mathcal{A}} = \frac{2^{-5}N^2}{2}\int_0^\infty dl \frac{1}{\eta^8}(V_8 - S_8)(il) \ , \tag{2.23}$$

$$\tilde{\mathcal{M}} = \frac{2\epsilon N}{2}\int_0^\infty dl \frac{1}{\hat{\eta}^8}(\hat{V}_8 - \hat{S}_8)(il+1/2) \ , \tag{2.24}$$

where we have referred all the amplitudes to a common double-covering torus with modular parameter $\tau = il$. In the limit $l \to \infty$ the transverse amplitudes are dominated by the contribution of the massless states and therefore factorize as a product of massless propagators and one-point functions for closed string states in front of boundaries and crosscaps. Since these one-point functions are non-zero, considered individually $\tilde{\mathcal{K}}$, $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{M}}$, diverge due to the divergence of the massless propagator. In the direct amplitudes this infrared divergence manifests itself as an ultraviolet divergence caused by the integration over the modulus t which extends until $t = 0$, since the divergent region is not removed by the requirement of modular invariance as it was the case for the torus. However the divergences can be removed if one combines in a suitable way the three amplitudes. In the factorization limit we can write

$$\tilde{\mathcal{K}} + \tilde{\mathcal{A}} + \tilde{\mathcal{M}} \sim \frac{1}{2}\left(2^5 + 2^{-5}N^2 - 2\epsilon N\right) = \frac{2^{-5}}{2}\left(2^5 - \epsilon N\right)^2 \ , \tag{2.25}$$

and therefore the divergences cancel for $\epsilon = 1$ and $N = 32$. The gauge group is thus completely fixed by the requirement of tadpole cancellation. The resulting model, the open descendant of Type IIB theory, is the Type I string theory, a theory of unoriented open and closed strings whose low energy limit is the $D = 10$, $N = 1$ supergravity coupled to $D = 10$, $N = 1$ super Yang-Mills with gauge group $SO(32)$. The model is chiral but anomaly free [8].

In the previous example we have been able to identify a suitable open string sector, namely open strings with Neumann boundary conditions, for completing the construction of the open descendant. The construction of open descendants for more involved closed string models, would require a general understanding of the conformal boundary conditions allowed by the internal bulk CFT. This is the subject of boundary conformal field theory and will be discussed in some detail in the next Chapter. Here we want to emphasize that the relevance of Boundary CFT for string theory is not related exclusively to the construction of Type I models.

Indeed, as first recognized by Polchinski [10], open string sectors can be used to describe the perturbative dynamics of intrinsic solitonic excitations of string theory, the so-called D-branes. The simplest way for introducing the D-branes is to quantize open strings with Neumann boundary conditions for the first $p+1$ coordinates X^α and with Dirichlet boundary conditions for the remaining $9-p$, X^i. Translation invariance is broken by the Dirichlet boundary conditions, which do not modify the usual spectrum of the open strings but force their momentum to lie along the Neumann directions. Dirichlet boundary conditions then describe open strings whose ends are confined on hyperplanes. In [11] these hyperplanes were interpreted as dynamical objects, and the open strings were considered as their perturbative excitations. Therefore open string sectors with Dirichlet boundary conditions along $9-p$ directions correspond to space-time defects which extends in $p+1$ dimensions, namely to Dp-branes. Similar objects were already familiar from the study of the low-energy supergravity approximation to string theory [9]. The bosonic sector of a supergravity action contains in general the space-time metric coupled with scalars and antisymmetric form potential of various ranks. There is however a simple consistent truncation which includes the metric, the dilaton and a $(p+1)$-form C

$$\mathcal{S}_{sugra} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2(p+1)!}e^{\alpha\phi}F^2\right) \ , \tag{2.26}$$

where $F = dC$, α is a constant and $\kappa_{(10)}$ is the ten-dimensional gravitational coupling. The equations of motion that follow from the action (2.26) allow for a variety of solutions and in particular for p-brane solutions, *i.e.* solutions with $(\text{Poincaré})_{p+1} \times SO(9-p)$ symmetry describing $(p+1)$-dimensional hyperplanes embedded in the ambient ten-dimensional spacetime. Moreover they preserve half of the supersymmetries and are charged with respect to the RR form potential.

It is therefore natural to consider the D-branes as a stringy description of supergravity solutions. This identification was confirmed by Polchinski's calculation of the charge ρ_p and the tension T_p of a Dp-branes [10]. Let us consider a system of two parallel Dp-branes. If we consider the D-branes as solitons of the low-energy supergravity, we can calculate their interaction energy, due to the exchange of dilatons, gravitons and antisymmetric tensors using field theory methods. The result is a good approximation when their separation is large compared to the string scale. From the string theory point of view, their interaction through the exchange of closed string modes is described by a transverse annulus amplitude, similar to that in (2.23). Comparing the field theory calculation with the factorization limit of the annulus amplitude, when the dominant contribution comes form the massless states, one obtains

$$T_p^2 = \rho_p^2 = \frac{\pi}{\kappa_{(10)}^2}(4\pi^2\alpha')^{3-p} \ . \tag{2.27}$$

From (2.27) we can see that the tension and the charge of a D-brane can be calculated unambiguously and expressed in terms of the string length $\sqrt{\alpha'}$ and of the string coupling g_s, confirming that they are intrinsic excitations of the theory. As a consequence of the equality between their charge and their tension, D-branes are BPS solitons, that is they leave half of the space-time supersymmetry unbroken. In particular there is no force between BPS configurations of D-branes and they can be moved freely around the space and also brought one on the top of the other. They also satisfy the generalized Dirac quantization condition for extended objects in higher dimensions

$$2\kappa_{(10)}^2 \rho_p \rho_{(6-p)} = 2\pi \ , \tag{2.28}$$

where we used the fact that in ten dimensions the object dual to a p-brane is a $(6-p)$-brane. As a first important application, D-branes were used to tests the duality conjectures. Actually, due to their status of RR charged BPS states, D-branes provided exactly the necessary degrees of freedom for filling the multiplets required for the validity of the non-perturbative duality relations between different string theories.

Having discussed the D-branes as gravitational solitons, let us now turn to a discussion of the low energy excitations living on their world-volume. The massless modes for a single D-brane are given by the dimensional reduction of a vector multiplet from 10 to $p+1$ dimensions and in particular comprise a massless vector field A^μ with $\mu = 0, \ldots, p$ and $9-p$ scalar fields X^i, $i = p+1, \ldots, 9-p$. The low-energy physics is then given by a $U(1)$ gauge theory. If we consider N D-branes, a new interesting phenomenon takes place [12]. Actually in dealing with a configuration containing more than just one brane, one has to take into account not only the open string states with both the ends on the same D-brane, but also open strings stretched between different branes. The ground state energy for such open strings is proportional to the distance between the branes and vanishes when they coincide, giving rise to new massless degrees of freedom on the brane worldvolume. These new degrees of freedom cause an enhancement of the gauge symmetry and the resulting low-energy field theory is an $SU(N)$ super Yang-Mills theory. The physics of D-branes then involves both gravity and gauge theories, a relation that follows from the two possible interpretations of the annulus amplitude, either as the exchange of closed string modes or as a loop of open strings.

We note however that the effective action for the world-volume theory of a D-brane is given by the Yang-Mills action only in a first approximation. A better approximation which takes into account higher α' corrections and the coupling to arbitrary supergravity backgrounds is given by the sum of the Dirac-Born-Infeld and the Wess-Zumino lagrangians [13]

$$S_p = \int d^{p+1}\xi (L_{DBI} + L_{WZ}) \ , \tag{2.29}$$

with

$$\begin{aligned} L_{DBI} &= T_p e^{-\phi} \sqrt{-\det(\hat{g}_{\alpha\beta} + \hat{B}_{\alpha\beta} + 2\pi\alpha' \hat{F}_{\alpha\beta})} \ , & (2.30) \\ L_{WZ} &= T_p \hat{C} \wedge e^{2\pi\alpha' F} \wedge \sqrt{\frac{\mathcal{A}(T)}{\mathcal{A}(N)}} \ , & (2.31) \end{aligned}$$

where a hat denotes the pull-back on the worldvolume of the corresponding spacetime field and C is the sum over all RR form potentials. Finally $\mathcal{A}$ denotes the *roof genus* familiar from anomaly calculations and T and N are respectively the tangent and the normal bundle to the brane. The Dirac-Born-Infeld action generalizes the Yang-Mills action while the Wess-Zumino term generalizes the Polchinski coupling of a p-brane to a RR $(p+1)$-form.

The interpretation of the open strings as describing fluctuations of space-time defects allows us to give an intuitive picture of the vacuum configuration of the open descendants. Open string descendants are also called *orientifolds* in the literature, since their construction strictly parallels an orbifold construction where the orbifold group also contains the worldsheet parity operation Ω which exchanges the left and the right moving modes. First of all we note that the Ω projection of the closed spectrum induces tadpoles for the closed string states, as one can see from the amplitude in eq. (2.22). We have interpreted the one-point functions in front of a boundary as couplings between the space-time fields and the D-branes. In the same way now we interpret the one-point functions in front of a crosscap as couplings between the space-time fields and a new object, called an *orientifold plane*. From a geometric point of view the orientifold planes are the fixed loci for the combined action of Ω and other spacetime involutions. Orientifold planes, as compared to D-branes, are non-dynamical objects, at least in string perturbation theory. Their net effect is to introduce in the vacuum a cosmological constant and a non-vanishing RR charge density. In a compact manifolds the charge density has to be canceled and therefore one has to introduce in the vacuum a brane configuration with the appropriate charge. This global neutrality condition is precisely the tadpole cancellation condition we met before. Let us stress that charge neutrality only requires the cancellation of the RR tadpoles. Actually, non-vanishing RR tadpoles signal an inconsistency of the model and lead to the appearance of anomalies [14]. On the other hand, NS-NS tadpoles need not to be imposed for the consistency of the model, they simply signal that we are not performing the perturbative expansion around the true vacuum of the theory. In supersymmetric compactifications NS-NS tadpoles cancel as a consequence of RR tadpole cancellations, thus assuring the perturbative stability of the vacuum. However in non-supersymmetric constructions, as those we will consider in Chapter 5, after the RR tadpoles have been canceled, non-vanishing NS-NS tadpoles can be still present and signal that flat space-time is no more a solution to the equations of motion. We can also obtain an intuitive picture for the Chan-Paton factors: the indexes associated to the ends of an open string specify to which D-branes the string is attached. The vacuum configuration of Type I string theory thus contains space-filling $D9$ branes, which account for the open spectrum of the model, together with an $O9$ plane, which accounts for the Ω projection of the closed and open spectrum.

Summarizing, in string theory D-branes can appear in two different contexts. In orientifold constructions they enter as part of a perturbative vacuum configuration of string theory. The vacuum configuration is anomaly-free due to the cancellation of R-R tadpoles and it is stable if also NS-NS tadpoles vanish. In this context, the backreaction of space-time to the presence of the branes is encoded in the orientifold planes, whose tension and charges exactly cancel the tension and the charges of the D-brane configuration. However we can also use D-branes as probes in different string theory vacua and study their dynamics in a fixed background. In both cases, D-branes correspond to open string sectors and therefore to conformally invariant boundary conditions. We then seen that the conformal field theories not only provide us with the classical space of vacua of string theory but also describe the physics of solitonic excitations, if we allow for the presence of boundaries. In the first case CFT techniques are used to describe fluctuations around the background space-time, *i.e.* closed string states, in the second case boundary CFT techniques are used to describe fluctuations around space-time defects, *i.e.* open string states. These space-time defects on their turn can be used as probes for investigating the small scale structure of the space-time and their dynamics is so rich and interesting that they can be used to model such different systems as black-holes [15] and moduli spaces of supersymmetric gauge theories [16]. The most fascinating aspect of D-branes physics is that they live so to say at the intersection between gravity and gauge theories. The holographic correspondence between string/gravity theories and gauge theories uncovered in the last few years [17], as well as the similarity between representation of the conformal algebra and boundary conditions that we will describe in the next chapter, seem both to point at a deeper relation between gauge theories, string theory and gravity.

Chapter 3

Boundary conformal field theory

3.1 Conformal field theory

Conformal invariance is a powerful symmetry in two dimensions [18] because the algebra of infinitesimal conformal transformations is infinite-dimensional. In complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, the conformal transformations are represented by analytic and anti-analytic functions of z. A generic infinitesimal conformal transformation can then be written as $z \mapsto z + \epsilon(z)$. Since correlation functions in a CFT factorize as a sum of products of holomorphic and anti-holomorphic functions

$$\langle \phi_1(z_1, z_1^*) ... \phi_n(z_n, z_n^*) \rangle = \sum_k f_k(\{z_i\}) g_k(\{z_i^*\}) \ , \tag{3.1}$$

it is convenient to consider the variable $\bar{z}$ as an independent variable, eventually identified with z^*, and to think of a CFT as composed by two "halves", a chiral part which depends only on z and an anti-chiral part which depends only on $\bar{z}$. On the other hand the group of global conformal transformations is the finite group $PSL_2(\mathbb{C})$, consisting of the maps

$$z \mapsto \frac{az + b}{cz + d} \ , \qquad a, b, c, d \in \mathbb{C} \ . \tag{3.2}$$

Fields that under general conformal tansformations $z \mapsto f(z)$ transform as

$$\phi(z, \bar{z}) \mapsto (\partial f)^h \left(\bar{\partial} \bar{f}\right)^{\bar{h}} \phi(f, \bar{f}) \ , \tag{3.3}$$

are called primary fields of conformal dimension $(h, \bar{h})$. The combination $\Delta = h + \bar{h}$ and $s = h - \bar{h}$ are the scaling dimension and the spin of the field ϕ since they control its behaviour under rotations and under global scale transformations. For infinitesimal transformation $z \mapsto z + \epsilon(z)$ Eq. (3.3) can be rewritten

$$\delta\phi(z, \bar{z}) = \left(h\epsilon(z) + \epsilon(z)\partial + \bar{h}\epsilon(\bar{z}) + \epsilon(\bar{z})\bar{\partial}\right) \phi(z, \bar{z}) \ . \tag{3.4}$$

Fields that transform as in Eq. (3.4) only for maps in $PSL_2(\mathbb{C})$ are called quasi-primary fields.

Let us recall the derivation of the conformal Ward identity for a CFT defined on the complex plane and later extend it in presence of a boundary. We define the energy-momentum tensor as

$$T_{\mu\nu}(x) = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}(x)} \ , \tag{3.5}$$

where $g_{\mu\nu}$ is a two-dimensional metric. The complex components are

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{xx} - T_{yy}) + \frac{1}{4i}(T_{xy} + T_{yx}) \ , \\ T_{z\bar{z}} &= \frac{1}{4}(T_{xx} + T_{yy}) - \frac{1}{4i}(T_{xy} - T_{yx}) \ . \end{aligned} \tag{3.6}$$

We also define $T = 2\pi T_{zz}$ and $\bar{T} = 2\pi T_{\bar{z}\bar{z}}$. The energy-momentum tensor is symmetric, conserved and traceless. In particular $\bar{\partial} T = \partial \bar{T} = 0$ and therefore T is an analytic function of z and so are the correlators containing $T(z)$ when z does not coincide with the insertion point of another operator. Let us consider the expectation value $\langle X \rangle$ of a collection of primary fields $X = \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)$ inserted at points $z_1, ..., z_n$ in the complex plane. We then make a coordinate transformation $z \mapsto \epsilon(z)$ that vanishes outside a compact domain containing all the fields and that is conformal inside it. From the variation of the path-integral, using the definition of the energy-momentum tensor, the transformation rule (3.4) and the invariance of the action with respect to conformal transformations, one can obtain

$$\langle \frac{1}{2\pi i}\left(\oint_{\mathcal{C}} \epsilon(z) T(z) - \oint_{\mathcal{C}} \epsilon(\bar{z}) \bar{T}(\bar{z})\right) X \rangle = \sum_{i=1}^{n} \langle \left(h_i \partial\epsilon(z_i) + \epsilon(z_i)\partial + \bar{h}_i \bar{\partial}\epsilon(\bar{z}_i) + \epsilon(\bar{z}_i)\right) X \rangle \ , \tag{3.7}$$

where the left hand side of (3.7) comes from the boundary $\mathcal{C}$ of the compact domain. From Cauchy's formula and restricting ourselves to the chiral part of the correlator we obtain

$$\langle T(z)\phi_1(z_1)...\phi_n(z_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_i\right)\langle \phi_1(z_1)...\phi_n(z_n)\rangle \ . \tag{3.8}$$

In particular it follows from Eq. (3.8) that correlators involving the stress-energy tensor can always be related to correlators involving only primary fields. T itself is not a primary field but only a quasi-primary field. Indeed the general form of the OPE of two energy-momentum tensors is

$$T(z)T(w) \sim \frac{c}{2(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T}{z-w} \ , \tag{3.9}$$

and the corresponding transformation rule under $z \mapsto f(z)$ reads

$$T(z) \mapsto (\partial f)^2\, T(z) + \frac{c}{12}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right) , \tag{3.10}$$

where the second term is the Schwarzian derivative. The constant c appearing in (3.10) is the central charge of the CFT.

When the theory is formulated on the upper-half plane [19] we can proceed in a similar way, except that we have to impose that the transformation $z \mapsto \epsilon(z)$ leaves the boundary invariant. Transformations that do not fix the boundary will be used in the sequel to relate correlators evaluated in different geometries. As a consequence, $\epsilon(z)$ has to satisfy $\epsilon(z) = \epsilon^*(z)$ when $z = \bar{z}$ or, in real coordinates, $\epsilon_y(x, 0) = 0$. As before $\epsilon(z)$ is conformal in a semi-disc containing all the operator insertions and vanishes outside. We require that the action is invariant not only in the semi-disc but also on a segment $[-a, a]$ of the real axis.

In the contour integral then appear an additional contribution $\int_{\mathbb{R}-[-a,a]} \epsilon_x(x, 0) T_{xy}$ that must be set to zero. This implies the non-trivial restriction $T_{xy}(x, 0) = 0$ or, in complex coordinates, $T(z) = \bar{T}(\bar{z})$ for $z = \bar{z}$. This condition means that there is no energy flow across the boundary. The energy-momentum tensor T is now integrated only along a semicircle around the insertions. However, since $T(z)$ and $\bar{T}(\bar{z})$ coincide on the real axis due to the boundary condition, we can use the Schwarz reflection principle to analytically continue $T(z)$ in the lower half plane and therefore we can write

$$\begin{aligned}
&\langle T(z)\phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle = &(3.11)\\
&\left(\sum_{j=1}^{n} \frac{h_j}{(z - z_j)^2} + \frac{1}{z - z_j}\partial_j + \sum_{j=1}^{n} \frac{\bar{h}_j}{(z - \bar{z}_j)^2} + \frac{1}{z - \bar{z}_j}\bar{\partial}_j\right) \langle \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle \ .
\end{aligned}$$

A first important consequence of (3.12) is that a $2n$-point correlator of chiral fields on the full plane

$$\langle T(z)\phi_1(z_1)...\phi_n(z_n)\bar{\phi}_1(\bar{z}_1)...\bar{\phi}_n(\bar{z}_n)\rangle \ , \tag{3.12}$$

where ϕ_i has weight $(h_i, \bar{h}_i)$ and $\bar{\phi}_i$ $(\bar{h}_i, h_i)$ satisfies the same Ward identity than an n-point correlator on the upper-half plane

$$\langle T(z)\phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n) \ , \tag{3.13}$$

From this observation and inspired by the method of images in classical electrodynamics, Cardy introduced a doubling trick for the calculation of boundary correlators: instead of calculating an n-point correlator on the upper-half plane, one can calculate a chiral correlator on the full plane containing for any field insertion $\phi_i(z_i)$ with $Im(z_i) > 0$ an image $\bar{\phi}_i(\bar{z}_i)$ with $Im(\bar{z}_i) < 0$.

Let us now discuss the operator formalism of CFT's. We use the radial quantization scheme, that is we take as evolution operator the generator of scale transformations and time-ordering take place on concentric circles around the point chosen as origin. If we expand the fields in the theory in a Laurent series around the origin, the previous OPE's can be reformulated in terms of commutation relations. For a primary field ϕ of weight h we write

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} , \qquad \phi_n = \frac{1}{2\pi i} \oint \phi(z) z^{n+h-1} . \tag{3.14}$$

and in particular for the energy-momentum tensor $T(z) = \sum_n L_n z^{-n-2}$. The commutation relations of the modes L_n define the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{c}{12}m(m^2-1) . \tag{3.15}$$

The OPE of T with a primary field ϕ becomes

$$[L_n, \phi(z)] = \big(h(n-1)z^n + z^{n+1}\partial\big)\,\phi(z) , \tag{3.16}$$

or equivalently

$$[L_n, \phi_m] = (n(h-1)-m)\phi_{n+m} . \tag{3.17}$$

Representations of the Virasoro algebra are generated acting with the modes L_{-n}, $n > 0$ on a highest weight state $|h, \bar{h}\rangle$ which is an eigenstate of L_0 and $\bar{L}_0$

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle , \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle , \tag{3.18}$$

and which is annihilated by all the modes L_n, $\bar{L}_n$ with $n > 0$. States of the form $L_{n_1}...\bar{L}_{m_1}...|h,\bar{h}\rangle$ are called descendants of level N where $N = \sum_i n_i + \sum_j m_j$. An highest weight state together with all its descendants form a Verma module. However not all the states so obtained are linearly independent, a linear relation between them at some level implying the reducibility of the representations, due to the presence of a null vector. Null vectors are a particularly useful feature of CFT's since they lead to partial differential equations for the correlation functions which allow their explicit computation. The $SL_2(\mathbb{Z})$-invariant vacuum $|0\rangle$ is a primary state annihilated also by L_0, $\bar{L}_0$ and we assume that it is unique. At any state in the Hilbert space it is possible to associate a field, and vice versa, according to

$$|\phi\rangle = \lim_{z\to 0} \phi(z,\bar{z})|0\rangle . \tag{3.19}$$

In particular to any highest weight state it corresponds a primary field. The descendant states can be obtained starting from the primary field by applying the differential operators $L_n(z)$. A primary field ϕ_i together with all its independent descendants form a conformal family $[\phi_i]$ which is a representation of the Virasoro algebra.

In the case of a boundary CFT, we can chose as origin for radial quantization either a point in the bulk or a point on the boundary. The two formulations are equivalent but the interpretation of the states in the resulting Hilbert spaces are different. In the first case, the Hilbert space structure is the same as for the theory formulated on the full complex plane, since the relation between T and $\bar{T}$ is non-local and hence two copies of the Virasoro algebra still act on $\mathcal{H}$. The presence of the boundary is accounted by the introduction of an out-state $|B\rangle$ called boundary state. The boundary state must solve a linear equation that originates form the constraint $T(z) = \bar{T}(\bar{z})$ for $z = \bar{z}$. If we map the upper-half plane with coordinate z to the unit disc with coordinate u using the function $z = -i(u-1)/(u+1)$, the constraint becomes $u^2 T = \bar{u}^2 \bar{T}$ for $u = e^{i\theta}$ and in terms of the modes we obtain

$$(L_n - \bar{L}_{-n})|B\rangle = 0 \ . \tag{3.20}$$

A basis of solutions for this equation is given by the Ishibashi states $|B_{ij}\rangle\rangle$ [20]. Let us decompose the Hilbert space in Virasoro irreducible representations: $\mathcal{H} = \bigoplus_{h_i,\bar{h}_j} \mathcal{H}_{h_i} \otimes \bar{\mathcal{H}}_{\bar{h}_j}$. We can solve Eq.(3.20) separately in each component of $\mathcal{H}$ as

$$|B_{ij}\rangle\rangle = \sum_{n,\bar{m}} a_{n,\bar{m}} |h_i, n\rangle \otimes |\bar{h}_j, \bar{m}\rangle \ , \tag{3.21}$$

since the $\mathcal{H}_{h_i}$ are representations of the Virasoro algebra. The sum over n and $\bar{m}$ represents all possible descendant states. Using the inner product in $\mathcal{H}_{h_i}$, we can equivalently consider the boundary state $|B_{ij}\rangle\rangle$ as a linear operator $B_{ij} : \mathcal{H}_{h_j} \to \mathcal{H}_{h_i}$ and from (3.20) it follows that B_{ij} is an intertwiner between two Virasoro irreducible representations. Therefore Schur's lemma implies that $B_{ij} = 0$ for $j \neq i^*$ while it is proportional to the identity for $j = i^*$, where i^* is the representations conjugate to i. Choosing the constant of proportionality equal to one, each of the B_{ii^*} becomes a projection operator on the component $\mathcal{H}_{h_i} \otimes \bar{\mathcal{H}}_{\bar{h}_{i^*}}$ of the Hilbert space and as a consequence there are as many Ishibashi states as self-conjugate pairs. Given a primary field $\phi_i(z)\bar{\phi}_{i^*}(\bar{z})$ the corresponding Ishibashi state reads

$$|\phi_i\rangle\rangle = \sum_n |h, n\rangle \otimes |\bar{h}, \bar{n}\rangle \ . \tag{3.22}$$

The Ishibashi states do not correspond to consistent boundary conditions since they do not give rise to consistent spectra for the boundary operators. Any consistent boundary state can however be expanded in this distinguished basis

$$|B\rangle = \sum_i B_{a,i} |\phi_i\rangle\rangle \ , \tag{3.23}$$

where the components $B_{a,i}$, called boundary coefficients, have to satisfy further constraints, as we will explain in the next Section. The previous argument can be directly extended to a general chiral algebra $\mathcal{A}$.

Let us now consider the other possibility and take as origin for radial quantization a point on the boundary. In this case the boundary conditions specified on the real axis influence the definition of the Hilbert space. Let a be the boundary condition on the positive real line and b the boundary condition on the negative real line. Only one Virasoro algebra acts on $\mathcal{H}_{ab}$ since on the real axis $T = \bar{T}$ and therefore $L_n = \bar{L}_n$. The field-state correspondence relates states in $\mathcal{H}_{ab}$ which are defined on semidisc around the origin to the so called boundary changing fields $\psi^{ab}(x)$, that is, we interpret the transition from the boundary condition b on the negative axis to the boundary condition a on the positive axis as due to the insertion of a boundary operator. From the point of view of string theory, boundary operators describe open string excitations.

The constraints imposed by conformal invariance are very powerful and a natural question is to what extent they completely determine a conformal field theory, or in other words how many quantities we need to specify in order to completely define the theory. Let us start considering theories defined on surfaces without boundaries. In this case we have certainly to specify the central charge and the operator content. A first consistency condition comes form the closure of the OPE, whose general form is

$$\phi_i(z)\phi_j(w) = \sum_k C_{ij}^k (z-w)^{h_k - h_i - h_j} \left[\phi_k(w) + \text{descendants}\right] \ , \qquad (3.24)$$

whit a similar expression for the anti-holomorphic part. The coefficients in front of the descendant fields are completely fixed by the Virasoro algebra and therefore the only free parameters are the OPE coefficients C_{ij}^k. In particular they give a set of selection rules: if C_{ij}^k vanishes for some k, no field in the conformal family $[\phi_k]$ can appear in the OPE of any field in $[\phi_i]$ and $[\phi_j]$. Furthermore, conformal invariance completely fixes the correlation functions with less than four external legs. They can be parameterized as follows

$$\begin{aligned} v &= \langle 1 \rangle = \langle 0|0 \rangle \ , \\ \langle \phi_i(z_i)\phi_j(z_j) \rangle &= \delta_{ij} \frac{\kappa_i}{(z_i - z_j)^{2h_i}} \ , \qquad (3.25) \\ \langle \phi_i(z_i)\phi_j(z_j)\phi_k(z_k) \rangle &= C_{ijk} z_{ij}^{h_k - h_i - h_j} z_{ik}^{h_j - h_i - h_k} z_{jk}^{h_i - h_j - h_k} \ , \end{aligned}$$

$$(3.26)$$

where $z_{ij} \equiv z_i - z_j$, $\kappa_i = v C_{ii}^0$, $C_{ijk} = v C_{jk}^i C_{ii}^0$ and the index 0 labels the identity field. For the sequel we set $v = \kappa_i = 1$ and therefore also $C_{ii}^0 = 1$.

We then see that a CFT on the complex plane can be completely specified by its central charge c, by the conformal dimensions of its operators $(h_i, \bar{h}_i)$ and by the OPE coefficients C_{ij}^k. Further constraints on these data are imposed by the requirement that they allow the definition of consistent correlation functions on Riemann surfaces of arbitrary genus. Higher genus surfaces can always be reduced by suitable sequences of cuts to a set of elementary surfaces. Different sequences of cuts correspond to different factorization limits of the same correlators and therefore we require that all of them lead to the same expression. In this way it is possible to derive a set of powerful constraints, called *sewing constraints* [21]. In particular, the independent constraints for theories involving only closed oriented surfaces are the duality of the four-point functions on the sphere, equivalent to the associativity of the OPE, and the modular invariance of the one-point functions on the torus. We illustrate in some detail only the constraint coming from the vacuum diagram, *i.e.* the torus without any operator insertion, since this constraint amounts to a completeness condition for the spectrum of bulk operators.

It is instructive to derive the torus amplitude starting from the theory defined on the plane. We can map the complex plane to a cylinder of circumference R through the map $z = \exp(2\pi i w/R)$. In the new geometry the energy-momentum tensor is

$$T_{cyl}(w) = -\frac{4\pi^2}{R^2}\left(T_{pl}(z)z^2 - \frac{c}{24}\right) , \tag{3.27}$$

With $w = x + iy$, evolution along the cylinder is generated by $H_{cyl} = \int_0^R dx T_{yy}$ and by (3.10) $H_{cyl} = \frac{2\pi}{R}\left[L_0 + \bar{L}_0 - \frac{c}{12}\right]$. If we identify periodically the y direction $y \sim y + iL$, the cylinder becomes a torus and the amplitude describing the propagation along the cylinder turns into the partition function

$$Z(R, L) = Tr q^{L_0 + \bar{L}_0 - \frac{c}{12}} , \tag{3.28}$$

where $q = \exp(-2\pi L/R)$. We can obtain the partition function also reversing the previous steps: we first make the transformation $z = \exp(-2\pi w/L)$ where $w \sim w + iL$, then we propagate the states along the x direction using the Hamiltonian $H_{cyl} = \int_0^L T_{xx}$ and finally we identify $x \sim x + R$. The resulting partition function reads

$$Z(R, L) = Tr \tilde{q}^{L_0 + \bar{L}_0 - \frac{c}{12}} , \tag{3.29}$$

where $\tilde{q} = \exp(-2\pi R/L)$. The theory is modular invariant if the two expressions coincide. More precisely, as we have explained in Chapter 2, a torus is completely specified by a modular parameter τ taking values in a fundamental region for the action of the group $SL_2(\mathbb{Z})$ on the upper-half plane, generated by the two transformations $S : \tau \mapsto -1/\tau$ and $T : \tau \mapsto \tau + 1$. The requirement of modular invariance then amounts to the conditions $Z(\tau) = Z(-1/\tau) = Z(\tau + 1)$. Any set of fields whose torus amplitude is not modular invariant, even if closed under the operator product, should be consistently enlarged by the inclusion of other fields. The importance of this completeness condition will become apparent when we will consider the construction of new conformal field theories through the orbifold method.

A particularly important class of conformal field theories is represented by the so-called rational conformal field theories. They are characterized by an infinite dimensional symmetry algebra $\mathcal{A}$, containing the Virasoro algebra and such that the Hilbert space decomposes in a finite number of representation of $\mathcal{A}$, $\mathcal{H} = \bigoplus_{i=1}^{N} \mathcal{H}_i$. The content of each representation is neatly summarized by its character

$$\chi_i(\tau) = Tr_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}} \ , \qquad q = e^{2\pi i \tau} \ . \tag{3.30}$$

Similarly the partition function decomposes in a finite sesquilinear sum of characters $Z = \sum_{ij} Z_{ij} \chi_i \bar{\chi}_j$ that is modular invariant. The action of the modular group on the characters is represented by two unitary matrices, S and T, as

$$\chi_i(\tau+1) = T_{ij}\chi_j(\tau) \ , \qquad \chi_i(-1/\tau) = S_{ij}\chi_j(\tau) \ . \tag{3.31}$$

Moreover correlation functions on an arbitrary surface with n operator insertion can be expressed as finite sums of holomorphic times anti-holomorphic functions of the moduli of surfaces with n punctures. These holomorphic building blocks for the correlators are the so called conformal blocks. Transformations between different basis of conformal blocks are performed with the aid of two matrices, the braid and the fusion matrices B and F. Since changes of basis correspond to different factorization limits for the correlators, the matrices B and F are necessary for the formulation of the sewing constraints.

We can now introduce the fusion ring. We define $\mathcal{N}_{ijk} = \dim(V_{ijk})$ where V_{ijk} is the space of the three-point conformal blocks on the sphere. The $\mathcal{N}_{ijk}$ count the number of distinct couplings between the representations $[\phi_i]$, $[\phi_j]$ and $[\phi_k]$, conventionally written as

$$\phi_i \phi_j = \mathcal{N}_{ij}{}^{k} \phi_k \tag{3.32}$$

The product in (3.32) is called the fusion product and if we consider the ϕ_i as abstract symbols, then the $\mathcal{N}_{ijk}$ represent the structure constants of an associative, commutative, semi-simple algebra with unity called the *fusion algebra*. For $k = 0$, the matrix $\mathcal{N}_{ij0} = C_{ij}$ define an order-two permutation and is called the charge-conjugation matrix. By definition, given a primary field ϕ_i, its conjugate ϕ_{i^*} is the only field whose fusion product with ϕ_i contains the identity. Indices labelling bulk representations are raised and lowered using the matrix C. An interesting consequence of the sewing constraints is the Verlinde formula [22] that relates the fusion algebra with the modular S matrix. It shows that

$$\mathcal{N}_{ij}^{k} = \sum_l S_{il} \frac{S_{jl}}{S_{1l}} \bar{S}_{lk} \ . \tag{3.33}$$

If we consider the $\mathcal{N}_{ijk}$ as a set of matrices $(\mathcal{N}_i)_j^k$, then eq. (3.33) says that these matrices are diagonalized by the S matrix.

In presence of a boundary, we need further data in order to completely specify the theory [23]. First of all we must consider the behaviour of a bulk field as it approaches the boundary

$$\phi(x+iy)|_a = C^k_{a\phi}(2y)^{h_k-2h_\phi}\left(\psi^{aa}_k(x) + a_1 y\left(L_{-1}\psi^{aa}_k\right)(x) + ...\right) . \tag{3.34}$$

where a is a given boundary condition. Furthermore, we have an OPE also for two boundary fields

$$\psi^{ab}_i(x)\psi^{bc}_j(y) = \sum_k C^{(abc)k}_{ij}(x-y)^{h_k-h_i-h_j}\left(\psi^{ac}_k(y) + a_1(x-y)(L_{-1}\psi^{ac}_k)(y) + ...\right) , \tag{3.35}$$

where $x > y$. Invariance under global conformal transformations fixes again the correlators with less than four operator insertions on the boundary

$$\begin{aligned}
\langle 1\rangle_a &= g_a , &(3.36)\\
\langle\psi^{ab}_i(x_i)\psi^{ba}_j(x_j)\rangle &= \delta_{ij^c}C^{aba0}_{ii}g_a(x_i-x_j)^{-2h_i} , &(3.37)\\
\langle\phi_i(x+iy)\rangle_a &= \delta_{h_i,\bar{h}_i}C^{a0}_k g_a(2y)^{-2h_i} , &(3.38)\\
\langle\psi^{ab}_i(x_i)\psi^{bc}_j(x_j)\psi^{ca}_k(x_k)\rangle &= C^{(bca)i}_{jk}C^{(aba)0}_{ii}g_a x^{h_k-h_i-h_j}_{ij}x^{h_j-h_i-h_k}_{ik}x^{h_i-h_j-h_k}_{jk} , &(3.39)\\
\langle\psi^{aa}_i(x_i)\phi_j(x_j+iy)\rangle_a &= C^{ai}_j C^{(aaa)0}_{ii}g_a(2y)^{-2h_j}\left(x^2_{ij}+y^2\right)^{-h_i} , &(3.40)
\end{aligned}$$

where $x_{ij} \equiv x_i - x_j$. We have assumed that there is only one boundary field with conformal dimension $h = 0$, namely the identity. Boundary conditions which fulfill this requirement are called pure boundary conditions.

The definition of a boundary CFT then requires, in addition to the central charge, the bulk operator content and the bulk OPE coefficients, also the boundary operator content and the OPE coefficient for two boundary operators and for a bulk operator approaching a boundary. As in the previous case, these data are subject to sewing constraint coming from factorization of the correlators on more complicated surfaces. The new independent constraints involve disc amplitudes with four boundary operators, with one bulk and two boundary operators or with one boundary and two bulk operators and annulus amplitudes with one boundary operator. As in the closed case, we restrict our consideration to the annulus without any operator insertion, since this amplitude plays a crucial role in the determination of the spectrum of boundary operators. We can construct the annulus following a procedure similar to the one we used for the torus. First of all we can map the upper-half plane to an infinite strip of width R using $z = \exp(i\pi w/R)$; the positive axis is mapped to the boundary of the strip in $x = 0$ while the negative axis is mapped to the boundary in $x = R$. The Hamiltonian reads

$$H_{strip} = \int_0^R T_{yy}dx = \frac{\pi}{R}\left(L_0 - \frac{c}{24}\right) . \tag{3.41}$$

The annulus is then constructed by identifying periodically the coordinate measuring the length of the strip. We have again two possible choices for the evolution operator: we can propagate states along the periodic direction (vertical time) or between the two boundaries (horizontal time). We notice that for the torus amplitude different choices of time were always interpreted as giving the partition functions for the bulk states. In contrast the interpretation of the annulus amplitude corresponding to the two choices of time are quite different. With vertical time, we are describing boundary excitations propagating in a loop and therefore the annulus amplitude is the partition function for the states living on the boundary and plays a similar role to the torus partition function. With horizontal time, we are describing the propagation of bulk states between two boundary states $|a\rangle$ and $|b\rangle$. Indeed the two choice of time relate the two Hilbert space descriptions we illustrated before

$$\langle a|e^{-\frac{2\pi R}{L}\left(L_0+\bar{L}_0-\frac{c}{12}\right)}|b\rangle = Tr_{\mathcal{H}_{ab}} e^{-\frac{\pi L}{R}\left(L_0-\frac{c}{24}\right)} \ . \tag{3.42}$$

The general form of the annulus partition function is

$$\mathcal{A}_{ab} = \sum_i A^i_{ab}\chi_i \ , \tag{3.43}$$

where the annulus coefficients A^i_{ab} are non-negative integers. If we expand the boundary states in the basis of the Ishibashi states as in Eq.(3.23) then Eq.(3.42) relates the boundary coefficients B_{ai} and the annulus coefficients A^i_{ab}

$$\sum_i B_{ai}B_{bi}S_{ij} = A^i_{ab} \ , \tag{3.44}$$

This constraint is known as Cardy's constraint [24].

3.2 Open descendants and completeness conditions for boundary states

In the previous Section we introduced some general features of conformal field theories defined on manifolds with boundaries. We now want to address the problem of classifying the conformal boundary conditions for a given CFT. The study of the space of conformally invariant boundary conditions for a given two-dimensional CFT coincides, from the point of view of string theory, with the problem of finding all the possible brane configurations in a given background. As stated the problem is too general to be solved. We start restricting ourselves to the class of rational conformal field theory and consider a RCFT characterized by a symmetry algebra $\mathcal{A}\otimes\bar{\mathcal{A}}$ and by a finite set of primary fields $\{\phi_{i,\bar{i}}\}$ with characters $\chi_i(\tau)$. We assume that $\mathcal{A}\sim\bar{\mathcal{A}}$

and that $\mathcal{A}$ is maximally extended (*i.e.* $Z_{ij} = \delta_{i,\sigma(j)}$ where σ is a permutation of the chiral labels) so that a field can be uniquely specified by its chiral index i. The bulk spectrum is encoded in the torus partition function

$$\mathcal{T} = \sum_{i,j} Z_{ij} \chi_i \bar{\chi}_j \ . \tag{3.45}$$

We have further to specify which bulk symmetries must be respected by the boundary conditions. For string theory applications, for instance, one could require that only the superconformal algebra is preserved. Again we restrict to boundary conditions preserving the full chiral algebra of the bulk theory. We will consider symmetry breaking boundaries only after having explained the orbifold construction and introduced the simple currents. When the torus partition function is the charge-conjugation modular invariant, a general solutions, giving all the boundary conditions preserving the bulk chiral algebra, was found by Cardy [24]. Before illustrating this solution, let us explain which constraints the boundary coefficients (3.23) have in general to satisfy and in which sense the Cardy set of boundary states can be considered a "complete" set.

In principle, in order to obtain the set of all possible boundaries together with a detailed knowledge of the OPE coefficients, we could solve the sewing constraints. However the explicit form of these constraints is known only in a few cases. We thus consider simpler constraints, which amount to the requirement of positivity and integrality of the partition functions, that should be satisfied by every boundary CFT. Therefore, in the sequel we will concentrate on the operator content of conformal field theories defined on Riemann surfaces with boundaries, which is encoded in the torus and in the annulus amplitudes. It is useful however to include from the start in the discussion also non-orientable surfaces, and to project both the bulk and the boundary operator spectrum using the worldsheet parity Ω, which exchanges the chiral and the anti-chiral part of the model. Therefore we have to include in our discussion the Klein bottle and the Möbius strip. The consideration of non-orientable surfaces is interesting not only from the string theory point of view, due to the fact that it gives the spectrum of Type I models, but also from the pure CFT point of view. Actually when the torus, the Klein bottle, the annulus and the Möbius strip are considered simultaneously, a quite elegant structure emerges and one can derive sum rules that relate the coefficients characteristic of each amplitude [25]. In order to obtain the spectrum of the theory we have to specify the set $\{m\}$ of Ishibashi labels, the set $\{a\}$ of boundary labels, a matrix B_{am} of boundary coefficients and a vector Γ_m of crosscap coefficients. We recall that there is an Ishibashi state for any primary field coupled to his charge conjugate in the torus partition function.

Let us now consider the Klein bottle [26]. As already explained, a non-orientable surface can be obtained from an orientable one by cutting a disc and identifying diametrically opposite points, that is by introducing a crosscap on the surface. The crosscap interchanges the chiral and the antichiral part of a field $\phi_{i,\bar{i}}$ and therefore fields whose chiral and antichiral parts are different combine in invariant pairs, while fields with $i = \bar{i}$ are left fixed and we have an involution $\Omega : \phi_{i,\bar{i}} \to \epsilon_i \phi_{i,\bar{i}}$ where $\epsilon_i = \pm 1$. Consistency with fusion requires that $\mathcal{N}_{ij}^{k} \epsilon_i \epsilon_j \epsilon_k \geq 0$. The Klein bottle amplitude can be written as follows

$$\mathcal{K} = \sum_i K^i \chi_i \ , \tag{3.46}$$

where the K^i are signed integers satisfying

$$|K^i| = Z_{ii} \bmod 2 \ , \tag{3.47}$$

in order to guarantee a consistent projection of the closed spectrum. From Eq.(3.47) we see that only characters paired diagonally in the torus amplitude can enter the Klein bottle in the direct channel. The bulk spectrum is then given by

$$\frac{1}{2} \left(\mathcal{T} + \mathcal{K} \right), \tag{3.48}$$

The transverse channel amplitude, obtained via an S modular transformation,

$$\tilde{\mathcal{K}} = \sum_i \Gamma_i^2 \chi_i \ , \tag{3.49}$$

describes the propagation of the closed spectrum between two crosscaps. Only characters coupled with their conjugate in the torus partition function can now have non vanishing crosscap coefficients Γ_i, proportional to the one-point functions of bulk fields in front of the crosscap. Comparing (3.46) and (3.49) we obtain

$$K^i = \sum_m S_{im} \Gamma_m \Gamma_m \ . \tag{3.50}$$

The boundary operator content is described by the annulus amplitude. Since a general boundary condition is given by a superposition of pure boundary conditions, we consider the following generating function for pure annulus amplitudes

$$\mathcal{A} = \sum_{a,b} n^a n^b A_{ab}^i \chi_i \ , \tag{3.51}$$

where the indices a, b run over pure boundary conditions and n^a, n^b are non-negative integers. In string theory these boundary multiplicities correspond to non-trivial Chan-Paton factors and are at the origin of non-abelian gauge symmetry groups in open string models. We would like to find for the possible boundary states a constraint analogous to modular invariance for the bulk states. This constraint is provided by the *completeness conditions* [27]

$$\begin{aligned}\sum_b A_{ia}{}^b A^c_{jb} &= \sum_k \mathcal{N}^k_{ij} A^c_{ka} \ , \\ \sum_i A_{iab} A^i{}_{cd} &= \sum_i A_{iac} A^i{}_{bd} \end{aligned} \tag{3.52}$$

From the first equation, it follows that A_{0ab} is an order-two permutation. It plays the role of a boundary conjugation matrix and it is used for raising and lowering the boundary indices. Upper and lower boundary indices have to be distinguished whenever complex charges n_a are present, as we will see in some example later. The condition $A_{0a}{}^b = \delta_{a,b}$ is consistent with the interpretation of a change in the boundary condition as due to the insertion of a non-trivial boundary operator. The first equation in (3.52) also means that the matrices A_i give a non-negative integer representation of the fusion algebra, with $A_0 = 1$ and $A_i^T = A_i$. An heuristic derivation of the completeness conditions can be given considering two different way of calculating the two-point function of bulk operators, either using the bulk OPE or inserting a complete set of boundary states. From the transverse channel annulus amplitude

$$\tilde{\mathcal{A}}_{ab} = \sum_i n^a n^b B_{ai} B_{bi} \chi_i \ , \tag{3.53}$$

we can read

$$A^i_{ab} = B_{aj} B_{bj} S_{ji} \ . \tag{3.54}$$

If we introduce the reflection coefficients $R_{ai} = \sqrt{S_{i0}} B_{ai}$, where as usual the label a is reserved for the boundary conditions and the label i for the representations of the chiral algebra, a sufficient condition for the validity of (3.52) is

$$\begin{aligned}\sum_i R_{ai} \bar{R}_{bi} &= \delta_{a,b} \ , \\ \sum_a R_{ai} \bar{R}_{aj} &= \delta_{i,j} \ . \end{aligned} \tag{3.55}$$

From this point of view, the completeness conditions require that the number of Ishibashi states equals the number of boundary states.

The Ω projection is implemented on the open spectrum by the Möbius strip. Since in the transverse channel this amplitude describes the propagation of the closed spectrum between a boundary state $|a\rangle$ and the crosscap, its form can be obtained up to a sign as the geometric mean of Eq.(3.49) and Eq.(3.53),

$$\tilde{\mathcal{M}}_a = \pm \sum_i \Gamma_i B_{ai} \hat{\chi}_i \tag{3.56}$$

where the basis of hatted characters has been introduced in eq.(2.21). In the direct channel, the general form of the Möbius amplitude is

$$\mathcal{M} = \pm \sum_{a,i} n^a M_a^i \; \hat{\chi}_i \; . \tag{3.57}$$

This amplitude must define a consistent projection of the open spectrum and therefore one should require that the M_a^i are integers satisfying the two conditions

$$|M_a^i| \leq A_{aa}^i \; , \qquad M_a^i = A_{aa}^i \bmod 2 \; . \tag{3.58}$$

A P modular transformation, where $P = T^{1/2} S T^2 S T^{/2}$, relates the two channels and allows to express the Möbius coefficients in terms of the boundary and crosscap coefficients

$$M_a^i = \sum_m B_{am} \Gamma_m P_{im} \; , \tag{3.59}$$

When the torus partition function is the charge-conjugation modular invariant,

$$\mathcal{T} = \sum_{i=0}^{N} \chi_i \bar{\chi}_{i^*} \; , \tag{3.60}$$

a general ansatz giving a complete set of boundary states has been presented by Cardy [24] and then extended to include the crosscap coefficients in [28]. According to the previous discussion, all the bulk fields can reflect on the boundaries and there is one Ishibashi state $|i\rangle\rangle$ for every representation of the chiral algebra. According to (3.52) we should be able to find a set of N consistent boundary states. The Cardy ansatz is

$$B_{ai} = \frac{S_{ai}}{\sqrt{S_{0i}}} \; . \tag{3.61}$$

Indeed the conditions in (3.55) are obviously satisfied since S is unitary. Using these boundary coefficients, eq.(3.52) coincide with the Verlinde's formula and the resulting annulus coefficients are given by the fusion coefficients

$$A_{ab}^i = \mathcal{N}_{ab}^i \; , \tag{3.62}$$

and are manifestly integers. In order to give compact expressions for the K^i and the M_a^i we introduce the tensor [28]

$$Y_{ij}^k = \sum_l \frac{S_{il} P_{jl} \bar{P}_{kl}}{S_{0l}} \; . \tag{3.63}$$

It has been shown that this tensor has integer components [29] and that Y_{00}^i is the Frobenius-Schur indicator ν_i for the representation χ_i of the chiral algebra [30], *i.e.*

$\nu_i = 0$ if the representation is complex and $\nu_i = \pm 1$ if the representation is real or pseudo-real. Using this tensor the ansatz reads

$$
\begin{aligned}
K^i &= Y^i_{00} \ , && (3.64)\\
M^i_a &= Y^i_{a0} \ , && (3.65)\\
\Gamma^i &= \pm \frac{P_{0i}}{\sqrt{S_{0i}}} \ . && (3.66)
\end{aligned}
$$

The crosscap coefficients in (3.66) can be simply derived [31] by noticing that the boundary coefficients for the Cardy state associated to the identity of the chiral algebra are all different from zero, since $S_{0i} > 0$ for all i. From the corresponding annulus coefficients $A^i_{00} = \delta_{i,0}$ we see that $M^i_0 = \pm\delta_{i,0}$ and therefore

$$
\sum_m \sqrt{S_{0m}} \Gamma_m P_{im} = \pm \delta_{i,0} \ , \qquad (3.67)
$$

from which we obtain the expression in (3.66). We will see in the sequel that the Cardy ansatz can be simply modified if one permutes the boundaries according to the action of some simple current [31].

3.3 Orbifolds

In this Section we introduce the orbifold method for the construction of new conformal field theories exploiting the discrete symmetries of a given one. As we will explain in more detail in the next Section, in this way it is also possible to obtain new modular invariants for the chiral algebra of the original model. The essential point is that the orbifold construction can be extended to the boundary states in a quite efficient way and that by orbifold techniques it is possible to obtain boundary states for modular invariants different from charge-conjugation as well as boundary states preserving only a part of the original symmetry of the model. In this Chapter we will illustrate these techniques presenting as examples the free bosons and the $SU(2)$ WZW models. In the next Chapter we will use them for our study of the boundary states for RCFT with $c = 1$ and $c = 3/2$.

Let us start recalling some general aspects of the orbifold construction [32]. Orbifolds where first introduced in string theory as simple modifications of toroidal compactifications aimed to obtain an explicit CFT description of string theory vacua preserving less than maximal supersymmetry. In geometric terms they can be defined as quotient of a manifold by some subgroup G of its isometry group; the singularities associated with the presence of points left fixed by the action of the orbifold group are avoided in string theory due to the introduction of suitable twisted sectors, *i.e.*

string states which close up to a twist by a group element $g \in G$

$$X(0,t) = gX(2\pi,t) \ . \tag{3.68}$$

In general, given any conformal field theory $\mathcal{C}$ we can study the quotient $\mathcal{C}/G$ where G is a discrete group of symmetry of the theory, *i.e.* a subgroup of the endomorphism group of the operator algebra, commuting with the Virasoro algebra and respecting the left-right decomposition of the Hilbert space. Due to the lack of a geometric understanding some of the properties of general orbifolds, such as the contribution to the partition function coming from the twisted sectors, can only be deduced by the requirement of modular invariance. As a first step in the construction of the partition function of an orbifold model, we have to project out from the original spectrum all the states that are not G-invariant. This can be accomplished by inserting a projection operator in the trace over the Hilbert space of states

$$\Pi^G = \frac{1}{|G|}\sum_{g\in G} g \ , \tag{3.69}$$

where $|G|$ is the order of the orbifold group. Consider the two non-trivial cycles **a** and **b** of the torus and denote by $Z[{}^g_h]$ the vacuum amplitude with the boundary conditions for the fields twisted by an element $g \in G$ along the cycle **a** and by an element $h \in G$ along the cycle **b**. The modular transformations properties of the functions $Z[{}^g_h]$ are given by

$$S : Z[{}^g_h] \mapsto Z[{}^{h^{-1}}_{g}] \ , \qquad T : Z[{}^g_h] \mapsto Z[{}^{hg^{-1}}_{h}] \ . \tag{3.70}$$

Since a trace with the operator h inserted corresponds to $Z[{}^0_h]$, it is clear from (3.70) that the requirement of modular invariance for the partition function will force us to introduce also terms of the form $Z[{}^h_0]$. These are the so called twisted sectors. Further modular transformations lead to the general expression for the orbifold partition function

$$Z = \frac{1}{|G|}\sum_{g,h\in G} Z[{}^g_h]\epsilon(g,h) \ , \qquad ghg^{-1}h^{-1} = 1 \ . \tag{3.71}$$

Some explanations are in order. First of all the phase $\epsilon(g,h)$ in (3.71) reflects an additional freedom present in orbifold models and called discrete torsion [33]. In general the set of the $Z[{}^g_h]$ decomposes in disjoint orbits under the action of the modular group and therefore the partition function is fixed by one-loop modular invariance up to phases multiplying these orbits. Imposing the constraint of modular invariance at higher loops, one obtains the following restrictions on the phases $\epsilon(g,h)$

$$\begin{aligned} \epsilon(g,1) &= \epsilon(1,g) = \epsilon(g,g) = 1 \ , \\ \epsilon(g,h) &= \epsilon(h,g) \ , \\ \epsilon(g,h_1h_2) &= \epsilon(g,h_1)\epsilon(g,h_2) \ . \end{aligned} \tag{3.72}$$

The previous relations imply that the non-trivial possibilities for discrete torsion are classified by $H^2(G, U(1))$, the second cohomology group of the orbifold group. This fact suggests that discrete torsion should be related to the possibility of having projective representations of the orbifold group. Indeed this is the case since the orbifold action on representations different from the vacuum representation is usually only projective. The simplest example is given by the D_2 orbifold of the affine $SU(2)$ WZW model at level $k = 1$. In this case there are two primary field, χ_1 and χ_2 (spin 0 and spin $\frac{1}{2}$), and the action of the orbifold group on χ_2 is projective and corresponds to an irreducible representation of the universal central extension of D_2, the quaternion group [34]. More examples of this will be provided by our discussion of the orbifolds of the affine $SU(2)$ WZW model at level $k = 1$ by the symmetry groups of the regular polyhedra in $\mathbb{R}^3$. These groups are A_4, S_4 and A_5 and we will see that their action on the χ_2 primary field is projective and it is given by irreducible representations of the corresponding universal central extensions, the groups $SL_2(\mathbf{Z}_3)$, $GL_2(\mathbf{Z}_3)$ and $SL_2(\mathbf{Z}_5)$. We will encounter examples of orbifolds with discrete torsion first in the course of our analysis of the superconformal field theory at $c = 3/2$ in Chapter 4 and then in the construction of open string vacua with brane supersymmetry breaking in Chapter 5.

Returning to eq.(3.71), the restriction to commuting elements $ghg^{-1}h^{-1} = 1$ comes from the fact that the cycle $\mathbf{a}\mathbf{b}\mathbf{a}^{-1}\mathbf{b}^{-1} = 1$ is contractible. As a result, states in a twisted sector $\mathcal{H}_g$ are projected only by the stabilizer $\mathcal{S}_g$ of g, where $\mathcal{S}_g = \{h \in G | hgh^{-1} = g\}$. Moreover the partition function contains a twisted contribution for every conjugacy class of G and not for every element of G. To see this, consider a state $|s\rangle \in \mathcal{H}_g$. By definition $|s\rangle \mapsto g|s\rangle$ under parallel transport along the $\mathbf{a}$-cycle and therefore $h : \mathcal{H}_g \mapsto \mathcal{H}_{hgh^{-1}}$. As a consequence $h|s\rangle$ and $|s\rangle$ are orthogonal unless h and g commute and we can conclude that sectors twisted by group elements in different conjugacy classes are orthogonal while sectors twisted by group elements in the same conjugacy class are related via the action of G and thus are projected into a single G-invariant sector in the orbifold theory. This is summarized by the relations $Z[{}^g_h] = 0$ if $ghg^{-1}h^{-1} \neq 1$ and $Z[{}^{tgt^{-1}}_{tht^{-1}}] = Z[{}^g_h]$.

There is a clear Hamiltonian interpretation of the partition function (3.71): $Z(g, h)$ is the contribution to the partition function coming from the trace on the g-twisted sector with an insertion of h. For each $g \in G$ we have the Hilbert space $\mathcal{H}_g$ of states in the g-twisted sector and we have to project it onto $\mathcal{S}_g$-invariant states. Recall that $|G| = |\mathcal{S}_g||C_g|$ where C_g is the conjugacy class of g. Then we can restrict the sum to the conjugacy classes A and chose a representative $g \in A$

$$Z = \sum_A \frac{1}{|\mathcal{S}_A|} \sum_{h \in \mathcal{S}_A} Z(g, h) = \sum_A Tr_{\mathcal{H}_A} \Pi^A q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \ , \tag{3.73}$$

and it is easy to verify that (3.71) and (3.73) coincide. Here

$$\Pi^A = \frac{1}{|\mathcal{S}_A|} \sum_{h \in \mathcal{S}_A} h \ , \tag{3.74}$$

is a projection operator onto $\mathcal{S}_A$-invariant states (we recall that the stabilizer groups for different elements in the same conjugacy class are isomorphic).

The whole construction furnishes a nice example of the power of modular invariance as a completeness condition for the closed string states. In sharp contrast with a theory of point particle, the restriction to the invariant states is not sufficient in order to obtain a well-defined model but one has to introduce new degrees of freedom.

The simplest example of an orbifold construction is given by a free scalar field X compactified on a circle of radius R. The partition function reads

$$Z(R) = \frac{1}{|\eta(q)|^2} \sum_{n,m \in \mathbf{Z}} q^{\frac{\alpha'}{4}p^2} \bar{q}^{\frac{\alpha'}{4}\bar{p}^2} \ , \tag{3.75}$$

where $p = \frac{n}{R} + \frac{mR}{\alpha'}$, $\bar{p} = \frac{n}{R} - \frac{mR}{\alpha'}$ and n, m are respectively the momentum and winding numbers. We can construct an orbifold of this model using the symmetry $\mathcal{P} : X \mapsto -X$ which connects the circle to the interval of length πR. We first have to project the spectrum in (3.75). Since a state $|n, m\rangle$ is mapped by $\mathcal{P}$ to $|-n, -m\rangle$ only states at the origin of the lattice contribute to $Z[{}^{0}_{\mathcal{P}}]$. These states are obtained acting with the modes α_n and $\bar{\alpha}_n$ of the field X on the vacuum state $|0, 0\rangle$ and therefore at each level N their contribution has a sign $(-1)^N$. We can write

$$Z[{}^{0}_{\mathcal{P}}] = \left| \frac{2\eta(q)}{\theta_2(q)} \right| \ . \tag{3.76}$$

An S modular transformation then gives the $\mathbb{Z}_2$-twisted sector

$$Z[{}^{\mathcal{P}}_{0}] = \left| \frac{2\eta(q)}{\theta_3(q)} \right| \ , \tag{3.77}$$

that is then projected onto invariant states by a T transformation

$$Z[{}^{\mathcal{P}}_{\mathcal{P}}] = \left| \frac{2\eta(q)}{\theta_4(q)} \right| \ . \tag{3.78}$$

The complete orbifold partition function reads

$$Z_o = \frac{1}{2} \left(\frac{1}{|\eta(q)|^2} \sum_{n,m \in \mathbf{Z}} q^{\frac{\alpha'}{4}p^2} \bar{q}^{\frac{\alpha'}{4}\bar{p}^2} + \left| \frac{2\eta(q)}{\theta_2(q)} \right| + \left| \frac{2\eta(q)}{\theta_3(q)} \right| + \left| \frac{2\eta(q)}{\theta_4(q)} \right| \right) \ . \tag{3.79}$$

Note that the factor of two in (3.76) is necessary for obtaining an integer number of states while in (3.77) it signals the presence of two degenerate $\mathbb{Z}_2$-twisted sectors, corresponding to the two endpoints of the interval.

Since we are mainly interested in the study of boundary conditions, we would like to have a more detailed information on the chiral content of an orbifold model than is apparent from (3.71). As stressed in [34] many basic properties of the orbifold chiral algebra are for a large part dictated by the group structure and depend very little on what the underlying CFT is, so that the orbifold can be described in terms of finite group theory. Let us start with a chiral algebra $\mathcal{A}$ and decompose the Hilbert space into chiral blocks

$$\mathcal{H} = \sum_{j,\bar{j}} [\phi_j] \otimes [\bar{\phi}_{\bar{j}}] \ , \tag{3.80}$$

where the $[\phi_j]$ are irreducible representations of the chiral algebra. Under the action of the orbifold group, a representation of the chiral algebra can be mapped to a different representation and as a result the two are identified

$$i \mapsto g(i) \ , \qquad \mathcal{N}_{ijk} = \mathcal{N}_{g(i)g(j)g(k)} \ , \tag{3.81}$$

or the representation can be mapped to itself and therefore the corresponding Hilbert space of states decomposes in irreducible representations of the orbifold group causing a splitting of the original representation and the emergence of twisted sectors.

In particular the original chiral algebra $\mathcal{A}$ decomposes according to $\mathcal{A} = \bigoplus_\alpha \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ contains the states that transform in the irreducible representation r_α of G. The chiral algebra of the orbifold model is $\mathcal{A}_0 = \mathcal{A}/G$, *i.e.* the G-invariant subalgebra of $\mathcal{A}$. Each $\mathcal{A}_\alpha$ is a representation of $\mathcal{A}_0$, in general reducible because G acts in $\mathcal{A}_\alpha$ and commutes with $\mathcal{A}_0$. We can then decompose each $\mathcal{A}_\alpha$ according to

$$\mathcal{A}_\alpha = [\phi_\alpha] \otimes r_\alpha \ . \tag{3.82}$$

From this decomposition and from the fact that the three point couplings of the original model are G invariant, it follows that the number of independent chiral couplings in $\phi_\alpha \phi_\beta = \mathcal{N}_{\alpha\beta\gamma}\phi_\gamma$ equals the number of independent Clebsh-Gordan coefficients in the tensor product of r_α and r_β . We can conclude that the representation algebra is always realized as a subalgebra of the untwisted fusion algebra.

Since the orbifold subalgebra $\mathcal{A}_0$ is G-invariant, in order to complete the construction we have to include operators that are local with respect to $\mathcal{A}$ up to the action of an element $g \in G$. These operators are the twisted operators and since twisted boundary conditions are uniquely defined up to conjugation, we have a correspondence between twisted operators and conjugacy classes of G.

In $\mathcal{A}^g$ acts the stabilizer $\mathcal{S}_g$ of g. Proceeding in the same way as for the untwisted sector, we consider the decomposition

$$\mathcal{A}^g = \bigoplus_\alpha \mathcal{A}^g_\alpha \ , \qquad \mathcal{A}^g_\alpha = [\phi^g_\alpha] \otimes r^g_\alpha \ . \tag{3.83}$$

where now α labels irreducible representations of $\mathcal{S}_g$.

Let us now concentrate on the modular properties at genus one, from which we can obtain the operator content of the model and its fusion algebra. In particular the characters of the orbifold theory associated to the decomposition $\mathcal{H}_g = \bigoplus_\alpha [\phi^g_\alpha] \otimes r^g_\alpha$ can be written as combinations of the traces

$$z\left[{g \atop h}\right] = Tr_{\mathcal{H}_g} h q^{L_0 - \frac{c}{24}} \ , \qquad \left|z\left[{g \atop h}\right]\right|^2 = Z\left[{g \atop h}\right] \ , \tag{3.84}$$

as follows

$$\chi^g_\alpha(q) = \frac{1}{|S_g|} \sum_{h \in S_g} \rho^g_\alpha(h^{-1}) \ z\left[{g \atop h}\right] \ , \tag{3.85}$$

where $\rho^g_\alpha(h^{-1})$ are characters of $\mathcal{S}_g$. The characters $\chi^g_\alpha(q)$ have obviously a q-expansion with positive integers coefficients. The inverse relation is

$$z\left[{g \atop h}\right] = \sum_\alpha \rho^g_\alpha(h) \chi^g_\alpha(q) \ , \tag{3.86}$$

We have used the completeness and orthogonality properties of the group characters

$$\frac{1}{|G|} \sum_{g \in G} \rho_\alpha(gh) \rho_\beta(h^{-1}) \ = \ \delta_{\alpha\beta} \frac{\rho_\alpha(h)}{d_\alpha} \ , \tag{3.87}$$

$$\frac{1}{|G|} \sum_\alpha \rho_\alpha(g) \rho_\beta(h^{-1}) \ = \ \frac{\delta(g,h)}{|C_g|} \ , \tag{3.88}$$

where $d_\alpha = \rho_\alpha(1)$ is the dimension of the representation and $\delta(g,h)$ is a delta function on the conjugacy classes.

In order to apply the previous expressions in a simple example, we consider again the $\mathbf{Z}_2$-orbifold of the free boson. We first recall that for rational values of the radius $R = \sqrt{\alpha' k}$, $k \in \mathbb{N}$ the free boson is a RCFT whose chiral algebra is a $U(1)_k$ affine algebra. It contains $2k$ primary fields whose characters are

$$\chi_r = \frac{1}{\eta(q)} \sum_{n \in \mathbf{Z}} q^{k(n - \frac{r}{2k})^2} \ , \qquad r = -k+1, ..., k \ , \qquad h_r = \frac{r^2}{4k} \tag{3.89}$$

and transform with the S matrix

$$S_{rr'} = \exp(-i\pi r r'/k)/\sqrt{2k} \ . \tag{3.90}$$

The chiral algebra of the $\mathbf{Z}_2$-orbifold of the free boson at rational radius can be obtained using the previous formulas [34]. It contains $(k+7)$ primary fields whose characters read

$$\begin{array}{lll} u_\pm & = \frac{1}{2}\left(\chi_0 \pm \sqrt{\frac{2\eta}{\theta_2}}\right) \ , & h = 0, 1 \ , \\ \phi_\pm & = \frac{1}{2}\chi_k \ , & h = \frac{k}{4} \ , \\ \chi_r \ , & & h = \frac{r^2}{4k} \ , \\ \sigma_i & = \frac{1}{2}\left(\sqrt{\frac{\eta}{\theta_4}} + \sqrt{\frac{\eta}{\theta_3}}\right) \ , & h = \frac{1}{16} \ , \\ \tau_i & = \frac{1}{2}\left(\sqrt{\frac{\eta}{\theta_4}} - \sqrt{\frac{\eta}{\theta_3}}\right) \ , & h = \frac{9}{16} \ , \end{array} \tag{3.91}$$

where the χ_r are the circle characters in (3.89). The fields χ_r, $r = 1, ..., k-1$ arise from the identification between the primaries χ_r and χ_{-r}. The fields $u_\pm$, $\phi_\pm$ arise from the splitting of the two fixed points at $r = 0, k$

$$u_\pm = Tr_{\chi_0}\left[\frac{1 \pm \mathcal{P}}{2} q^{L_0 - \frac{c}{24}}\right] , \tag{3.92}$$

$$\phi_\pm = Tr_{\chi_k}\left[\frac{1 \pm \mathcal{P}}{2} q^{L_0 - \frac{c}{24}}\right] , \tag{3.93}$$

where the signs $\pm$ label the two irreducible representations of $\mathbf{Z}_2$. Finally there are four twisted fields for the two fixed points, which are obtained by considering for each point the combinations

$$\frac{1}{2}\left(z\left[\begin{smallmatrix}\mathcal{P}\\1\end{smallmatrix}\right] \pm z\left[\begin{smallmatrix}\mathcal{P}\\\mathcal{P}\end{smallmatrix}\right]\right) . \tag{3.94}$$

in agreement with (3.85). The S matrix for this chiral theory can be found in [34].

3.4 Simple current modular invariants

In this Section we introduce a very useful tool for the construction of modular invariants for a rational conformal field theory, the so called simple currents [35]. In particular they allow to define a procedure, known as a simple current extension, that is the inverse operation to abelian orbifold constructions. As we will show with some examples in this Chapter and more throughly in the next Chapter, by a combination of simple current and orbifold methods one can describe boundary states for a large class of modular invariants. Let us first recall some general aspects of modular invariant partition functions. A general modular invariant can be written as

$$\mathcal{T} = \sum_{i \in I} Z_{ij} \chi_i \bar{\chi}_j , \tag{3.95}$$

where Z_{ij} is a non-negative integer matrix commuting with the matrices S and T, which represent the action of the modular group on the characters χ_i. We label the characters with indices in a set I with $|I| = N$, where N is the number of primary fields in the given RCFT. There are two modular invariants generically present for any RCFT: the charge-conjugation modular invariant $Z_{ij} = \delta_{i,j^*}$ and the diagonal modular invariant $Z_{ij} = \delta_{i,j}$. They coincide when all the characters are self-conjugate, *i.e.* when the modular S matrix is real. The modular invariants can be organized in two classes: automorphism modular invariants and extension modular invariants [36]. Automorphism modular invariants are of the form $Z_{ij} = \delta_{i,\sigma(i^*)}$ where σ is an automorphism of the fusion rules, namely

$$\mathcal{N}_{ij}^{k} = \mathcal{N}_{\sigma(i)\sigma(j)}^{\sigma(k)} , \qquad \forall i, j, k \in I . \tag{3.96}$$

For instance, the diagonal modular invariant is an automorphism modular invariant associated to the automorphism $\sigma(i) = i^*$. On the other hand, extension modular invariants have the following general form

$$\sum_i n_i \left| \sum_l m_{il} \chi_l \right|^2 , \tag{3.97}$$

where m_{il} and n_i are non-negative integers. We assume that the identity χ_0 appears exactly once, *i.e.* $m_{00} = n_0 = 1$. Modular invariants belonging to this class deserve their name because they suggest an extension of the underlying chiral algebra, that is they can be interpreted as diagonal modular invariants for an extended chiral algebra. From this point of view each term $\sum_l m_{il}\chi_l$ represents the contribution of an irreducible representation of the extended algebra.

Automorphism and extension modular invariants already appear for the $SU(2)_k$ WZW model [37]. Let us recall some general properties of these models. At level k there are $k+1$ integrable representations whose characters χ_j have conformal dimension $h_j = \frac{(j^2-1)}{4(k+2)}$, where $j = 2s+1$ and the spin $s = 0, \frac{1}{2}, 1, ..., \frac{k}{2}$. The S matrix of the model is real

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi i j}{k+2} , \tag{3.98}$$

and therefore all the representations are self-conjugate. Finally the fusion rules are

$$\chi_{2r+1}\chi_{2s+1} = \sum_{q=|r-s|}^{min(r+s,k-r-s)} \chi_{2q+1} . \tag{3.99}$$

The possible modular invariant partition functions for this class of models have been classified in [38]. They follow an ADE pattern, *i.e.* they are organized in two infinite series and three exceptional cases that can be labeled respectively by the two infinite series of simply-laced Lie algebras, A_{k+1} and D_{k+3} with $k \geq 1$, and by the three exceptional simply laced Lie algebras E_6, E_7 and E_8. For any level k we have the diagonal modular invariant, which correspond to the A-series

$$\mathcal{T}_{A_{k+1}} = \sum_{j=1}^{k+1} |\chi_j|^2 . \tag{3.100}$$

At even level there is another generic modular invariant, which corresponds to the D-series. Its explicit form is for $k = 4l$

$$\mathcal{T}_{D_{2l+2}} = {\sum_{j=1}^{2l-1}}' |\chi_j + \chi_{4l+2-j}|^2 + 2\,|\chi_{2l+1}|^2 , \tag{3.101}$$

and for $k = 4l+2$

$$\mathcal{T}_{D_{2l+3}} = {\sum_{j=1}^{4l+3}}' |\chi_j|^2 + |\chi_{2l+2}|^2 + {\sum_{j=2}^{2l}}'' \left(\chi_j \bar{\chi}_{4l+4-j} + \chi_{4l+4-j}\bar{\chi}_j\right) , \tag{3.102}$$

where $\sum'$ and $\sum''$ are sums restricted to odd and even integers respectively. The D_{even} modular invariants (3.101) are extension modular invariants while the D_{odd} modular invariants (3.102) are automorphism modular invariants. Finally for $k = 10$, 16 and 28 we have the three exceptional modular invariants

$$\begin{aligned}
\mathcal{T}_{E_6} &= |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2 \ , && (3.103)\\
\mathcal{T}_{E_7} &= |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + |\chi_9|^2 \\
&+ \chi_9 \left(\bar{\chi}_3 + \bar{\chi}_{15}\right) + \left(\chi_3 + \chi_{15}\right) \bar{\chi}_9 \ , && (3.104)\\
\mathcal{T}_{E_8} &= |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2 \ . && (3.105)
\end{aligned}$$

New modular invariants can be obtained whenever between the primary fields of a rational CFT there is a simple current. A simple current J [35] is a primary field whose fusion product with any other primary field contains only one term, that is $J\phi_i = \phi_{J(i)}$. As a consequence, the simple currents form an abelian group $\mathcal{G}$, called the center of the conformal field theory. To every primary field ϕ_i we can associate a charge with respect to a simple current $J \in \mathcal{G}$, as the monodromy exponent of the $J\phi_i$ OPE modulo integers:

$$Q_J(i) = h_i + h_J - h_{J(i)} \mod 1 \ , \tag{3.106}$$

This charge is conserved in the fusion product

$$Q_J(\phi_i \phi_k) = Q(\phi_i) + Q(\phi_k) \mod 1 \ . \tag{3.107}$$

The order N of the simple current J is the minimal integer such that $J^N = 1$. As a consequence $Q_J(J) = \frac{r}{N}$ where r is defined modulo N and also

$$h_J = \frac{N-1}{2}\, Q_J(J) \mod 1/2 \ . \tag{3.108}$$

Finally it is possible to prove that S-matrix entries related by fusion with a simple current satisfy the following identity

$$S_{Ji,k} = e^{2\pi i Q_J(k)} S_{i,k} \ . \tag{3.109}$$

Simple current modular invariants arise through an orbifold-like construction in which one exploits the abelian discrete symmetry of the conformal field theory implied by their presence and generated by the operator $\exp(2\pi i Q_J)$.

As a first instance of a simple current modular invariant we can consider the D-series (3.101) and (3.102) for the $SU(2)_k$ WZW models. Between the primary fields of an $SU(2)_k$ WZW model there is a single non-trivial simple current $J = \chi_{k+1}$. This current has order two, conformal dimension $k/4$, and its fusion with the other primaries is given by $J\chi_i = \chi_{k+2-i}$. The modular invariants associated to this current will be of extension type for $k = 4l$ and of automorphism type for $k = 4l+2$, according to eqs.(3.101) and (3.102). The monodromy charge follows from (3.106)

$$Q_J(i) = \frac{i-1}{2} \bmod 1 \ , \tag{3.110}$$

and we can see that primaries χ_i with odd or even i have monodromy charge 0 or 1/2 respectively. The discrete symmetry associated to the simple current is the isospin $\chi_i \mapsto (-1)^{i+1}\chi_i$ and in the sequel we will refer to characters with even or odd i as fermionic or bosonic characters. This symmetry can also be identified with an isometry of the group manifold S^3, the antipodal reflection, and the resulting orbifold model is thus again a WZW model but for the non-simply connected Lie group $SO(3)$. If we perform explicitly the orbifold construction, we first have to project out all the fermionic characters. The resulting $\mathbb{Z}_2$-twisted sector, obtained by an S transformation, contains the off-diagonal combinations $\chi_i\bar{\chi}_{k+2-i} + \chi_{k+2-i}\bar{\chi}_i$. For $k = 4l$ the T transformation then removes all the fermionic twisted terms, leading to the modular invariant in eqs.(3.101), while for $k = 4l+2$ it removes all the bosonic twisted terms, leading to the modular invariant in eqs.(3.102). From this point of view, the presence of two copies of the character $\chi_{\frac{k}{2}+1}$ in eqs.(3.101) takes into account the contribution of the twisted sector and in the same spirit also the fermionic term $|\chi_{\frac{k}{2}+1}|^2$ in eqs.(3.102) comes from the twisted sector.

In principle, one could follow the same steps and build case by case the simple current modular invariants. However, due to the peculiar features of simple currents and to the relation (3.109), it is possible to display the general form of the modular invariant resulting from the orbifold construction [39]. We first organize the representations of the chiral algebra in orbits for the action of the simple current group and introduce a stabilizer $\mathcal{S}_i$ for every representation i. A representation satisfying $Ji = i$ is a fixed point for the action of the current J. The length of the orbit containing i is given by $|\mathcal{G}|/|\mathcal{S}_i|$.

The general form of a simple current modular invariant depends on the order and on the spin of the current. It can be completely specified by a subgroup $\mathcal{G}$ of the center of the RCFT and by a matrix X. Let $\mathcal{G} = \otimes_s \mathbb{Z}_{n_s}$ and let J_s the current that generates the cyclic group $\mathbb{Z}_{n_s}$. In order to define the matrix X we associate to these generators a monodromy matrix $R_{st} = Q_s(J_t)$ where $Q_s(i)$ is the monodromy charge of the primary fields χ_i with respect to J_s. The diagonal entries of R are fixed only modulo two in such a way that Eq.(3.108) is valid modulo integers and not only modulo half-integers. The matrix X, defined modulo integers, is fixed by the requirement

$$X + X^T = R \tag{3.111}$$

and by a constraint on its antisymmetric part. Inequivalent choices for the antisymmetric part of X are classified by $H^2(\mathcal{G}, U(1))$ and therefore they are in one-to-one correspondence with the different possibilities for discrete torsion in abelian orbifold constructions.

Consider now two currents $J, K \in \mathcal{G}$ and let $J = J_1^{n_1}...J_k^{n_k}$, $K = K_1^{n_1}...K_k^{n_k}$ their expression in terms of the generators; we then define $X(K,J) = n_s X_{st} m_t$. The modular invariant corresponding to $\mathcal{G}$ and X is given by the product of charge conjugation and a matrix $Z(\mathcal{G}, X)$ whose entries Z_{ij} count the number of solutions to the conditions

$$\begin{aligned} j &= J(i)\ , \quad J \in \mathcal{G} & (3.112)\\ Q_K(i) + X(K,J) &= 0 \bmod 1 \quad \forall K \in \mathcal{G}\ . & (3.113)\end{aligned}$$

In the following we will consider two particular cases of this general construction: modular invariants generated by an order two simple current of half-integer spin and modular invariants generated by integer spin simple currents. In the first case the modular invariant has the same structure as the D_{odd} modular invariants of $SU(2)$ (3.102) and they will be discussed in the next Section. In the second case the simple current modular invariant belongs to the class of the extension modular invariants, and it is of the general form

$$\mathcal{Z} = \sum_{Q_a=0} |\mathcal{S}_a| \left| \sum_{J\in\mathcal{G}/\mathcal{S}_a} \chi_{J(a)} \right|^2 . \tag{3.114}$$

In the extended theory the vacuum representation is given by $\tilde{\mathcal{H}}_0 = \bigoplus_{J\in\mathcal{G}} \mathcal{H}_J$ where, as usual, $\mathcal{H}_i$ is the Hilbert space associated to the primary field ϕ_i. We can now illustrate in which sense the operation of extending the chiral algebra by a simple current is the inverse operation to orbifolding the theory by an abelian group. Indeed a general feature of abelian orbifolds is that they always contain a set of simple currents, stemming from the decomposition of the identity of the chiral algebra of the original theory, that form a group isomorphic to the orbifold group G. This currents allow for reconstructing the original theory as a simple current extension of the orbifold. Actually the original vacuum representation can be characterized as the orbifold subalgebra of $\tilde{\mathcal{H}}_0$ with respect to the orbifold group $G = \mathcal{G}^*$, *i.e.* the dual to the simple current group whose action is defined according to $R(\psi)|_{\mathcal{H}_J} = \psi(J)\mathrm{id}_{\mathcal{H}_J}$, where $\psi \in G$ is a character of the abelian group.

Muliplicities $|\mathcal{S}_a|$ greater than one in eq. (3.114) can have a double meaning. Either they have to be absorbed in the definition of the extended characters in order to account for the correct degeneracy of the ground state, or they imply that the extended theory has more than one representation whose characters $\sum_{J\in\mathcal{G}/\mathcal{S}_a} \chi_{J(a)}$ are identical functions of τ. When characters of different representations coincide, one has to face the problem of disentangle their S modular transformation properties. The problem of establishing the precise number of degenerate characters and their modular transformation properties is generally known as the fixed point resolution problem.

Let us assume that the extension modular invariant corresponds to an extended chiral algebra and let us try to derive the modular transformation matrix $\tilde{S}$ for the extended algebra. It has to satisfy a number of non-trivial constraints. It must be unitary, symmetric and together with the matrix T it must form a representation of the modular group, *i.e.* $\tilde{S}^4 = 1$ and $(\tilde{S}T)^3 = \tilde{S}^2$. Furthermore, one has to require that the matrix $\tilde{S}$ gives non-negative integer fusion rules when inserted in the Verlinde's formula

$$\tilde{N}_{ijk} = \sum_l \frac{\tilde{S}_{il}\tilde{S}_{jl}\tilde{S}_{kl}}{\tilde{S}_{0l}} . \tag{3.115}$$

Starting from the S and T matrices of the unextended theory one can derive the matrix T and the elements of the matrix $\tilde{S}$ between characters that are not fixed points. The main difficulty is then to find appropriate matrices S^J acting only on the fixed points of the current J. It has been shown in [40] that $\tilde{S}$ fulfills the requirement for a consistent S matrix if also the S^J are unitary, symmetric and together with the restriction of the matrix T to the set of fixed points give a representation of the modular group. For this reason the matrices S^J are considered as modular S matrices for a fixed point conformal field theory. In the particular case of WZW model the fixed point CFT's are still WZW models and explicit expressions for the matrices S^J are known. Let us consider a specific example, the D-invariant of $SU(2)_4$

$$|\chi_1 + \chi_5|^2 + 2\,|\chi_3|^2 . \tag{3.116}$$

The characters of the extended algebra are $\rho_1 = \chi_1 + \chi_5$, $\rho_2 = \chi_3$ and $\rho_3 = \chi_3$. The characters ρ_2 and ρ_3 are degenerate and we can parameterize the matrix $\tilde{S}$ as follows

$$\tilde{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2}+\epsilon & -\frac{1}{2}-\epsilon \\ 1 & -\frac{1}{2}-\epsilon & -\frac{1}{2}+\epsilon \end{pmatrix} , \tag{3.117}$$

The requirement of unitarity and the condition $(\tilde{S}T)^3 = \tilde{S}^2$ fix $\epsilon = i\sqrt{3}/2$. The resulting $\tilde{S}$ matrix coincides with the S matrix of $SU(3)_1$.

The general solution to the fixed point resolution problem for simple current extensions has been given in [40] where it was shown that in order to count correctly the number of resolved fixed points one has to introduce the so called *untwisted stabilizer*, which keeps track of the action of a simple current K on the fixed point set of another simple current J.

3.5 Boundary states for simple current modular invariants

We now turn to the discussion of the boundary states for simple current modular invariants [42, 41, 31, 43]. If we consider only boundary conditions which preserve the extended chiral algebra, the boundary states for an extension modular invariant can be discussed in the framework of Cardy ansatz using as reflection coefficients the entries of the matrix $\tilde{S}$. However we are interested also in the study of boundary conditions which preserve only a subset of the bulk symmetries, in particular an orbifold subalgebra of the original chiral algebra. In this respect simple currents are very useful since, as we have seen, they allow to reconstruct the original model staring from its orbifold, at least if one deals with abelian orbifold groups. The general theory of symmetry breaking boundaries has been developed in [41]. Here we explain how one can obtain symmetry breaking boundary conditions by applying the extension procedure to the boundary states associated to the twisted sectors of an orbifold theory. Moreover we would like to show how these symmetry breaking boundaries can also be interpreted as symmetry preserving boundary for the modular invariant associated to the automorphism induced by the orbifold group. This is better appreciated with the aid of a simple example, the rational boson and its $\mathbb{Z}_2$ orbifold.

For the compactified free boson, as we already explained in the second Chapter, there are two natural boundary conditions, namely Neumann and Dirichlet boundary conditions. Let us consider the model at a rational point $R^2 = \alpha' k$, $k \in \mathbb{N}$ with charge-conjugation modular invariant. The chiral content of the theory has been summarized in (3.89). The $2k$ Cardy states can be interpreted as D0 branes sitting at distinguished point along the circle $x_r = \pi R r/k$, $r = -k+1, \ldots, k$.

If we consider instead the diagonal partition function $Z = \sum_r |\chi_r|^2$, there exist only two Ishibashi states, relative to the two self-conjugate characters with $r = 0, k$, and as a consequence there are only two boundary states given by

$$|\pm\rangle = \left(\frac{k}{2}\right)^{\frac{1}{4}} (|0\rangle\rangle \pm |k\rangle\rangle) \ . \tag{3.118}$$

The Cardy boundary states of the orbifold, which are in one-to-one correspondance with the primary fields listed in (3.91), have a simple interpretation within the same framework. The states $|\chi_r\rangle$, $r = 1, ..., k-1$ come from the identification of branes sitting at opposite points along the circle. In their spectrum there is an exactly marginal operator that allows displacements of the branes along the circle. The states $|u_+\rangle$, $|u_-\rangle$, $|\phi_+\rangle$ and $|\phi_-\rangle$ describe fractional branes sitting at the fixed points of the interval. Indeed the fields in the twisted sector have non-vanishing one-point functions in front of these boundaries and since there are not marginal deformations they are forced to live on the fixed point unless they combine with the other fractional brane. Finally the Cardy states related to the twisted sector primary fields $|\sigma_i\rangle$, $|\tau_i\rangle$ with $i = 1, 2$ can be interpreted as fractional $D1$-branes with suitable Wilson lines.

Let us now perform a simple current extension by the integer-spin simple current u_-. The extension modular invariant thus obtained is the circle theory. We can follow the transformation of the boundary states: $|u_+\rangle$ and $|u_-\rangle$ combine giving $|\chi_0\rangle$ while $|\phi_+\rangle$ and $|\phi_-\rangle$ give $|\chi_k\rangle$; the fixed points $|\chi_r\rangle$ split giving the two boundary states $|\chi_r\rangle$ and $|\chi_{-r}\rangle$.

From the twisted sector we obtain two more boundary states

$$|+\rangle = \frac{1}{\sqrt{2}}(|\sigma_1\rangle + |\tau_1\rangle) = \left(\frac{k}{2}\right)^{1/4} (|u_+\rangle - |u_-\rangle + |\phi_+\rangle - |\phi_-\rangle) \ , \tag{3.119}$$

and similarly for $|-\rangle = (|\sigma_1\rangle + |\tau_1\rangle)/\sqrt{2}$. They preserve only a $U(1)_k/\mathbf{Z}_2$ orbifold subalgebra of the full $U(1)_k$ symmetry of the circle [41] and can be interpreted as $D1$ branes, namely they correspond to Neumann boundary conditions for the circle theory. This is confirmed by the annulus amplitude between these new states and the original $D0$ states of the circle

$$A_{+r} = \sigma_1 + \tau_1 \ , \tag{3.120}$$

which is the usual partition function for a free boson on a strip with Neumann boundary conditions on one side and Dirichlet boundary conditions on the other. Actually the Ishibashi states $|u_+\rangle\rangle - |u_-\rangle\rangle$ and $|\phi_+\rangle\rangle - |\phi_-\rangle\rangle$ are precisely those associated to the Neumann gluing condition $\Omega' = \Omega\mathcal{P}$. According to the general formalism for symmetry breaking boundaries of Ref. [41], we can move the automorphism $r \mapsto -r$ from the gluing condition to the modular invariant and consider these boundary states as boundary states for the diagonal modular invariant.

We can generalize this result as follows. Let us consider a conformal field theory $\mathcal{C}$ with a chiral algebra $\mathcal{A}$, and an orbifold of this theory by some discrete group G. We call σ the fusion rule automorphism induced by the orbifold action and we assume that the standard gluing condition Ω has been chosen for the chiral algebra. We now construct the boundary states for $\mathcal{C}$ by a simple current extension from the orbifold theory. Boundary states in the orbifold theory associated to untwisted representations become symmetry preserving boundaries in the original theory while boundary states corresponding to twisted representations give rise to new boundary states which only preserve the orbifold subalgebra $\mathcal{A}/G$ and are therefore associated to the gluing $\Omega \cdot \sigma$. Moreover, if we chose for $\mathcal{C}$ a modular invariant specified by an automorphism π, $Z_{i\pi(j*)}$ and use another automorphism σ to specify a gluing condition $\Omega' = \Omega{\cdot}\sigma$, only the difference $\pi\sigma^{-1}$ between the two automorphisms matters for the construction of the boundary states [41]. We can conclude that the symmetry breaking boundaries coming from the orbifold twisted sectors can also be interpreted as symmetry preserving boundaries for the modular invariant specified by σ, $Z_{i\sigma(j*)}$.

We now turn to modular invariants of automorphism type and discuss in some detail a case that will occur quite often in our subsequent description of boundary CFT at $c = 1$ and $c = 3/2$. As we have already explained, the boundary conditions for the charge conjugation modular invariant are controlled by the fusion algebra. A similar structure has been uncovered in [42] for the case of simple current modular invariants generated by a current J of order two and half-integer spin. We observe that the charge of a field fixed under the action of J is $Q = 1/2$. The primary fields of the model are then organized in three group: a set of N_0 fields of charge $Q = 0$, a set of N_1 fields of charge $Q = 1/2$ that are not fixed points and a set of N_f fixed points. The resulting modular invariant is

$$Z = \sum_{i\ |Q(i)=0} \chi_i \bar{\chi}_{i^*} + \sum_{i\ |Q(i)=1/2} \chi_i \overline{\chi}_{J(i^*)} \tag{3.121}$$

In particular the D_{odd}-series modular invariants for the $SU(2)$ WZW models are of this type. In [42] a complete set of boundaries for this class of modular invariants was constructed considering the algebra satisfied by the reflection coefficients of the bulk states, called the *classifying algebra*. Actually the boundary conditions are interpreted as one-dimensional representations of this algebra. While for the charge conjugation modular invariant the classifying algebra coincides with the fusion algebra, for modular invariants of the type (3.121) it should be $(N_0 + N_f)$-dimensional, since only $(N_0 + N_f)$ bulk fields can couple to the boundary. On the other hand if we count the number of expected boundary conditions looking at the action of the simple current on the representations of the chiral algebra, we should expect $\frac{1}{2}(N_0+N_1)+2N_f$ irreducible representations. We have therefore the sum rule

$$\frac{1}{2}(N_0 + N_1) + 2N_f = N_0 + N_f \ , \tag{3.122}$$

and it is possible to show that it is always satisfied. The structure constants of the classifying algebra were linked, through Verlinde's formula, to the fixed points S matrix $\hat{S}$

$$\hat{N}_{fg\nu} = \sum_{h\in N_f} \frac{\hat{S}_{fh}\hat{S}_{fh}S_{\nu h}}{S_{0h}} \ . \tag{3.123}$$

We have to distinguish between two kinds of representations. Orbits α of length two give a single representation R_α

$$R_\alpha(\phi_\lambda) = \begin{cases} \frac{S_{\alpha\lambda}}{S_{0\alpha}} & Q(\lambda) = 0 \\ 0 & J\lambda = \lambda \end{cases} \tag{3.124}$$

while a fixed point f gives two irreducible representations $R_{f^\pm}$

$$R_{f^\pm}(\phi_\lambda) = \begin{cases} \frac{S_{f\lambda}}{S_{0f}} & Q(\lambda) = 0 \\ \pm\frac{\hat{S}_{f\lambda}}{S_{0f}} & J\lambda = \lambda \end{cases} \tag{3.125}$$

The corresponding annulus coefficients read

$$\begin{aligned} A^\mu_{\alpha\beta} &= \frac{1}{2}\left(N_{\alpha\beta\mu} + \hat{N}_{J(\alpha)\beta\mu}\right) \ , \\ A^\mu_{\alpha f^\pm} &= N_{\alpha f\mu} \ , \\ A^\mu_{f^\pm g^\pm} &= \frac{1}{2}\left(N_{fg\mu} + \hat{N}_{fg\mu}\right) \ , \\ A^\mu_{f^\pm g^\mp} &= \frac{1}{2}\left(N_{fg\mu} - \hat{N}_{fg\mu}\right) \ . \end{aligned} \tag{3.126}$$

and one can explicitly verify that they are non-negative integers and satisfy the completeness conditions. We can illustrate the intuitive meaning of this formal construction with the aid of the D-series modular invariants for the $SU(2)_k$ WZW models.

Since there is a simple orbifold relation between the A-series and the D-series modular invariants, one is led to think that a similar simple relation should exist also for the possible boundary states. Indeed under an orbifold operation, boundary conditions behaves in a way that strictly mimics the behavior of the representations of the chiral algebra. Let us see how does it work concretely. First of all we should form combinations of boundary states that are invariant under the orbifold operation. This is easily achieved considering the states

$$|i\rangle_D = \frac{1}{\sqrt{2}}\left(|i\rangle_A + |k+2-i\rangle_A\right) \ , \tag{3.127}$$

for $i = 1, ..., \frac{k}{2}$. In (3.127) the subscript A stands for boundary states for the A-series and the subscript D stands for boundary states for the D-series. We note that only the bulk states with i odd can reflect on the boundaries in (3.127). The state $|f\rangle_A$ where $f = (k+2)/2$ behaves in a different way. Since it is fixed by the orbifold action, it gives rise to two different boundary states. These two states are distinguished from the "untwisted states" (3.127) since they have a non-vanishing reflection coefficient for the twisted field $\chi_{\frac{k+2}{2}}$. They can be considered, using a string terminology, as fractional boundary states. These two fractional boundaries read

$$|f,\pm\rangle_D = \frac{1}{\sqrt{2}}\,|f\rangle_A \pm \frac{R}{\sqrt{S_{1f}}}\,|f\rangle\!\rangle \ . \tag{3.128}$$

The coefficient R in front of the Ishibashi states for the representation $\chi_{\frac{k+2}{2}}$ is fixed by requiring that the sets of boundaries so obtained satisfies the completeness conditions outlined before: $R = 1/\sqrt{2}$. The general pattern of the boundary states for pure automorphism modular invariant generated by a $\mathbb{Z}_2$ current of half-integer spin is already apparent in the previous example. There are "untwisted" boundaries in one-to-one correspondence with length-two orbits of the simple current which we label with an index α

$$|\alpha\rangle = \frac{S_{\alpha,i} + S_{J(\alpha),i}}{\sqrt{2S_{0,i}}}|i\rangle\!\rangle \ . \tag{3.129}$$

These states are characterized by having vanishing reflection coefficients for all the fields i with $Q_J(i) \neq 0$. In addition there are "twisted" boundary states, two for each fixed point of the simple current, characterized by non-vanishing reflection coefficients for those fields that have $Q_J = 1/2$ and are also fixed by J. Actually we can think of this field as "twisted" fields since in an orbifold construction of the modular invariant the corresponding representation are projected out from the untwisted spectrum but reappear in the twisted sector. Their boundary states read

$$|f_\pm\rangle = \frac{R_{f_\pm,i}}{\sqrt{S_{0,i}}}|i\rangle\!\rangle \quad , \tag{3.130}$$

where the boundary coefficients with respect to the Ishibashi states $|i\rangle\!\rangle$ satisfying $J(i) \neq i$ are simply given by the S matrix of the model $R_{f\pm i} = \frac{S_{fi}}{\sqrt{2}}$. On the other hand the boundary coefficients with respect to Ishibashi states corresponding to fixed points of the simple current can be expressed in terms of a suitable fixed point S matrix that we have denoted by $\hat{S}$ in (3.125). The reflection coefficients are

$$R_{f\pm g} = \pm\frac{\hat{S}_{fg}}{\sqrt{2}} \ . \tag{3.131}$$

A general ansatz for the boundary and crosscap coefficients for modular invariants given by symmetric simple current modifications of the conjugation modular invariant was presented in [43]. The resulting expression is similar to Cardy formula but again involves the fixed point matrices S^J and the characters of a suitable subgroup of the stabilizer.

We close this section mentioning that using a simple current J one can also modify the Ω projection and obtain new classes of open descendants [31]. Cardy solution can be easily generalized in order to include also these cases and the boundary and crosscap coefficients read

$$B^{(J)}_{ai} = \frac{S_{ai}}{\sqrt{S_{Ji}}} \ , \qquad \Gamma^{(J)}_i = \frac{P_{ai}}{\sqrt{S_{0i}}} \ . \tag{3.132}$$

The corresponding coefficients for the direct channel amplitudes are

$$K^i = Y^i_{JJ^*} \ , \tag{3.133}$$
$$A^i_{ab} = \mathcal{N}^{Ji}_{ab} \ , \tag{3.134}$$
$$M^i_a = Y^i_{J^*(a)J} \ . \tag{3.135}$$

When $J \neq 1$ what is altered with respect to the Cardy case are the boundary conjugation properties and therefore the assignment of real and complex charges. Indeed the simple current acts as a permutation of the boundary states. In the next Section we will show that there is an interesting geometric interpretation for these modified Ω projections.

3.6 Branes on group manifolds

Until now we have characterized the conformal boundary conditions using algebraic notions. The structure of the chiral algebra and of its orbifold subalgebras have been key concepts in order to classifying the possible boundary states and to understand which part of the bulk symmetries they preserve. However, when a conformal field theory has a description as a σ-model with target space M we can try to find the geometric interpretation of the various boundary conditions in terms of submanifolds of the target space and of vector bundles on them. In the second Chapter we have seen what is the interpretation of the boundary states in a flat space-time. The most natural setting for trying a geometric interpretation of the boundary conditions in a curved background is represented by the WZW models since in this case both the geometric and the CFT description are under control. The target space of a WZW is a Lie group manifold G; in particular we assume G to be real, compact and simply connected. In the course of the analysis we have to remember that geometric techniques can provide reliable information only in the large volume limit. In the stringy regime only the algebraic boundary CFT description outlined in the previous Section is available.

In flat space the picture of D-branes as hyperplanes on which open strings can end followed by inspection of the boundary conditions imposed on the fields X^i, $i = 0, ..., D-1$ that describe the embedding of the string worldsheet Σ in the target space. In the same way we can try to gain a first understanding of the geometry of branes on group manifolds looking at the boundary conditions for the fields $g(z, \bar{z}) : \Sigma \mapsto G$ that appear in the WZW model Lagrangian.

The left and right affine algebras are generated by the currents $j(z) = kg^{-1}\partial g$ and $\bar{j}(\bar{z}) = -k\bar{\partial}gg^{-1}$. Boundary conditions that preserve the affine chiral symmetry are obtained imposing [44]

$$j(z) = \Omega\bar{j}(\bar{z}) \ , \qquad \text{for } z = \bar{z} \ , \tag{3.136}$$

where Ω is an automorphism of the affine algebra $\hat{\mathcal{G}}$. The most general automorphism is given by the composition of an outer automorphism ω, related to a symmetry of the Dynkin diagram, and an inner automorphism Ad_g, given by conjugation by an element $g \in G$. Since inner automorphisms can be interpreted as rigid translations induced from the left action of g on the group manifold, we will restrict ourselves to the consideration of outer automorphisms. Let us assume $\omega = 1$ for the moment. The boundary conditions (3.136) can be rewritten as

$$[1 - Ad(g)]\, g^{-1}\partial_t g = [1 + Ad(g)]\, g^{-1}\partial_x g \ , \qquad (3.137)$$

where $Ad(g)h \equiv hgh^{-1}\ \forall g, h \in G$. If we split the tangent space T_gG in two subspaces, one tangent and the other orthogonal, with respect to the Killing form, to the orbit of the adjoint action of G in g, $T_gG = (T_gG)_{||}\,(T_gG)_{\perp}$, we obtain a form for the boundary conditions which is reminiscent of the boundary conditions in flat space

$$\begin{aligned} \left(g^{-1}\partial_x g\right)_{\perp} &= 0 \ , \\ \left(g^{-1}\partial_t g\right)_{||} &= \frac{1 + Ad(g)}{1 - Ad(g)} \left(g^{-1}\partial_x g\right)_{||} \ . \end{aligned} \qquad (3.138)$$

From the previous equations, we can see that the ends of the open strings are confined on conjugacy classes supporting a non-trivial B field. The appearance of a non-trivial 2-form field is expected in a curved geometry since the string equations of motions enforce $H \neq 0$ when the metric curvature does not vanish. In the more general case $\omega \neq 1$, the brane worldvolume spans a twined conjugacy class, defined as

$$C_u^{\omega} = \{hu\omega(h^{-1}) | h \in G\} \ . \qquad (3.139)$$

Generically twined conjugacy classes are diffeomorphic to G/T^{ω} where T^{ω} is the image in the Lie group G of the elements of the Cartan subalgebra fixed under ω. Their generic dimension is $\text{dim}C_u^{\omega} = \text{dim}G - \text{dim}T^{\omega}$.

A better understanding of the shape of the D-branes can be obtained from the study of one-point functions of bulk fields in the presence of the brane [45]. From these one-point functions it is possible to extract the couplings of the brane to the bulk fields and in particular its tension and its charges with respect to the RR forms. Moreover the location of the brane can be extracted form the coordinate dependence of the one-point functions. It is not surprising that almost all relevant data are contained in the one-point functions, since due to factorization, all the information on a boundary condition that can be obtained by use of bulk fields is obtainable from correlators involving a single bulk field.

We proceed now to illustrate the previous geometric interpretation of the boundary states in WZW models studying again the simplest case: the $SU(2)_k$ WZW model.

For an $SU(2)$ WZW at level k the group manifold is a three-sphere S^3. The metric and the Neveu-Schwarz three form backgrounds are

$$ds^2 = k\alpha' \left[d\psi^2 + \sin^2\psi \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \ , \qquad (3.140)$$

$$H = 2k\alpha' \sin^2\psi \sin\theta \ d\psi d\theta d\phi \ . \qquad (3.141)$$

By a suitable choice of gauge, the 2-form potential can be made proportional to the volume form of the two-sphere spanned by (θ, ϕ),

$$B = k\alpha' \left(\psi - \frac{\sin 2\psi}{2}\right) \sin\theta d\theta d\phi \ . \tag{3.142}$$

The singularity at $\psi = \pi$ is unobservable due to the quantization of the level k. The conjugacy classes of $SU(2)$ are two-spheres S^2. A D2-brane wrapped around a (θ, ϕ) two-sphere, at a fixed value of ψ, breaks the $SU(2)_L \times SU(2)_R$ symmetry to a diagonal $SU(2)$. Consistently with this residual symmetry, its worldvolume can support a uniform flux

$$F = dA = -\frac{n}{2} \sin\theta d\theta d\phi \ . \tag{3.143}$$

which is crucial for the brane stability. Indeed if the dominant contributions to their low energy effective action were contained in the gravitational part, the branes would shrink to a point. Performing a semiclassical analysis [46] one can reproduce correctly the boundary CFT results for the mass, the charge and the spectrum of small fluctuations around the brane. In general, for compact Lie groups, there is complete agreement between the semiclassical analysis based on the Born-Infeld action and the exact boundary CFT results, up to a shift of the weight by the Weyl vector and of the level by the dual Coxeter number [47].

We can extend the semiclassical analysis also to the orientifolds of $SU(2)$ [48, 49]. Let us represent a generic element g of the group $SU(2)$ as

$$g = \frac{1}{R} \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ x_3 - ix_4 & x_1 - ix_2 \end{pmatrix} \ . \tag{3.144}$$

and use the following parameterization

$$x_1 = R\cos\psi \ , \qquad x_2 = R\sin\psi\cos\theta \ , \qquad x_3 + ix_4 = R\sin\psi\sin\theta e^{i\gamma} \ , \tag{3.145}$$

where $\psi, \theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, We want to determine those orientifold projections that preserve a diagonal affine $SU(2)$ algebra. For that purpose we need isometries of the group manifold S^3 that combined with the operation of worldsheet parity $\Omega : \sigma \mapsto 2\pi - \sigma$ preserve the background antisymmetric 2-form B. Since its field strength H is proportional to the volume form and since Ω reverses the orientation of S^3, Ω alone would project out the two-form B. In order to have an orientifold projection consistent with the background 2-form flux we need an isometry that reverse the orientation. There are, modulo rotations, two distinct choices

$$h_0 : g \mapsto g^\dagger \qquad (\psi, \theta, \phi) \ \mapsto \ (\psi, \pi - \theta, \pi + \phi) \tag{3.146}$$

$$h_2 : g \mapsto -g^\dagger \qquad (\psi, \theta, \phi) \ \mapsto \ (\pi - \psi, \theta, \phi) \tag{3.147}$$

$$\tag{3.148}$$

The first Ω projection acts on the group manifold as a reflection with respect to the axis of rotation, the second as a reflection with respect to the equator. These two different geometric actions are reflected neatly by the charge assignments for the D-branes. In the first case all the boundary states are self-conjugate and carry orthogonal or symplectic gauge groups. In the second case boundary states wrapped around two-spheres sitting at ψ and $\pi - \psi$ respectively are paired and carry complex charges. Only the brane wrapped around the equator, which exists only for k even, has a real Chan-Paton multiplicity.

This is confirmed by the construction of the open descendants. Complete set of boundaries and various Klein bottle and Möbius projections were first considered for the $SU(2)_k$ WZW models in [28, 50, 27]. For the A series there are two possible Ω projections,

$$\mathcal{K}_r = \frac{1}{2}\sum_{i=1}^{k+1}(-1)^{i+1}\chi_i \ , \tag{3.149}$$

which leads to an annulus amplitude involving only real charges and

$$\mathcal{K}_c = \frac{1}{2}\sum_{i=1}^{k+1}\chi_i \ , \tag{3.150}$$

which leads to an annulus amplitude with only complex charges for odd k and with only complex charges but the one related to $\chi_{\frac{k}{2}+1}$ for even k.

We display explicitly the annulus and the Möbius amplitudes corresponding to the two choices for the A_7 model, in order to compare them later with the D_5 model. The first Ω projection leads to

$$\begin{aligned}
\mathcal{A}_r &= \frac{1}{2}\left(n_1^2+n_2^2+n_3^2+n_4^2+n_5^2+n_6^2+n_7^2\right)\chi_1 \\
&+ \left(n_1n_2+n_2n_3+n_3n_4+n_4n_5+n_5n_6+n_6n_7\right)\chi_2 \\
&+ \left(\frac{n_2^2+n_3^2+n_4^2+n_5^2+n_6^2}{2}+n_1n_3+n_2n_4+n_3n_5+n_4n_6+n_5n_7\right)\chi_3 \\
&+ \left(n_2n_3+n_1n_4+n_3n_4+n_2n_5+n_4n_5+n_3n_6+n_5n_6+n_4n_7\right)\chi_4 \\
&+ \left(\frac{n_3^2+n_4^2+n_5^2}{2}+n_2n_4+n_1n_5+n_3n_5+n_2n_6+n_4n_6+n_3n_7\right)\chi_5 \\
&+ \left(n_3n_4+n_2n_5+n_4n_5+n_1n_6+n_3n_6+n_2n_7\right)\chi_6 \\
&+ \left(\frac{n_4^2}{2}+n_3n_5+n_2n_6+n_1n_7\right)\chi_7 \ ,
\end{aligned} \tag{3.151}$$

$$\begin{aligned}
\mathcal{M}_r &= \pm\frac{1}{2}[\left(n_1-n_2+n_3-n_4+n_5-n_6+n_7\right)\chi_1 \\
&+ \left(n_2-n_3+n_4-n_5+n_6\right)\chi_3+\left(n_3-n_4+n_5\right)\chi_5+n_4\chi_7] \ .
\end{aligned} \tag{3.152}$$

We see that all the branes are self-conjugate and therefore carry orthogonal or symplectic groups, in agreement with the geometric action of the involution since it leaves their worldvolumes fixed. The annulus and Möbius amplitudes for the other choice for Ω are

$$\begin{aligned}
\mathcal{A}_c &= \left(n_1\bar{n}_1 + n_2\bar{n}_2 + n_3\bar{n}_3 + \frac{n_4^2}{2}\right)\chi_1 \\
&+ \left(n_3n_4 + n_2\bar{n}_3 + n_4\bar{n}_3 + n_1\bar{n}_2 + n_3\bar{n}_2 + n_2\bar{n}_1\right)\chi_2 \\
&+ \left(\frac{n_3^2 + \bar{n}_3^2 + n_4^2}{2} + n_1\bar{n}_3 + n_2n_4 + n_3\bar{n}_3 + n_2\bar{n}_2 + n_4\bar{n}_2n_3\bar{n}_1\right)\chi_3 \\
&+ \left(n_2n_3 + n_1n_4 + n_3n_4 + n_2\bar{n}_3 + n_4\bar{n}_3 + n_3\bar{n}_2 + \bar{n}_3\bar{n}_2 + n_4\bar{n}_1\right)\chi_4 \\
&+ \left(\frac{n_2^2 + n_3^2 + n_4^2 + \bar{n}_2^2 + \bar{n}_3^2}{2} + n_1n_3 + n_2n_4 + n_3\bar{n}_3 + n_4\bar{n}_2 + \bar{n}_3\bar{n}_1\right)\chi_5 \\
&+ \left(n_1n_2 + n_2n_3 + n_3n_4 + n_4\bar{n}_3 + \bar{n}_3\bar{n}_2 + \bar{n}_2\bar{n}_1\right)\chi_6 \\
&+ \frac{1}{2}\left(n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2 + n_7^2\right)\chi_7 \;, \qquad (3.153) \\
\mathcal{M}_c &= \pm\frac{1}{2}[n_4\chi_1 + (n_3 + n_4 + \bar{n}_3)\chi_3 + (n_2 + n_3 + n_4 + \bar{n}_3 + \bar{n}_2)\chi_5 \\
&+ (n_1 + n_2 + n_3 + n_4 + \bar{n}_3 + \bar{n}_2 + \bar{n}_1) \;. \qquad (3.154)
\end{aligned}$$

In this case all the branes are paired by the involution and carry complex charges except for the brane at the equator which is fixed.

Also for the D series there are two choices for the Ω projection. The resulting annulus amplitude contains real charges for all the boundaries except for the two fractional boundaries whose charges can either be both real or form a complex pair. The possibility of introducing one complex charge in the D series is then related to the implementation of the Ω projection on the two fractional branes arising from the splitting of the $|\chi_{k/2+1}\rangle$ Cardy state. The explicit amplitudes for D_5 are, in the real case

$$\begin{aligned}
\mathcal{A}_r^D &= \frac{1}{2}\left(n_1^2 + n_2^2 + n_3^2 + n_+^2 + n_-^2\right)\chi_1 \\
&+ \left(n_1n_2 + n_2n_3 + n_3n_+ + n_3n_-\right)\chi_2 \\
&+ \left(\frac{n_2^2 + 2n_3^2}{2} + n_1n_3 + n_2n_+ + n_2n_-\right)\chi_3 \\
&+ \left(n_1n_+ + n_1n_- + 2n_2n_3 + n_3n_+ + n_3n_-\right)\chi_4 \\
&+ \left(\frac{n_2^2 + 2n_3^2 + n_+^2 + n_-^2}{2} + n_1n_3 + n_2n_+ + n_2n_-\right)\chi_5 \\
&+ \left(n_1n_2 + n_2n_3 + n_3n_+ + n_3n_-\right)\chi_6 \\
&+ \left(\frac{n_1^2 + n_2^2 + n_3^2}{2} + n_1n_2 + n_+n_-\right)\chi_7 \;, \qquad (3.155) \\
\mathcal{M}_r^D &= \pm\frac{1}{2}[(-n_1 + n_2 - n_3 + n_+ + n_-)\chi_1 + (-n_2 + 2n_3)\chi_3 \\
&+ (n_2 + n_+ + n_-)\chi_5 + (n_1 + n_2 + n_3)\chi_7] \;, \qquad (3.156)
\end{aligned}$$

and in the complex case

$$\mathcal{A}_c^D = \frac{1}{2}\left(n_1^2 + n_2^2 + n_3^2 + 2n_1n_2 + 2n\bar{n}\right)\chi_1$$

$$
\begin{aligned}
&+ && (n_1 n_2 + n_2 n_3 + n_3 n + n_3 \bar{n})\, \chi_2 \\
&+ && \left(\frac{n_2^2 + 2n_3^2 + n^2 + \bar{n}^2}{2} + n_1 n_3 + n_2 n + n_2 \bar{n} \right) \chi_3 \\
&+ && (n_1 n + n_1 \bar{n} + 2 n_2 n_3 + n_3 n + n_3 \bar{n})\, \chi_4 \\
&+ && \left(\frac{n_2^2 + 2n_3^2}{2} + n_1 n_3 + n_2 n + n_2 \bar{n} \right) \chi_5 \\
&+ && (n_1 n_2 + n_2 n_3 + n_3 n + n_3 \bar{n})\, \chi_6 \\
&+ && \left(\frac{n_1^2 + n_2^2 + n_3^2 + n^2 + \bar{n}^2}{2} + n_1 n_2 \right) \chi_7 \ , \qquad (3.157)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_c^D &= && \pm\frac{1}{2}[(n_1 - n_2 + n_3)\, \chi_1 + (n_2 + n + \bar{n})\, \chi_3 \\
&+ && (n_2 + 2n_3)\, \chi_5 + (n_1 + n_2 + n_3 + n + \bar{n})\, \chi_7] \ . \qquad (3.158)
\end{aligned}
$$

Note that there are only five types of boundaries and that the algebra of the boundary operators is different from the algebra of the bulk operators, in particular it contains multiplicities greater than one, as it is typical for an extended algebra. Indeed the boundary chiral algebra is extended by the simple current χ_{k+1} which, having half-integer spin, can not extend the bulk chiral algebra [27].

A geometric interpretation for the orientifold planes in general WZW models has been presented in [49]. The result of the analysis is that the modified Klein bottle projections introduced at the end of the previous Section are in one-to-one correspondence with involutions of the group manifold given by the composition of Ω with a non-trivial element R_n of the center of the group

$$
\Omega_n = \Omega R_n \ , \qquad R_n : g \mapsto \gamma^n g^{-1} \ , \qquad (3.159)
$$

where we assumed that the center is cyclic. Indeed, with only one exception, the center of the chiral algebra of a WZW model and the center of the corresponding Lie group coincide. The orientifold planes are the set of points satisfying $\{g \in G | g = \gamma^n g^{-1}\}$ and it is easy to see that they are localized on conjugacy classes. For $SU(2)$ the first Ω projection corresponds to two $O0$-planes sitting at the north and at the south pole while the second Ω projection corresponds to an $O2$-plane sitting at the equator.

The boundary states for the $SU(2)$ WZW can be also derived using the adjacency matrices of ADE graphs as explained in [51].

3.7 Superconformal field theories

In this Section we introduce the superconformal algebra [52], as another example of a chiral algebra containing the Virasoro algebra. Besides the energy-momentum tensor, this algebra contains a new generator $G(z)$ of conformal dimension $h = 3/2$, the supercurrent. The algebra can be defined through the OPEs of the two generators T and G which read

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} , \tag{3.160}$$

$$T(z)G(w) \sim \frac{3G(z)}{2(z-w)^2} + \frac{\partial G(w)}{z-w} , \tag{3.161}$$

$$G(z)G(w) \sim \frac{\hat{c}}{(z-w)^3} + \frac{2T(w)}{z-w} , \tag{3.162}$$

where $\hat{c} = 2c/3$. Actually there are two superconformal algebras which differ for the boundary conditions imposed on the supercurrent. If $G(z)$ satisfies periodic boundary conditions, the algebra is called the Ramond (R) algebra while if $G(z)$ satisfies antiperiodic boundary conditions, the algebra is called the Neveu-Schwarz (NS) algebra. Using the mode expansions

$$T(z) = \sum_{n \in Z} L_n z^{-n-2} , \tag{3.163}$$

$$G(z) = \sum_{r \in Z} G_r z^{-r-\frac{3}{2}} , \tag{3.164}$$

we obtain the following commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} , \tag{3.165}$$

$$[L_n, G_r] = \left(\frac{n}{2} - r\right) G_{n+r} , \tag{3.166}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0} \tag{3.167}$$

where the index r takes values in $\mathbf{Z}$ for the R algebra and in $\mathbf{Z}+1/2$ for the NS algebra. From (3.167) it follows that $G_0^2 = L_0 - c/24$ and therefore global supersymmetry is not broken when the ground states in the Ramond sector have dimension $h = c/24$. The partition function of a superconformal model can be written in general as

$$Z = \frac{1}{2}\left(Z_{NS} + Z_{\widetilde{NS}} + Z_R \pm Z_{\tilde{R}}\right) , \tag{3.168}$$

where the four terms correspond to antiperiodic (a) and periodic (p) boundary conditions for the supercurrent G along the two non-trivial cycles of the torus: respectively, (a,a), (p,a), (a,p) and (p,p). The last term is the Witten index [53] $Tr_R(-1)^F$ and the two choices of sign in front of it are related by $(-1)^F$, a $\mathbb{Z}_2$ symmetry generically present in superconformal models which takes the value $+1$ on states in the (NS,NS) sector and the value -1 on states in the (R,R) sector. Indeed one theory is the orbifold of the other by $(-1)^F$.

Chapter 4

Boundary States of $c = 1$ and $3/2$ Rational Conformal Field Theories

In this Chapter we study the boundary states for the rational conformal theories at $c = 1$ and for the superconformal theories at $c = 3/2$ [**toi**]. We discuss in particular the isolated points on the moduli space which can be obtained as orbifold by the symmetry groups of regular polyhedra in $\mathbb{R}^3$ and whose existence was first pointed out in [55, 56]. In the course of this analysis, we will apply extensively the formalism developed for the study of boundary states in orbifolds and simple current modular invariants [42, 43] and for the construction of symmetry breaking boundaries [41] reviewed in the previous Chapter.

4.1 Boundary states at c=1

Conformal field theories with $c < 1$ have been completely classified and are represented by the well-known Virasoro minimal models [57]. For any allowed value of $c < 1$ there are only few distinct models, associated to different modular invariant partition functions. At $c = 1$ the space of conformal field theories has a non-trivial structure and it is possible to connect continuously different models using exactly marginal deformations. For instance, if we consider the simplest realization of a model with $c = 1$, a free scalar field X compactified on a circle of radius R, it is clear that the model is conformal for any value of R. We recall its partition function, already presented in (3.75)

$$Z_c(R) = \frac{1}{|\eta|^2} \sum_{n,m\in\mathbb{Z}} \Gamma_{nm} \ , \qquad \Gamma_{nm} = q^{\frac{\alpha'}{4}p^2} \overline{q}^{\frac{\alpha'}{4}\overline{p}^2} \ . \tag{4.1}$$

We can interpolate between different values of R using the operator $\partial X \bar{\partial} X$.

In [55] a general method was introduced for the investigation of the moduli space of conformal field theories with a given central charge c and applied to conformal field theory with $c = 1$. According to [55], one starts from a known theory and studies its exactly marginal deformations, covering a first region of the moduli space. Then one uses the discrete symmetries of these models for reaching new branches of the moduli space by an orbifold construction. If these symmetries are present for generic values of some modulus, one obtains a new family of models. However if the symmetry used for taking the quotient exists only for a particular value of the moduli, the resulting orbifolds will be isolated points on the moduli space. The last step is to look for multicritical points, in order to find eventual intersection points between the different branches of the moduli space. Once a set of marginal deformations has been individuated in a point of the moduli space, one has to verify that they really correspond to inequivalent direction of deformation before concluding that that point is a multicritical point.

Let us analyze in this way the space of conformal field theories at $c = 1$, starting from the circle line (4.1). The reflection symmetry $\mathcal{P} : X \mapsto -X$ is present for any value of the radius and therefore we obtain a second line of models, the orbifold line we have already discussed in (3.79).

We now look for points with enhanced symmetry. We recall that the circle line partition function has the property $Z_c(R) = Z_c(\alpha'/R)$ and that at the self-dual radius the model displays an $\widehat{SU(2)}_1$ affine algebra. We can then take the quotient of the $Z_c(\sqrt{\alpha'})$ model by any discrete subgroup of $SU(2)$.

Also for the discrete subgroups of $SU(2)$ there exists an ADE classification. The first series of discrete subgroups is represented by the cyclic groups $\mathbf{C}_n$ whose elements $g_{l/n}$, $l = 0, ..., n-1$ act by rotation of an angle $\frac{2\pi l}{n}$ around the J^3 axis, where J_i, $i = 1, 2, 3$ are the three generators of the group. On the scalar field X, this action simply amounts to the shift $X \mapsto X + 2\pi\sqrt{\alpha'}l/n$. The orbifold partition function reads

$$Z_n \equiv Z(\mathbf{C}_n) = \frac{1}{n}\sum_{k,l=0}^{n-1} Z\begin{bmatrix} g_{k/n} \\ g_{l/n} \end{bmatrix} , \tag{4.2}$$

where

$$Z\begin{bmatrix} g_{k/n} \\ g_{l/n} \end{bmatrix} = \frac{1}{|\eta(q)|^2} \sum_{p\in\mathbf{Z},\ m\in\mathbf{Z}+\frac{k}{n}} e^{\frac{2\pi i l p}{n}} q^{\frac{(p+m)^2}{4}} \bar{q}^{\frac{(p-m)^2}{4}} . \tag{4.3}$$

Actually it is easy to verify that the resulting theory coincides with the free scalar compactified at radius $R^2 = \frac{\alpha'}{n^2}$ and therefore we do not obtain a new model but only particular points along the circle line.

The second series of discrete subgroups is represented by the dihedral groups $\mathbf{D}_n$ and, as it is natural to expect, the resulting orbifolds give points along the orbifold line at radius $R^2 = \frac{\alpha'}{n^2}$. Indeed the $\mathbf{D}_n$ groups can be obtained adding the order two element $e^{i\pi J_1}$ to the $\mathbf{C}_n$ groups and the action of this new element on the scalar field X is precisely the reflection $\mathcal{P} : X \mapsto -X$.

The three exceptional discrete groups correspond to the three symmetry group of the regular polyhedra in $\mathbb{R}^3$: the tetrahedral group $\mathbf{T}$ (the symmetry group of the tetrahedron), the octahedral group $\mathbf{O}$ (the symmetry group of the cube and of the octahedron) and the icosahedral group $\mathbf{I}$ (the symmetry group of the dodecahedron and of the icosahedron). These three groups are non-abelian and give rise to three new isolated points on the $c = 1$ moduli space. The construction of the partition function of these models and of their chiral algebra is an interesting application of the orbifold construction illustrated in Chapter 3 and will be discussed in the next Section.

We now consider the possible existence of multicritical points. At the self-dual point $Z_c(\sqrt{\alpha'})$ there are nine marginal deformation $J_i\bar{J}_j$ with $i, j = 1, 2, 3$ but all of them are equivalent to the radius deformation $J_3\bar{J}_3$ due to the $SU(2) \times SU(2)$ symmetry. The point $R = 2\sqrt{\alpha'}$ along the circle line is a truly multicritical point; there are four marginal deformation, one which generate the usual radius deformation and other three, identified by the $U(1) \times U(1)$ symmetry of the model, which move along the orbifold line. Therefore the circle and the orbifold line intersect each other when $R_c = 2\sqrt{\alpha'}$ and $R_o = \sqrt{\alpha'}$ and in particular $Z_c(2\sqrt{\alpha'}) = Z_o(\sqrt{\alpha'})$. We note that the intersection point is the self-dual point of the orbifold model. Other apparently multicritical points are those at $R = \sqrt{\alpha'}n$, $n \geq 3$ along the circle line [58] since they have three marginal operators; all of them are however equivalent to the radius deformation. Before concluding we recall that this ADE series of orbifolds of the circle at the self-dual point, gives all the *rational* $c = 1$ conformal field theories [59].

Let us now turn to the discussion of the boundary states. Boundary states for the circle and for the orbifold lines have been already discussed in the previous Chapter. Here we complete the construction of the open descendants by considering the possible Klein and Möbius amplitudes. Consider first the circle line at radius $R^2 = \alpha' k$, $k \in \mathbb{N}$ and take the charge-conjugation modular invariant. In this case only two representations can appear in the Klein bottle, those corresponding to the characters χ_0 and χ_k which are self-conjugate and hence appear diagonally in the torus partition function. There are two natural Klein bottle projection: the usual Ω and $\Omega \cdot \delta_2$, where δ_2 is the shift $X \mapsto X + \pi R$

$$\mathcal{K}_\pm = \frac{1}{2}(\chi_0 + (-1)^k \chi_k) \ . \qquad (4.4)$$

Note that the two projections are really distinct only for k odd. Indeed it is instructive to display explicitly the amplitudes for $k = 3$. The Klein bottle $\mathcal{K}_+$ leads to a model with all complex charges except for the two related to χ_0 and χ_3

$$\begin{aligned}
\mathcal{A}_r &= \frac{1}{2}\left[\chi_0\left(n_0^2+n_3^2+2n\overline{n}+2m\overline{m}\right)+\chi_1\left(2n_0n+2m\overline{n}+2n_3\overline{m}\right)\right. \\
&\quad +\chi_2\left(n^2+\overline{m}^2+2n_0m+2n_3\overline{n}\right)+\chi_3\left(2n_0n_3+2nm+2\overline{n}\overline{m}\right) \\
&\quad \left.+\chi_4\left(m^2+\overline{n}^2+2nn_3+2n_0\overline{m}\right)+\chi_5\left(2n_0\overline{n}+2n\overline{m}+2\overline{n}m\right)\right] , \\
\mathcal{M}_r &= \pm\frac{1}{2}\left[\chi_0(n_0-n_3)+\chi_2(-n+m)+\chi_4(-\overline{n}+\overline{m})\right] .
\end{aligned} \tag{4.5}$$

The Klein bottle $\mathcal{K}_-$ leads to a model with only complex charges

$$\begin{aligned}
\mathcal{A}_c &= \frac{1}{2}\left[\chi_0\left(2l\overline{l}+2n\overline{n}+2m\overline{m}\right)+\chi_1\left(2l\overline{m}+2n\overline{l}+m^2+\overline{n}^2\right)\right. \\
&\quad +\chi_2\left(2l\overline{n}+2n_0\overline{m}+2m_3\overline{l}\right)+\chi_3\left(l^2+\overline{l}^2+2n\overline{n}+m\overline{m}\right) \\
&\quad \left.+\chi_4\left(2ln+2m\overline{n}+2\overline{l}\overline{m}\right)+\chi_5\left(2lm+2\overline{l}\overline{n}+n^2+\overline{m}^2\right)\right] , \\
\mathcal{M}_c &= \pm\frac{1}{2}\left[\chi_1(m+\overline{n})+\chi_3(l+\overline{l})+\chi_5(n+\overline{m})\right] .
\end{aligned} \tag{4.6}$$

We can see neatly how to different choices for the Ω projection correspond different boundary conjugation properties. When a geometric interpretation is available, the boundary conjugation properties simply reflect the action of Ω on the boundary states. In the first case Ω maps $r \mapsto -r$ and therefore all the branes sitting at opposite position form conjugate pairs of branes except for the branes sitting at $r = 0$ and $r = k$ that are fixed under Ω. In the second case $\Omega \cdot \delta_2 : r \mapsto -r + k$. For k even it results only in a relabelling of the branes since there are still two fixed points. For k odd there are not fixed points and all the charges are complex. In a string theory context, the Klein bottle $\mathcal{K}_-$ is a simple example of an orientifold that does not require the introduction of an open sector [60]. Indeed in the transverse channel there are no massless tadpoles and imposing the tadpole conditions will force all the boundary multiplicity to zero.

Let us now consider the diagonal modular invariant. In this case all the characters can appear in the Klein bottle and the two projections read

$$\mathcal{K}_r = \sum_{i=0}^{2k-1} \chi_i , \qquad \mathcal{K}_c = \sum_{i=0}^{2k-1} (-1)^i \chi_i . \tag{4.7}$$

The annulus in this case contains only the two boundaries in (3.118), obtained by the orbifold construction. The corresponding amplitudes are

$$\begin{aligned}
\mathcal{A}_r &= \frac{1}{2}\left(n_+^2+n_-^2\right)\sum_{i=0}^{k}\chi_{2i}+n_+n_-\sum_{i=0}^{k-1}\chi_{2i+1} , \\
\mathcal{M}_r &= \pm\frac{1}{2}\left(n_++\epsilon n_-\right)\sum_{i=0}^{k}\chi_{2i} ,
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_r &= n\bar{n}\sum_{i=0}^{k}\chi_{2i}+\frac{1}{2}\left(n^2+\bar{n}^2\right)\sum_{i=0}^{k-1}\chi_{2i+1}\ , \\
\mathcal{M}_r &= \pm\frac{1}{2}\left(n-\bar{n}\right)\sum_{i=0}^{k-1}\chi_{2i+1}\ .
\end{aligned} \tag{4.8}$$

It is easy to verify that these amplitudes are consistent also in the transverse channel.

For the orbifold model we have again various Klein bottles. All the charges for the annulus amplitudes are real except for those related to the twisted fields sitting at the fixed points that are real or complex depending on the choice for the Ω projection. This is exactly the same phenomenon we noticed for the open descendants of the D-series modular invariants for $SU(2)$.

4.2 T – O – I chiral algebras and boundary states

Let us now consider the orbifolds of the $\widehat{SU(2)}_1$ model by the symmetry groups of the regular solids in $\mathbb{R}^3$ [55]. Following the general prescription for the orbifold partition function, we have to sum over sector twisted by representatives of the different conjugacy classes of the group and project each sector under the action of its stabilizer. It is however useful to express the partition function as a sum over mutually commuting subgroups from which all overlappings are removed. In order to do that let us consider the mutually commuting elements of the A_4, S_4 and A_5 groups, easily visualized in terms of their action on the tetrahedron, on the cube and on the dodecahedron. For the tetrahedral group we have 4 $\mathbf{C}_3$ subgroups that acts by rotations around axes through the centers of the faces and a $\mathbf{D}_2$ generated by rotations of π around axes passing through the center of opposite edges. The partition function is then

$$Z(\mathbf{T})=\frac{1}{12}\Big(4\left(3Z_3-Z_1\right)+Z_o(2\sqrt{\alpha'})\Big)\ , \tag{4.9}$$

where the Z_n are defined as in (4.2). For the octahedral group we have 3 $\mathbf{C}_4$ acting about axes through the centers of opposite faces of the cube, 4 $\mathbf{C}_3$ acting about axes through antipodal vertices, one $\mathbf{D}_2$ containing the order two elements of the three $\mathbf{C}_4$ and three $\mathbf{D}_2$ generated by one order two element of the $\mathbf{C}_4$ and a π rotation around axes passing through the center of opposite edges. The resulting partition function is

$$Z(\mathbf{O})=\frac{1}{24}\Big(3(4Z_4-2Z_2)+4(3Z_3-Z_1)+Z_o(2\sqrt{\alpha'})+3(Z_o(2\sqrt{\alpha'})-2Z_2)\Big)\ . \tag{4.10}$$

For the icosahedral group we have 6 $\mathbf{C}_5$ acting about axes through the centers of the faces of the dodecahedron, 10 $\mathbf{C}_3$ acting about axes through antipodal vertices and 5 $\mathbf{D}_2$. The partition function reads

$$Z(\mathbf{I})=\frac{1}{60}\Big(6(5Z_5-Z_1)+10(3Z_3-Z_1)+5(Z_o(2\sqrt{\alpha'})-Z_1)+Z_1\Big)\ . \tag{4.11}$$

There is an interesting relation between the orbifold line at radius $R^2 = 4\alpha'$ (4-state Potts model or $\widehat{SU(2)}_1/\mathbf{D_2}$ model), the tetrahedron and the octahedron, that is due to the following chain of normal subgroups

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \subset \mathbf{T} \subset \mathbf{O} \tag{4.12}$$

with $\mathbf{O}/\mathbf{T} = \mathbb{Z}_2$ and $\mathbf{T}/\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}_3$. Accordingly these models are related among themselves by successive abelian orbifold operations and backward by simple current extensions. Moreover the $\mathbf{O}$ model can be considered as a non-abelian S_3 orbifold of the 4-states Potts model.

The following expression of the Z_n in terms of θ-functions will be useful for deriving the characters for the three Ginsparg points

$$Z_n = \frac{1}{n} \sum_{a=0}^{2n-1} \sum_{b=0}^{n-1} \left| \frac{\Theta\begin{bmatrix} a/2n \\ b/n \end{bmatrix}(q^2)}{\eta(q)} \right|^2 , \tag{4.13}$$

where $\eta(q)$ is the usual η function and the Θ function are now defined as

$$\Theta[{}^a_b]\,(q) = \sum_{m\in\mathbf{Z}} q^{\frac{1}{2}(m-a)^2} e^{-2\pi i m b} \tag{4.14}$$

For convenience we also define

$$\theta[{}^a_b]\,(q) = \frac{\Theta[{}^a_b]\,(q^2)}{\eta(q)} . \tag{4.15}$$

Using (4.13) we can expresses the traces $z\,[{}^g_h]$ in the various sectors of the orbifold in terms of θ functions (4.15); then from (3.85) and from the character tables for $SL_2(\mathbf{Z}_3)$, $GL_2(\mathbf{Z}_3)$ and $SL_2(\mathbf{Z}_5)$, we obtain the characters for the chiral algebra of the three Ginsparg models. We show here the characters for the tetrahedron model. The other cases are listed in Appendix A. The full field content of the tetrahedron model consists of 21 chiral fields [34], [**toi**]: 7 in the untwisted sector, 2 in the $\mathbb{Z}_2$-twisted sector and 6 for each of the two $\mathbb{Z}_3$-twisted sectors. For the untwisted sector we obtain ($i = 0, 1, 2$)

$$\begin{aligned}
u_i &= \tfrac{1}{12}\theta[{}^0_0] + \tfrac{\omega^i}{3}\theta\begin{bmatrix} 0 \\ 1/3 \end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix} 0 \\ 2/3 \end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} , && h = 0, 4, 4 , \\
j &= \tfrac{1}{4}\theta[{}^0_0] - \tfrac{1}{4}\theta\begin{bmatrix} 0 \\ 1/2 \end{bmatrix} , && h = 1 , \\
\phi_i &= \tfrac{1}{6}\theta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} - \tfrac{\omega^{i+2}}{3}\theta\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} - \tfrac{\overline{\omega}^{i+2}}{3}\theta\begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix} , && h = \tfrac{1}{4}, \tfrac{9}{4}, \tfrac{9}{4} ,
\end{aligned} \tag{4.16}$$

where $\omega = \exp(2i\pi/3)$. We can see that the identity representation of $\widehat{SU(2)}_1$ decomposes according to the representation of $A_4 \subset SL_2(\mathbf{Z}_3)$ while the spin one-half representation according to those $SL_2(\mathbf{Z}_3)$ representations that are projective A_4 representations. The two characters in the $\mathbb{Z}_2$-twisted sector

$$\begin{aligned}
\sigma &= \tfrac{1}{2}\theta\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} + \tfrac{1}{2}\theta\begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} , && h = \tfrac{1}{16} , \\
\tau &= \tfrac{1}{2}\theta\begin{bmatrix} 1/4 \\ 0 \end{bmatrix} - \tfrac{1}{2}\theta\begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} , && h = \tfrac{9}{16} ,
\end{aligned} \tag{4.17}$$

clearly reflect the structure of their $\mathbb{Z}_2$ stabilizer, as in the orbifold line. Finally the $\mathbb{Z}_3$-twisted characters ($i = 0, 1, 2$)

$$
\begin{aligned}
\omega_i^+ &= \tfrac{1}{3}\theta\begin{bmatrix}1/3\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/3\\2/3\end{bmatrix}\ , &\qquad h &= \tfrac{1}{9}, \tfrac{4}{9}, \tfrac{16}{9}\ ,\\
\omega_i^- &= \tfrac{1}{3}\theta\begin{bmatrix}2/3\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}2/3\\2/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}2/3\\2/3\end{bmatrix}\ , &\qquad h &= \tfrac{1}{9}, \tfrac{4}{9}, \tfrac{16}{9}\ ,\\
\theta_i^+ &= \tfrac{1}{3}\theta\begin{bmatrix}1/6\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/6\\2/3\end{bmatrix}\ , &\qquad h &= \tfrac{1}{36}, \tfrac{25}{36}, \tfrac{49}{36}\ ,\\
\theta_i^- &= \tfrac{1}{3}\theta\begin{bmatrix}5/6\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}5/6\\1/3\end{bmatrix} + \tfrac{\overline{w}^i}{3}\theta\begin{bmatrix}5/6\\2/3\end{bmatrix}\ , &\qquad h &= \tfrac{1}{36}, \tfrac{25}{36}, \tfrac{49}{36}\ ,
\end{aligned}
\tag{4.18}
$$

organize according to the representations of $\mathbb{Z}_3$. From the explicit form of the characters we can calculate the modular S matrix. The result (multiplied by $12\sqrt{2}$) is

	u_j	j	ϕ_j	σ	τ	ω_j^+	ω_j^-	τ_j^+	τ_j^-
u_i	1	3	2	6	6	$4\omega^i$	$4\overline{\omega}^i$	$4\omega^i$	$4\overline{\omega}^i$
j	3	9	6	-6	-6	0	0	0	0
ϕ_i	2	6	-4	0	0	$-4\omega^i$	$-4\overline{\omega}^i$	$4\omega^i$	$4\overline{\omega}^i$
σ	6	-6	0	$6\sqrt{2}$	$-6\sqrt{2}$	0	0	0	0
τ	6	-6	0	$-6\sqrt{2}$	$6\sqrt{2}$	0	0	0	0
ω_i^+	$4\omega^j$	0	$-4\omega^j$	0	0	$4\overline{\alpha}^2\overline{\omega}^{i+j}$	$4\alpha^2\omega^{i+j}$	$4\alpha\omega^{2i+j}$	$4\overline{\alpha\omega}^{2i+j}$
ω_i^-	$4\overline{\omega}^j$	0	$-4\overline{\omega}^j$	0	0	$4\alpha^2\omega^{i+j}$	$4\overline{\alpha}^2\overline{\omega}^{i+j}$	$4\overline{\alpha\omega}^{2i+j}$	$4\alpha\omega^{2i+j}$
τ_i^+	$4\omega^j$	0	$4\omega^j$	0	0	$4\alpha\omega^{i+2j}$	$4\overline{\alpha\omega}^{i+2j}$	$4\overline{\beta}\omega^{i+j}$	$4\beta\overline{\omega}^{i+j}$
τ_i^-	$4\overline{\omega}^j$	0	$4\overline{\omega}^j$	0	0	$4\overline{\alpha\omega}^{i+2j}$	$4\alpha\omega^{i+2j}$	$4\beta\overline{\omega}^{i+j}$	$4\overline{\beta}\omega^{i+j}$

(4.19)

where $\alpha = \exp(2i\pi/9)$ and $\beta = \exp(i\pi/9)$.

The complex S matrix implies a non-trivial conjugation for this model. Actually there are five self-conjugate fields corresponding to $\{u_0, j, \phi_0, \sigma, \tau\}$ while the fields $\{u_1, \phi_1, \omega_i^+, \theta_i^+\}$ are mapped to $\{u_2, \phi_2, \omega_i^-, \theta_i^-\}$. As a consequence there are both the charge-conjugation and the diagonal modular invariant. The fields u_i form a $\mathbb{Z}_3$ group of simple currents that allow the extension of the tetrahedron to the 4-state Potts model. Furthermore the untwisted field fusion rules coincide with the representation algebra of the group $\mathbf{T}$ as expected from the general discussion in Chapter 3.

The octahedron model contains 28 chiral fields: eight from the untwisted sector $\{u_\pm, u_f, j_\pm, \phi_\pm, \phi_f\}$, two from the $\mathbb{Z}_2$-twisted sector $\{\mu_r\}$, $r = 0, 1$, six from the $\mathbb{Z}_3$-twisted sector $\{\omega_i, \tau_i\}$, $i = 0, 1, 2$, and twelve from the $\mathbb{Z}_4$-twisted sector $\{\alpha_k, \beta_k, \sigma_\pm, \tau_\pm\}$, $k = 0, 1, 2, 3$. This notation reflects the $\mathbb{Z}_2$-orbifold relation with the $\mathbf{T}$ model. Actually the u_f and the ϕ_f characters originate from the identification of the u_1, u_2 and of the ϕ_1, ϕ_2 characters of the tetrahedron whereas $u_\pm$, $j_\pm$, $\phi_\pm$, $\sigma_\pm$ and $\tau_\pm$ arise from the splitting of the corresponding characters in the $\mathbf{T}$ model. To these five fixed points correspond ten new twisted fields and finally the fields in the two $\mathbb{Z}_3$-twisted sectors are pairwise identified, resulting in only one $\mathbb{Z}_3$-twisted sector. The simple current that extends the octahedron model back to the tetrahedron is the chiral field u_-. The octahedron S matrix (multiplied by $24\sqrt{2}$) is

	$u_\pm$	u_f	$j_\pm$	$\phi_\pm$	ϕ_f	μ_s	$\sigma_\pm$	$\tau_\pm$	ω_j	θ_j	α_l	β_l
$u_\pm$	1	2	3	2	4	± 12	6	6	8	8	± 6	± 6
u_f	2	4	6	4	8	0	12	12	-8	-8	0	0
$j_\pm$	3	6	9	6	12	± 12	-6	-6	0	0	∓ 6	∓ 6
$\phi_\pm$	2	4	6	-4	-8	0	0	0	-8	8	$\pm 6\sqrt{2}$	$\mp 6\sqrt{2}$
ϕ_f	4	8	12	-8	-16	0	0	0	8	-8	0	0
μ_r	± 12	0	± 12	0	0	$12\sqrt{2}\epsilon_{rs}$	0	0	0	0	0	0
$\sigma_\pm$	6	12	-6	0	0	0	$6\sqrt{2}$	$-6\sqrt{2}$	0	0	$\pm c_l$	$\pm s_l$
$\tau_\pm$	6	12	-6	0	0	0	$-6\sqrt{2}$	$6\sqrt{2}$	0	0	$\pm s_l$	$\mp c_l$
ω_i	8	-8	0	-8	8	0	0	0	a_{ij}	b_{ij}	0	0
θ_i	8	-8	0	8	-8	0	0	0	b_{ji}	d_{ij}	0	0
α_k	± 6	0	∓ 6	$\pm 6\sqrt{2}$	0	0	$\pm c_k$	$\pm s_k$	0	0	q_{kl}	r_{kl}
β_k	± 6	0	∓ 6	$\mp 6\sqrt{2}$	0	0	$\pm s_k$	$\mp c_k$	0	0	r_{lk}	s_{kl}

(4.20)

where $\epsilon_{rs} = (-1)^{r+s}$, $c_k = (-1)^k 12\cos(\pi/8)$, $s_k = (-1)^k 12\sin(\pi/8)$ and we have introduced the matrices

$$\begin{aligned} a_{ij} &= 16\mathrm{Re}\,(\overline{\alpha}^2\overline{\omega}^{i+j})\ , \qquad b_{ij} = 16\mathrm{Re}\,(\alpha\omega^{2i+j})\ , \qquad d_{ij} = 16\mathrm{Re}\,(\overline{\beta}\omega^{i+j})\ , \\ q_{kl} &= 12\mathrm{Re}\,\left(e^{-\frac{i\pi}{16}} i^{l+k}\right)\ , \quad r_{kl} = 12\mathrm{Re}\,\left(e^{-\frac{3i\pi}{16}} i^{l-k}\right)\ , \\ s_{kl} &= 12\mathrm{Re}\,\left(e^{-\frac{9i\pi}{16}} (-i)^{l+k}\right)\ . \end{aligned} \tag{4.21}$$

In this case the S matrix is real and there is only the diagonal modular invariant.

Finally the icosahedron field content amounts to 37 chiral fields whose characters are listed in Appendix A together with its S matrix. There are nine fields in the untwisted sector $\{u_i, \phi_j\}$, with $i = 0, ..., 4$ and $j = 1, ..., 4$, two fields in the $\mathbb{Z}_2$-twisted sector $\{\sigma, \tau\}$, six in the $\mathbb{Z}_3$-twisted sector $\{\omega_i, \theta_i\}$, $i = 0, 1, 2$ and twenty in the $\mathbb{Z}_5$-twisted sector $\{\pi_k, \rho_k, \lambda_k, \xi_k\}$, $k = 0, ..., 4$. Also in this case the S matrix is real and there is only the diagonal modular invariant. Moreover there are no simple currents between the chiral fields of the icosahedron model, according to the fact that it can not be obtained through a sequence of abelian orbifold operations as the **T** and the **O** models.

In the analysis of these orbifolds one should also allow for the presence of discrete torsion; the different possibilities are classified by $H^2(G, U(1))$ that, in our case, is non-trivial and equal to $\mathbf{Z}_2$ only for $G = \mathbf{D}_{2n}, \mathbf{T}, \mathbf{O}, \mathbf{I}$. Actually one can verifies that at $c = 1$ the models with or without discrete torsion are equivalent. However, as we will see in the sequel, discrete torsion does play a role in the superconformal case.

Having discussed the chiral algebra of the **T**, **O** and **I** orbifold models we can now turn to investigate their boundary states [**toi**]. The boundary coefficients pertaining to the conjugation modular invariant can be read from the respective S matrices according to the Cardy prescription.

We can do the following remarks. First of all we can observe that boundary states corresponding to a given twisted field are uncharged with respect to fields in different twisted sectors. Actually the S matrix vanishes on the corresponding block entries. The untwisted boundary states can again be interpreted as fractional branes. For instance in the tetrahedron the first four states, $|u_i\rangle$, $|j\rangle$, $i=0,1,2$, come from the splitting of the $D0$ brane at the north pole of $\widehat{SU(2)}_1$ while the next three, $|\phi_i\rangle$, come from the $D0$ brane at the south pole. The fractional nature of these branes is confirmed by the number of possible marginal deformations, respectively 0, 2 and 1, reflecting the possible direction of motion, that are lower than the 3 displacements of the original $D0$ brane in $SU(2)$. This can be clearly seen from the corresponding annulus amplitudes

$$\begin{aligned} A_{u_0,u_0} &= u_0 \ , \\ A_{j,j} &= u_0 + u_1 + u_2 + 2\ j \ , \\ A_{\phi_0,\phi_0} &= u_0 + j \ . \end{aligned} \tag{4.22}$$

The boundary states associated to the twisted fields $|\sigma\rangle$, $|\tau\rangle$, $|\omega_i^{\pm}\rangle$ and $|\theta_i^{\pm}\rangle$, $i=0,1,2$, should be interpreted as Neumann-like branes. A similar pattern is found for the Cardy states of the octahedron and icosahedron models.

Let us now discuss the boundary states for the $\mathbf{T}$ diagonal modular invariant. In this case there exist only five Ishibashi states and therefore we expect to find a set of five consistent boundary states. We use the same method illustrated for the diagonal modular invariant of the free boson, namely we obtain them from the octahedron by simple current extension, the relevant order-two simple current being $J=u_-$ which has integer spin. Indeed the $\mathbf{Z}_2$ symmetry $\mathcal{R}$ in $\mathbf{O}=\mathbf{T}/\mathcal{R}$ induces as automorphism of the tetrahedron fusion rules precisely the charge-conjugation. The boundaries for the tetrahedron with conjugation modular invariant that preserve only the $\mathbf{T}/\mathcal{R}$ orbifold subalgebra then coincide with the symmetry preserving boundaries for the tetrahedron with diagonal modular invariant.

Under the action of J, the ten boundary states that correspond to the fields in the $\mathcal{R}$-twisted sector combine in pairs. Explicitly J maps the boundaries $|\mu_0\rangle$, $|\alpha_0\rangle$, $|\alpha_1\rangle$, $|\beta_0\rangle$ and $|\beta_1\rangle$ respectively to $|\mu_1\rangle$, $|\alpha_2\rangle$, $|\alpha_3\rangle$, $|\beta_2\rangle$ and $|\beta_3\rangle$. The boundary coefficients are

$$B_{ai} = \frac{1}{\sqrt{2S_{0i}}}\left(S_{ai} + S_{J(a)i}\right) \ , \tag{4.23}$$

where S_{ij} is the S matrix of the octahedron. The resulting boundary coefficients are non zero only for the following five combinations of octahedron Ishibashi states

$$|u_+\rangle\rangle - |u_-\rangle\rangle \ , \qquad |j_+\rangle\rangle - |j_-\rangle\rangle \ , \qquad |\phi_+\rangle\rangle - |\phi_-\rangle\rangle \ ,$$

$$|\sigma_+\rangle\rangle - |\sigma_-\rangle\rangle \ , \qquad |\tau_+\rangle\rangle - |\tau_-\rangle\rangle \ , \tag{4.24}$$

that are precisely the $\mathbf{T}$ Ishibashi states resulting from the gluing $\Omega\,\mathcal{R}$. In this basis the boundary coefficients are

$$B = \frac{1}{\sqrt{24}} \begin{pmatrix} 6 & 2\sqrt{3} & 6 & \sqrt{6(2+\sqrt{2})} & \sqrt{6(2-\sqrt{2})} \\ 6 & 2\sqrt{3} & 6 & -\sqrt{6(2+\sqrt{2})} & -\sqrt{6(2-\sqrt{2})} \\ 6 & 2\sqrt{3} & -6 & \sqrt{6(2-\sqrt{2})} & -\sqrt{6(2+\sqrt{2})} \\ 6 & 2\sqrt{3} & -6 & -\sqrt{6(2-\sqrt{2})} & \sqrt{6(2+\sqrt{2})} \\ 12 & -4\sqrt{3} & 0 & 0 & 0 \end{pmatrix} \tag{4.25}$$

Moving the automorphism from the choice of gluing Ω to the choice of modular invariant, we can interpret these states as the five symmetry preserving boundaries for the diagonal modular invariant. We have explicitly verified that the annulus amplitudes for the five boundaries in (4.25) are consistent and that the coefficient $A^i_{\alpha\beta}$ give a five dimensional representation of the fusion algebra. We have also verified that the five boundaries in (4.25), interpreted as symmetry breaking boundaries for the tetrahedron with conjugation modular invariant, have consistent overlaps with the usual Cardy states and that the corresponding annulus amplitudes contain, as expected, the twisted characters of the octahedron. For instance two of the $\mathcal{A}_{i,a}$, $a = 1, ..., 5$ are

$$\mathcal{A}_{u_0,1} = \alpha_0 + \alpha_2 \ . \tag{4.26}$$

$$\mathcal{A}_{j,5} = 3\mu_0 + 3\mu_1 \ . \tag{4.27}$$

For all these models one can also complete the construction of the open descendants by adding suitable Klein bottle and Möbius amplitudes.

4.3 $c = 3/2$ superconformal field theories

The moduli space of $c = 3/2$ superconformal field theories has been investigated in [56] using exactly the same method we have illustrated for the $c = 1$ theories.

The simplest realization of superconformal symmetry at $c = 3/2$ is given by a free $N = 1$ superfield, which consists of a free scalar field X and a free Majorana fermion ψ , compactified on a circle of radius R. The partition function for this system is the product of the familiar lattice sum, which counts the momentum and winding modes of the compactified boson, and of the fermion partition function, summed over the different spin structures. The partition function is then

$$Z_c(R) = \sum_{n,m\in\mathbf{Z}} \Gamma_{n,m} \left(|o|^2 + |v|^2 + |s|^2\right) \ . \tag{4.28}$$

The fermion contribution is expressed in terms of the characters of the Ising model

$$\begin{aligned}
o &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}+\sqrt{\frac{\theta_4}{\eta}}\right) ,\\
v &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}-\sqrt{\frac{\theta_4}{\eta}}\right) ,\\
s &= \sqrt{\frac{\theta_2}{2\eta}} ,
\end{aligned} \tag{4.29}$$

whose modular transformation properties are encoded in the matrices

$$S=\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} , \qquad T=e^{-\frac{i\pi}{24}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & e^{\frac{i\pi}{8}} \end{pmatrix} . \tag{4.30}$$

The free superfield compactified on a circle represents a first family of superconformal field theories, parametrized by the radius R. In this model there is one R-R ground state and the Witten index vanishes. The models at radius R is T-dual to the model at radius α'/R as at $c=1$. At the self-dual radius $R=\sqrt{\alpha'}$ there is an $N=3$ superconformal algebra, resulting from the combination of the affine $SU(2)$ symmetry with the $N=1$ superconformal symmetry. For generic values of the circle radius the symmetries are the reflection

$$\mathcal{P}: \qquad X\mapsto -X\ , \qquad \psi\mapsto-\psi\ . \tag{4.31}$$

and the operator $(-1)^F$ introduced in Chapter 3. Furthermore one can identify X modulo translations by integer fractions of the compactification radius

$$\delta_n: \qquad X\sim X+\frac{2\pi R}{n}\ , \qquad \psi\mapsto\psi\ . \tag{4.32}$$

The resulting model is again the circle but at radius $\frac{R}{n}$. New models were obtained by combining δ_2 and the other two involutions. As shown in [56], the relevant cases are $\mathcal{P}$, $(-1)^F\mathcal{P}$ and $(-1)^F\delta_2$.

Orbifolding by $\mathcal{P}$ we obtain the free superfield compactified on an interval of length πR and described by the partition function

$$Z_o(R)=\frac{1}{2}\left(\sum_{n,m\in\mathbf{Z}}\Gamma_{n,m}+\left|\sqrt{\frac{2\eta}{\theta_2}}\right|^2+\left|\sqrt{\frac{2\eta}{\theta_4}}\right|^2+\left|\sqrt{\frac{2\eta}{\theta_3}}\right|^2\right)\left(|o|^2+|v|^2+|s|^2\right)\ . \tag{4.33}$$

We have the same T-duality relation as for the circle $Z_o(R)=Z_c\left(\sqrt{\alpha'}/R\right)$ and at the self-dual radius this line of models meet the circle line, $Z_o(\sqrt{\alpha'})=Z_c(\sqrt{2\alpha'})$. In this case there are three R-R ground states: u_+s, $\sigma_1 o$ and $\sigma_2 o$ and the Witten index is equal to three.

If we use $(-1)^F\mathcal{P}$ instead of $\mathcal{P}$ we obtain a very similar model that indeed is nothing else than the orbifold by $(-1)^F$ of the interval line. This model was called [56] the orbifold-prime and its partition function reads

$$\begin{aligned} Z_{o'}(R) &= \frac{1}{2}\left(\sum_{n,m\in\mathbf{Z}}\Gamma_{n,m}+\left|\sqrt{\frac{2\eta}{\theta_2}}\right|^2\right)\left(|o|^2+|v|^2\right)+\frac{1}{2}\left(\sum_{n,m\in\mathbf{Z}}\Gamma_{n,m}-\left|\sqrt{\frac{2\eta}{\theta_2}}\right|^2\right)|s|^2 \\ &+ \frac{1}{2}\left(\left|\sqrt{\frac{2\eta}{\theta_4}}\right|^2-\left|\sqrt{\frac{2\eta}{\theta_3}}\right|^2\right)(o\overline{v}+v\overline{o})+\frac{1}{2}\left(\left|\sqrt{\frac{2\eta}{\theta_4}}\right|^2+\left|\sqrt{\frac{2\eta}{\theta_3}}\right|^2\right)|s|^2 \,. \end{aligned} \tag{4.34}$$

In terms of the characters at the rational $R=\sqrt{\alpha' k}$ point the partition function can be rewritten as

$$\begin{aligned} Z_{o'}(k) &= \left(|u_+|^2+|u_-|^2+|\phi_+|^2+|\phi_-|^2\right)\left(|o|^2+|v|^2\right) \\ &+ \left(u_+\overline{u}_-+u_-\overline{u}_++\phi_+\overline{\phi}_-+\phi_-\overline{\phi}_+\right)|s|^2+\sum_{r=1}^{k-1}|\chi_r|^2\left(|o|^2+|v|^2+|s|^2\right) \\ &+ \sum_{i=1}^{2}\left[(\sigma_i\overline{\tau}_i+\tau_i\overline{\sigma}_i)(o\overline{v}+v\overline{o})+\left(|\sigma_i|^2+|\tau_i|^2\right)|s|^2\right] \,. \end{aligned} \tag{4.35}$$

From this expression it is clear that there are no R-R ground states since the characters u_+s, $\sigma_1 o$ and $\sigma_2 o$ now appear in off-diagonal combinations.

The twist by $(-1)^F\delta_2$ gives the so-called super-affine line whose partition function is

$$Z_{sa}(R) = \sum_{n,m\in\mathbf{Z}}\Gamma_{2n,m}\left(|o|^2+|v|^2\right)+\sum_{n,m\in\mathbf{Z}}\Gamma_{2n+1,m}|s|^2 \tag{4.36}$$

$$+ \sum_{n,m\in\mathbf{Z}}\Gamma_{2n+1,m+\frac{1}{2}}(o\overline{v}+v\overline{o})+\sum_{n,m\in\mathbf{Z}}\Gamma_{2n,m+\frac{1}{2}}|s|^2 \,. \tag{4.37}$$

At radius $R=\sqrt{2\alpha' m}$ we have a rational model. For m even the partition function is

$$\begin{aligned} Z_{sa}(\sqrt{2\alpha' m}) &= \sum_{r\ even}|\chi_r|^2\left(|o|^2+|v|^2\right)+\sum_{r\ odd}|\chi_r|^2|s|^2 \\ &+ \sum_{r\ even}\chi_r\overline{\chi}_{r-2m}|s|^2+\sum_{r\ odd}\chi_r\overline{\chi}_{r-2m}(o\overline{v}+v\overline{o}) \ , \end{aligned} \tag{4.38}$$

and for m odd it is

$$Z_{sa}(\sqrt{2\alpha' m}) = \sum_r|\chi_r o+\chi_{r-2m}v|^2+\sum_r|(\chi_r+\chi_{r-2m})s|^2 \ , \tag{4.39}$$

where $r=-2k+1,...,2k$. The difference between the two modular invariants is easily understood noticing that they are simple current modular invariant. The simple current is the field $\chi_{2m}v$ of the circle theory whose conformal dimension is $h=\frac{m}{2}$ and hence we have an automorphism or an extension modular invariant according to if m is even or odd.

These models display the duality $Z_{sa}(R)=Z_{sa}\left(\frac{\sqrt{2\alpha'}}{R}\right)$. At the self-dual point $R=\sqrt{2\alpha'}$ there is a super-affine $SO(3)\times SO(3)$ symmetry and the partition function can be rewritten in terms of the $\widehat{SO(3)}_1$ characters

$$
\begin{aligned}
O_3 &= \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^{\frac{3}{2}}+\left(\frac{\theta_4}{\eta}\right)^{\frac{3}{2}}\right] , \\
V_3 &= \frac{1}{2}\left[\left(\frac{\theta_3}{\eta}\right)^{\frac{3}{2}}-\left(\frac{\theta_4}{\eta}\right)^{\frac{3}{2}}\right] , \\
S_3 &= \frac{1}{\sqrt{2}}\left(\frac{\theta_2}{\eta}\right)^{\frac{3}{2}} .
\end{aligned}
\tag{4.40}
$$

Along this line there are not R-R ground states. The super-affine and the orbifold-prime line cross each other and at the intersection point their partition functions coincide: $Z_{sa}(\sqrt{4\alpha'})=Z_{o'}(\sqrt{\alpha'})$.

The fifth and final family of theories is obtained orbifolding the super-affine line by $\mathcal{P}$ and is called the super-orbifold line. The resulting models can also be considered as $\mathbf{Z}_2\times\mathbf{Z}_2$ orbifolds of the circle where the group is generated by $(-1)^F\delta_2$ and $\mathcal{P}$. Let us present the partition function directly at the rational points $R=\sqrt{2\alpha' m}$. They are again a simple current modular invariants, the relevant simple current being $\phi_+ v$ with conformal weight $h=(m+1)/2$. As for the super-affine line we have to distinguish between even m that gives an automorphism modular invariant

$$
\begin{aligned}
Z_{so}(\sqrt{2\alpha' m}) &= \left(|u_+|^2+|u_-|^2+|\phi_+|^2+|\phi_-|^2\right)\left(|o|^2+|v|^2\right) \\
&+ \left(u_+\overline{\phi}_+ +\phi_+\overline{u}_+ + u_-\overline{\phi}_- +\phi_-\overline{u}_-\right)|s|^2+\sum_{r=1}^{m-1}|\chi_{2r}|^2\left(|o|^2+|v|^2\right) \\
&+ \sum_{r=1}^{m}|\chi_{2r-1}|^2|s|^2+\sum_{r=1}^{m-1}\chi_{2r}\overline{\chi}_{2r+2m}|s|^2+\sum_{r=1}^{m}\chi_{2r-1}\overline{\chi}_{2r-1+2m}\left(o\overline{v}+v\overline{o}\right) \\
&+ \left(|\sigma_1|^2+|\tau_1|^2\right)\left(|o|^2+|v|^2+|s|^2\right) \\
&+ \left(\sigma_2\overline{\tau}_2+\tau_2\overline{\sigma}_2\right)\left(o\overline{v}+v\overline{o}\right)+\left(|\sigma_2|^2+|\tau_2|^2\right)|s|^2 ,
\end{aligned}
\tag{4.41}
$$

and odd m that gives an extension modular invariant

$$
\begin{aligned}
Z_{so}(\sqrt{2\alpha' m}) &= |u_+o+\phi_+v|^2+|\phi_+o+u_+v|^2+|u_-o+\phi_-v|^2+|\phi_-o+u_-v|^2 \\
&+ \sum_{r=1}^{m-1}|\chi_{2r}o+\chi_{2m-2r}v|^2+\sum_{r=1}^{\frac{m-1}{2}}|\chi_{2r-1}+\chi_{2m-2r+1}|^2|s|^2+2|\chi_m s|^2
\end{aligned}
$$

$$+ \quad |\sigma_2 o + \tau_2 v|^2 + |\tau_2 o + \sigma_2 v|^2 + 2(|\sigma_1|^2 + |\tau_1|^2)|s|^2 \ . \tag{4.42}$$

Along this line there is one R-R ground state and the Witten index is one. The super-orbifold line crosses both the orbifold-prime and the circle line. Indeed $Z_{so}(\sqrt{4\alpha'}) = Z_{o'}(\sqrt{4\alpha'})$ and $Z_{so}(\sqrt{2\alpha'}) = Z_c(\sqrt{2\alpha'})$.

The resulting picture is most easily understood from the point of view of the chiral algebras underlying the various family of models. There are two different chiral algebras: the direct product of the Ising model algebra with the $\widehat{U(1)}_k$ or the $\widehat{U(1)}_k/\mathbb{Z}_2$ algebra. The first one describes the circle and super-affine lines, the second applies to the orbifold, the orbifold-prime and the super-orbifold lines. Different lines for the same chiral algebra correspond to simple current modular invariants generated by simple currents of order two. The super-affine line is related to the circle line by $\chi_{2m} v$ whereas the orbifold-prime and the super-orbifold are related to the orbifold line by $u_- v$ and $\phi_+ v$, respectively.

Let us now consider orbifolds by discrete groups that exist only at particular points on the moduli space. One natural possibility is represented by the self-dual point along the circle line. Modding out $Z_c(\sqrt{\alpha'})$ by discrete subgroups of $SU(2)$ we obtain an ADE series of models with a twisted $N = 3$ superconformal algebra. Their partition functions are simply the product of the ADE partition functions discussed at $c = 1$ with the partition function of the free fermion. New $N = 1$ superconformal models can be constructed as ADE orbifolds of the super-affine line at $R = \sqrt{2\alpha'}$, where it display an affine $\widehat{SO(3)}_1$ symmetry. Taking the quotient by discrete subgroups of $SO(3)$ the superconformal symmetry is not broken since the supercurrent $G = -\frac{1}{12}\epsilon_{ijk}\psi^i\psi^j\psi^k$ is manifestly left invariant. $\mathcal{C}_n$-orbifolds result in points along the super-affine or along the circle line

$$Z(\mathbf{C}_{2n+1}) = Z_{sa}(2n+1) \ , \qquad Z(\mathbf{C}_{2n}) = Z_c(2n) \ . \tag{4.43}$$

where we have fixed $2\alpha' = 1$. In a similar way we see that

$$\begin{aligned} Z(\mathbf{D}_{2n+1}) &= Z_{so}(2n+1) \ , \\ Z(\mathbf{D}_{2n}) &= Z_o(2n) \ , \\ Z(\mathbf{D}'_{2n}) &= Z_{o'}(2n) \ . \end{aligned} \tag{4.44}$$

We obtain two different results for the $\mathbf{D}_{2n}$ orbifolds since for these groups there is the possibility of introducing a $\mathbb{Z}_2$ discrete torsion. Models with discrete torsion are distinguished by a prime. Finally there are the isolated points obtained, as in the conformal case, twisting by A_4, S_4 and A_5. Again, due to the possible presence of discrete torsion, we obtain in total six points. We postpone the discussion of their chiral algebra and boundary states to the next Section and turn now to the boundary states for the five lines of models we have just illustrated.

The boundary states for the circle line are easily obtained as combinations of the boundary states for the compactified scalar and the Majorana fermion. Restricting ourselves to the rational case $R^2 = \alpha' k$, we have

$$\begin{aligned}
|r,o\rangle_c &= \frac{1}{(8k)^{\frac{1}{4}}} \sum_{l=0}^{2k-1} e^{-\frac{i\pi rl}{k}} \left(|r,o\rangle\rangle + |r,v\rangle\rangle + \sqrt{2}|r,s\rangle\rangle\right) \ , \\
|r,v\rangle_c &= \frac{1}{(8k)^{\frac{1}{4}}} \sum_{l=0}^{2k-1} e^{-\frac{i\pi rl}{k}} \left(|r,o\rangle\rangle + |r,v\rangle\rangle - \sqrt{2}|r,s\rangle\rangle\right) \ , \\
|r,s\rangle_c &= \frac{1}{(2k)^{\frac{1}{4}}} \sum_{l=0}^{2k-1} e^{-\frac{i\pi rl}{k}} \left(|r,o\rangle\rangle - |r,v\rangle\rangle\right) \ ,
\end{aligned} \tag{4.45}$$

where $|r,o\rangle\rangle$, $|r,v\rangle\rangle$ and $|r,s\rangle\rangle$, are the products of Ishibashi states for the rational scalar and the Ising model. The three types of boundary states in (4.45) can be considered as positively charged, negatively charged and uncharged boundary states respectively, according to the value of the one point function of fields in the R-R sector in front of them. In total we have $6k$ boundary states.

The number of boundary states one expects for the super-affine line at $R^2 = \alpha' k$ with $k = 2m$ is $3m$. They can be obtained acting on the boundaries of the circle with the operation $(-1)^F \delta_2$ or equivalently with the simple current $J = \chi_{2m} v$, that is freely acting. Therefore, there are neither fixed points nor fractional branes and the invariant boundary states are made of pairs, as follows

$$\begin{aligned}
|r,o\rangle_{sa} &= \frac{1}{\sqrt{2}} \left(|r,o\rangle_c + |r+k,v\rangle_c\right) \ , \qquad r = 0, ..., 2k-1 \\
|r,s\rangle_{sa} &= \frac{1}{\sqrt{2}} \left(|r,s\rangle_c + |r+k,s\rangle_c\right) \ , \qquad r = 0, ..., k-1 \ .
\end{aligned} \tag{4.46}$$

For m odd the bulk modular invariant is of extension type and the boundaries $|r,o\rangle_{sa}$ with r odd are breaking the extended symmetry.

The Cardy boundary states of the orbifold line are again products of boundary states for the bosonic orbifold with states for the Ising model, for a total of $3(k+7)$ boundary states. They are labelled by the corresponding fields: in the untwisted NS sector, there are $u_\pm I, \phi_\pm I, \chi_r I,\ I = o, v$; in the untwisted R sector, $u_\pm s, \phi_\pm s, \chi_r s$; in the twisted NS sector, $\sigma_i s, \tau_i s,\ i = 0, 1$; finally, in the twisted R sector, $\sigma_i I, \tau_i I$, $I = o, v$. We can distinguish them according to their R-R charges, taking into account that in this case both untwisted and twisted charges appear. For instance, the states $|u_\pm I\rangle$, have non-vanishing both untwisted and twisted charges, while the states $|u_\pm s\rangle$ carry only twisted charges.

Starting from this set of boundary states and acting with the simple current $J = \phi_+ v$ we can obtain the boundary states for the super-orbifold line. From the partition function (4.41,4.42), we count $(3m+15)$ Ishibashi states at radius $R^2 = 2\alpha' m$. Under the action of the simple current, $6(m+3)$ boundaries are combined in pairs, while 3 of them are fixed, leading to the required $3m+15$ boundary states. In order to construct these boundary states we must know the orbits of the simple current: the three fixed points are $\chi_m s$, $\sigma_1 s$ and $\tau_1 s$; the representations

$$u_\pm o, \quad \phi_\pm o, \quad \chi_r o, \quad \sigma_1 o, \quad \tau_1 o, \quad \sigma_2 o, \quad \tau_2 o \ , \qquad r = 1, \ldots, 2m-1 \ , \tag{4.47}$$

are respectively paired with the representations,

$$\phi_\pm v, \quad u_\pm v, \quad \chi_{2m-r} v, \quad \sigma_1 v, \quad \tau_1 v, \quad \tau_2 v, \quad \sigma_2 v \ , \tag{4.48}$$

while the representations

$$u_\pm s, \quad \chi_r s, \quad \sigma_2 s \ , \qquad r = 1, ..., m-1 \ , \tag{4.49}$$

are paired with

$$\phi_\pm s, \quad \chi_{2m-r} s, \quad \tau_2 s \ . \tag{4.50}$$

Again the boundary states of the super-orbifold follow the general pattern described in [42] for pure automorphism modular invariant generated by a $\mathbb{Z}_2$ current of half-integer spin. In particular we have a set of $3m+9$ invariant boundaries in one-to-one correspondence with the length-two orbits of the simple current, as in Equation (3.129). In addition there are six fractional boundary states, two for each of the fixed points of the simple current. These boundary states have the form displayed in (3.130) with the fixed-point $\widetilde{S}$-matrix equal to that of the Ising model (4.30) in the basis $\{|\sigma_1 s\rangle\rangle, |\tau_1 s\rangle\rangle, |\chi_m s\rangle\rangle\}$.

Finally the boundary states for the orbifold-prime line can be obtained acting with the simple current $J = u_- v$ on the orbifold states. All the states labelled by NS fields are paired and give $(k+5)$ boundary states, while 12 states labelled by Ramond fields are paired and $(k-1)$ are fixed. The total number of boundaries is $(3k+9)$, in agreement with the number of Ishibashi states (see Eq.(4.35)). The $(k-1)$ fixed boundary states

$$\begin{aligned} |r, s\rangle_{o'} &= \frac{2}{(2k)^{1/4}} \Big[|u_+\rangle\rangle + |u_-\rangle\rangle + (-1)^r |\phi_+\rangle\rangle + (-1)^r |\phi_-\rangle\rangle \\ &+ \sum_{l=1}^{k-1} \sqrt{2} \cos\left(\frac{\pi r l}{k}\right) |l\rangle\rangle \Big] \left(|o\rangle\rangle - |v\rangle\rangle\right) \ , \qquad r = 1, ..., k-1 \end{aligned} \tag{4.51}$$

give rise to new boundaries that differ by their charges with respect to the $(k-1)$ RR fields $\chi_r s$, $r = 1, ..., k-1$

$$|r, s, \pm\rangle_{o'} = \frac{1}{(2k)^{1/4}} \Big[|u_+\rangle\rangle + |u_-\rangle\rangle + (-1)^r |\phi_+\rangle\rangle + (-1)^r |\phi_-\rangle\rangle$$

$$
\begin{aligned}
&+ \sum_{l=1}^{k-1} \sqrt{2}\cos\left(\frac{\pi r l}{k}\right)|l\rangle\!\rangle \Bigg] (|o\rangle\!\rangle - |v\rangle\!\rangle) \\
&\pm \frac{1}{(2k)^{1/4}} \sum_{l=1}^{k-1} 2\sin\left(\frac{\pi r l}{k}\right)|l\rangle\!\rangle |s\rangle\!\rangle \ , \qquad r = 1, ..., k-1 \ . \qquad (4.52)
\end{aligned}
$$

These states give consistent annulus amplitudes.

4.4 Superconformal T-O-I models and boundary states

In this Section we find the chiral algebras for the superconformal **T**-**O**-**I** models and then discuss their boundary states [**toi**]. The derivation strictly parallels that of the three $c = 1$ models and will be sketched in the following. We start from the self-dual point on the super-affine line and mod it by the symmetry groups A_4, S_4 and A_5. Taking into account the discrete torsion, one obtains three further models, $\mathbf{T}'$, $\mathbf{O}'$, $\mathbf{I}'$; they can also be realized as $(-1)^F$ orbifolds of the torsionless **T**, **O**, **I** models, in analogy with the relation between orbifold and orbifold-prime lines. The two triples notably differ in the number of RR ground states.

The **T** model can also be realized as a $\mathbb{Z}_3$ orbifold of the theory made by the product of three Ising models, that is found at $R = \sqrt{2\alpha'}$ on the orbifold line. From the chain of inclusions (4.12), it is then clear that **O** can be obtained as a permutation orbifold of the triple Ising theory, *i.e.* as a non-abelian $\mathbf{S}_3$ orbifold: $(\text{Ising})^{\otimes 3}/\mathbf{Z}_3 = \mathbf{T}$ and $(\text{Ising})^{\otimes 3}/S_3 = \mathbf{O}$. General expressions for permutation orbifolds have been derived in [61], and our findings agree with them.

The characters and S matrices for the isolated superconformal models are reported in Appendix A; they are obtained as in the bosonic case (4.9-4.11), by first expanding the partition functions in terms pertaining to the mutually commuting subgroups of $\mathbf{T}-\mathbf{O}-\mathbf{I}$ (4.43,4.44), and then by expressing the latter in terms of Θ functions, using the identities for $R^2 = 2\alpha' n$:

$$
\begin{aligned}
\sum_{p,w\in\mathbf{Z}} \frac{1\pm(-1)^p}{2}\Gamma_{p,w} &= \frac{1}{n|\eta(q)|^2}\sum_{r=0}^{2n-1}\sum_{s=0}^{2n-1}\frac{1\pm(-1)^r}{2}\left|\Theta\begin{bmatrix} r/2n \\ s/2n \end{bmatrix}(q)\right|^2 \ , \\
\sum_{p,w\in\mathbf{Z}} \frac{1\pm(-1)^p}{2}\Gamma_{p,w+\frac{1}{2}} &= \frac{1}{n|\eta(q)|^2}\sum_{r=0}^{2n-1}\sum_{s=0}^{2n-1}\frac{1\pm(-1)^r}{2}(-1)^s\left|\Theta\begin{bmatrix} r/2n \\ s/2n \end{bmatrix}(q)\right|^2 \ . \qquad (4.53)
\end{aligned}
$$

The field content of the superconformal **T** model consists of 35 fields: 22 of them belong to the NS sector and 13 to the R sector. There are several possible modular invariants: first of all the S matrix is complex and therefore the conjugation and diagonal invariants are distinct. Moreover, starting from the charge-conjugation modular invariant $Z(\mathbf{T}_c)$, we can obtain other four modular invariants using two $\mathbb{Z}_2$ operations. The first is the orbifold by $(-1)^F$: this is obtained through the action of the simple current ξ_0, that is the primary field containing the supercurrent, as usual. The second is the exchange $\sigma_v \leftrightarrow \tau_o$, an automorphism of the fusion rules. The result is:

$$\begin{aligned}
Z(\mathbf{T}_c) &= \left(|\sigma_o|^2 + |\sigma_v|^2 + |\tau_o|^2 + |\tau_v|^2\right) + \sum_i \chi_i \overline{\chi}_{i^*} \ , \\
Z(\mathbf{T}_{ca}) &= \left(|\sigma_o|^2 + \sigma_v \overline{\tau_o} + \tau_o \overline{\sigma_v} + |\tau_v|^2\right) + \sum_i \chi_i \overline{\chi}_{i^*} \ , \\
Z(\mathbf{T}'_c) &= \left(\sigma_o \overline{\tau_v} + \tau_v \overline{\sigma_o} + \sigma_v \overline{\tau_o} + \tau_o \overline{\sigma_v}\right) + \sum_i \chi_i \overline{\chi}_{i^*} \ , \\
Z(\mathbf{T}'_{ca}) &= \left(\sigma_o \overline{\tau_v} + \tau_v \overline{\sigma_o} + |\sigma_v|^2 + |\tau_o|^2\right) + \sum_i \chi_i \overline{\chi}_{i^*} \ .
\end{aligned} \tag{4.54}$$

In these expressions, the index i runs over all the fields not explicitly present in the first parenthesis, the prime indicates the presence of discrete torsion and the subscript a stands for the automorphism. The diagonal modular invariant $Z(\mathbf{T}_d)$ similarly generates three further partition functions, that are denoted by $Z(\mathbf{T}_{da})$, $Z(\mathbf{T}'_d)$ and $Z(\mathbf{T}'_{da})$; they differ from the expressions (4.54) by the substitution $\sum_i \chi_i \overline{\chi}_{i^c} \rightarrow \sum_i \chi_i \overline{\chi}_i$.

The boundary states for the model $Z(\mathbf{T}'_c)$ with discrete torsion are given by the action of the simple current ξ_0 on the Cardy boundaries of the $Z(\mathbf{T}_c)$ model. The expected number of 31 boundary states is reproduced, because the simple current ξ_0 acting on the tetrahedron chiral fields form 13 length-two orbits and 9 fixed points.

The boundary states for the $(\mathbf{T}_{da})$ model can be obtained by using the simple-current extension from the $\mathbf{O}$ model, as already done in the bosonic case; this will be discussed further below. The $(\mathbf{T}'_{da})$ model can also be analyzed by combining the previous two approaches.

The boundaries for the other four cases $Z(\mathbf{T}_{ca})$, $Z(\mathbf{T}'_{ca})$, $Z(\mathbf{T}_d)$ and $Z(\mathbf{T}'_d)$, are not obviously related to the action of a simple current.

We now turn to the discussion of the supersymmetric octahedron model. This possesses 49 primary fields, 30 belonging to the NS sector and 19 to the R sector, that are all self-conjugate. In this model, there also exist several modular invariants. Indeed its simple-current group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ and comprises the three currents (u_-, v_+, v_-). The first one has integer spin and give the extension of the octahedron to the tetrahedron; the second has half-integer spin and contains the supercurrent: the corresponding modular invariant coincide with the orbifold by $(-1)^F$, namely $\mathbf{O}'$. Finally, the current v_- is another half-integer spin simple current that gives a new automorphism modular invariant, that is named $Z(\widetilde{\mathbf{O}})$. The expressions of these modular invariants are, besides the diagonal one:

$$Z(\mathbf{O}') = (\mu_{0v}\overline{\mu_{1o}} + \mu_{0o}\overline{\mu_{1v}} + \sigma_{o+}\overline{\tau}_{v+} + \sigma_{o-}\overline{\tau}_{v-} + c.c.) + \sum_i |\chi_i|^2 \tag{4.55}$$

$$Z(\widetilde{\mathbf{O}}) = \big(\alpha_0\overline{\beta}_2 + \alpha_1\overline{\beta}_3 + \alpha_2\overline{\beta}_0 + \alpha_3\overline{\beta}_1 + \rho_+\overline{\rho}_- + \sigma_{o+}\overline{\tau}_{v+} + \sigma_{o-}\overline{\tau}_{v-} + c.c.\big) + \sum_i |\chi_i|^2 \tag{4.56}$$

$$Z(\widetilde{\mathbf{O}}') = (\alpha_0\overline{\beta}_2 + \alpha_1\overline{\beta}_3 + \alpha_2\overline{\beta}_0 + \alpha_3\overline{\beta}_1 + \rho_+\overline{\rho}_- + \mu_{0v}\overline{\mu}_{1o} + \mu_{0o}\overline{\mu}_{1v} + \sigma_{o+}\overline{\tau}_{v+} + \sigma_{o-}\overline{\tau}_{v-} + c.c.) + \sum_i |\chi_i|^2 \tag{4.57}$$

Let us discuss them in turn.

$Z(\mathbf{O}')$. In this model there are 41 Ishibashi states and the simple current v_+ has precisely 19 length-two orbits and 11 fixed points, corresponding to the chiral fields $\rho_\pm$, ρ, osv, Λ_i, and γ_i.

$Z(\widetilde{\mathbf{O}})$. There are 35 Ishibashi states: under the action of v_-, the octahedron fields form 21 orbits of length two and seven fixed points, that split developing charges with respect to the seven twisted characters ρ, osv, Λ_i, $i = 0, 1, 2$, and $\mu_{\alpha s}$, $\alpha = 0, 1$. The boundary coefficients R_{ai} of the resulting 14 boundaries, restricted to the twisted characters (see Eqs.(3.130,3.131), are

$$\begin{array}{c|cccc} & \rho & osv & \Lambda_j & \mu_{\beta s} \\ \hline |\rho,\pm\rangle & 0 & 0 & \pm 16\sqrt{3} & 0 \\ |osv,\pm\rangle & 0 & 0 & 0 & \pm 24\sqrt{2}(-1)^\beta \\ |\Lambda_i,\pm\rangle & \pm 16\sqrt{3} & 0 & \pm 32 s_{ij} & 0 \\ |\mu_{\alpha s},\pm\rangle & 0 & \pm 24\sqrt{2}(-1)^\alpha & 0 & \pm 24 \end{array} \tag{4.58}$$

where $s_{ij} = \mathrm{Im}\ (e^{2i\pi/9}\ \omega^{i+j})$, and $\omega = \exp(2i\pi/3)$.

$Z(\widetilde{\mathbf{O}}')$. There are 31 Ishibashi states and the corresponding boundary states can be obtained acting on the boundaries of the $\mathbf{O}$ model with the full simple-current group $\mathbf{Z}_2 \times \mathbf{Z}_2$.

The boundary states for the $(\mathbf{T}_{da})$ model are obtained from the $\mathbf{O}$ model, as follows. The $\mathbb{Z}_2$ symmetry $\mathcal{R}$ which relates the tetrahedron and the octahedron model according to $\mathbf{T}/\mathcal{R} = \mathbf{O}$, induces as automorphism of the fusion rules the charge-conjugation $(*)$ composed with the exchange $\sigma_v \leftrightarrow \tau_o$. The 9 boundary states for the $Z(\mathbf{T}_{da})$ modular invariant are obtained from the 18 boundary states of the octahedron corresponding to the $\mathcal{R}$-twisted sector, namely $\{\alpha_k, \beta_k, \gamma_k, \mu_{\alpha I}\}$, with $k = 0, \ldots, 3$, $\alpha = 0, 1$ and $I = o, v, s$. They are coupled by the simple current u_- as follows

$$(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1, \mu_{0s}, \mu_{0o}, \mu_{0o}) \mapsto (\alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_2, \gamma_3, \mu_{1s}, \mu_{1o}, \mu_{1o}). \tag{4.59}$$

Extending the $\mathcal{R}$-twisted sectors of the octahedron, we then obtain the boundary states for the $Z(\mathbf{T}_{da})$ model. In a similar way one can obtain the boundary states for the $Z(\mathbf{T}'_{da})$ model.

Chapter 5

Brane supersymmetry and brane supersymmetry breaking

In this Chapter we apply the method of the open descendants [7] to the study of total and partial supersymmetry breaking in Type I models.

When supersymmetry is broken, one loses the control of the string theory perturbative expansion and the stability of the vacuum is no more guaranteed. Actually the spectrum of non-supersymmetric string models usually contains a tachyon, as it is the case for the Type 0 models discussed in Chapter 2.

One of the first mechanism proposed for supersymmetry breaking in string theory [62] is through gaugino condensation in a hidden sector. This mechanism involves the strong coupling dynamics of the low-energy action and therefore does not have a perturbative worldsheet description. A mechanism for supersymmetry breaking that can be described in terms of the conformal field theory on the worldsheet is the Scherk-Schwarz mechanism [63]. Scherk-Schwarz deformations are generalized dimensional reductions where the fields are periodic along the compact directions up to symmetry transformations. In order to induce the breaking of supersymmetry one must allow for a different behaviour of fermionic and bosonic fields in the internal space. The Scherk-Schwarz mechanism was extended to closed string models in [64] and, as in the field theory case, the breaking scale was shown to be tied to the compactification scale. Moreover in [65], Scherk-Schwarz deformations were shown to be equivalent to freely-acting orbifolds construction.

The study of Scherk-Schwarz deformations in open string models was initiated in [66] where it was pointed out that the presence of D-branes in the vacuum gives rise to a new phenomenon. Branes with world-volume parallel to the breaking direction feel supersymmetry breaking at tree level and the mass splittings are still proportional to the compactification scale. The massless spectrum of branes orthogonal to the breaking direction remains however supersymmetric at tree level and the breaking of supersymmetry is induced through radiative corrections from the brane massive states or from the gravitational sector. This phenomenon is called *brane supersymmetry* [67] and could have interesting applications in phenomenological models with a low string scale [68] or in *brane world* models [69].

In open string models however one has at his disposal another interesting possibility. As we emphasized in Chapter 2, RR tadpole cancellation conditions may be regarded as global neutrality conditions for the RR charges. For some chiral four-dimensional model it is apparently impossible to satisfy all the RR tadpole conditions. Two typical examples of Type IIB orbifolds where the problem of unsolvable tadpoles was first encountered [70, 71] are the $T^6/Z_2 \times Z_2$ orbifold with discrete torsion and the T^6/Z_4 orbifold. However if we do not require that the brane configuration be supersymmetric, solutions actually can be found. In the resulting constructions supersymmetry is broken at the string scale on some collection of branes while it is preserved at lowest order in the bulk and possibly on other branes. This mechanism was introduced in [72] and called *brane supersymmetry breaking.* Similar constructions were considered in [73, 74]. The non-vanishing NS-NS tadpoles, generically present in this class of models, signal that flat space-time is no more a solution to the equations of motion. It would be very interesting to understand how the background is altered by the breaking of supersymmetry. Some result in this direction have been reported in [75]. Another interesting feature of these models is that supersymmetry is non-linearly realized on the non-supersymmetric branes and this is crucial for the consistent coupling of the non-supersymmetric gauge theory on the branes to the bulk supergravity theory [76].

Models with brane supersymmetry breaking represent just a first step in the attempt of understanding string theory vacua without supersymmetry. Since supersymmetry is broken only in a localized region of spacetime one can hope to have a better control of the system than in models where supersymmetry is broken from the start in the whole spacetime. From this point of view they are similar to recent study of tachyon condensation in unstable brane systems [77] or at orbifold singularities [78].

Finally we mention that supersymmetry breaking in open string model can be induced also by a magnetic field on the brane, since particles of different spin couple differently to the magnetic field and then acquire different masses [79].

In this Chapter we first illustrate some general feature of the $T^6/Z_2 \times Z_2$ orbifold, since it will be at the center of our discussion of both brane supersymmetry and brane supersymmetry breaking. Then we study a rich class of Scherk-Schwarz deformations given by $Z_2 \times Z_2$ freely-acting orbifolds [**shift**], where the conventional projections are accompanied by Z_2-valued momentum or winding lattice shifts. Their open descendants exhibit $N = 4 \to N = 1$ or $N = 2 \to N = 1$ partial supersymmetry breaking and involve interesting D-brane configurations. We describe in details one of this model and briefly comment on the main features of the others. In particular we discuss a model where all massive brane modes, not only the massless ones have extended supersymmetry. We note that branes in these shift orbifolds can be considered as multiplets of branes of the original $Z_2 \times Z_2$ model. This is the same phenomenon we met during the study of boundary states for general modular invariants in the third Chapter and it is due to the combination of the original boundary states in invariant configurations. In our case the branes can arrange themselves either in doublets or in quadruplets of branes. We then illustrate how to describe systems of branes and anti-branes and finally present two examples, the $T^6/Z_2 \times Z_2$ orientifold with discrete torsion and the T^6/Z_4 orientifold, of chiral four-dimensional models for which the brane content required for consistency by the closed parent model is non-supersymmetric [**bsb**].

5.1 $Z_2 \times Z_2$ orientifolds and discrete torsion

Let us consider the type IIB string theory compactified on $T^6/Z_2 \times Z_2$. Aside from the identity, that we denote by o, the three non-trivial elements contained in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group are denoted by g, f and h. If we decompose T^6 in the product of three internal two-tori T_{45}, T_{67} and T_{89}, and label a generic point with three complex coordinates (z_1, z_2, z_3), the three orbifold operations act as

$$g : (+, -, -) \quad , \qquad f : (-, +, -) \quad , \qquad h : (-, -, +) \quad . \tag{5.1}$$

The $Z_2 \times Z_2$ orbifold, is the simplest case where one has the freedom of introducing discrete torsion [33], that corresponds to associating a sign $\epsilon = \pm 1$ to the independent modular orbit containing all terms that are twisted and projected by two different orbifold operations. Omitting for brevity the contributions of the space-time bosons, the torus amplitudes of the two $Z_2 \times Z_2$ models are

$$\begin{aligned}
\mathcal{T} = \frac{1}{4}\Big\{ & |T_{oo}|^2 \Lambda_1 \Lambda_2 \Lambda_3 + |T_{og}|^2 \Lambda_1 \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 + |T_{of}|^2 \Lambda_2 \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 + |T_{oh}|^2 \Lambda_3 \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 \\
+ \quad & |T_{go}|^2 \Lambda_1 \left| \frac{4\eta^2}{\vartheta_4^2} \right|^2 + |T_{gg}|^2 \Lambda_1 \left| \frac{4\eta^2}{\vartheta_3^2} \right|^2 + |T_{fo}|^2 \Lambda_2 \left| \frac{4\eta^2}{\vartheta_4^2} \right|^2 + |T_{ff}|^2 \Lambda_2 \left| \frac{4\eta^2}{\vartheta_3^2} \right|^2 \\
+ \quad & |T_{ho}|^2 \Lambda_3 \left| \frac{4\eta^2}{\vartheta_4^2} \right|^2 + |T_{hh}|^2 \Lambda_3 \left| \frac{4\eta^2}{\vartheta_3^2} \right|^2 \\
+ \quad & \epsilon \left(|T_{gh}|^2 + |T_{gf}|^2 + |T_{fg}|^2 + |T_{fh}|^2 + |T_{hg}|^2 + |T_{hf}|^2 \right) \left| \frac{8\eta^3}{\vartheta_2 \vartheta_3 \vartheta_4} \right|^2 \Big\} \quad ,
\end{aligned} \tag{5.2}$$

where the Λ_k are lattice sums for the three internal tori and we have set $\alpha' = 2$. The choice $\epsilon = 1$ defines the model without discrete torsion, while the choice $\epsilon = -1$

defines the model with discrete torsion. We have expressed the torus amplitude in terms of the 16 quantities ($k = o, g, h, f$)

$$
\begin{aligned}
T_{ko} &= \tau_{ko} + \tau_{kg} + \tau_{kh} + \tau_{kf} \quad , \qquad T_{kg} = \tau_{ko} + \tau_{kg} - \tau_{kh} - \tau_{kf} \quad , \\
T_{kh} &= \tau_{ko} - \tau_{kg} + \tau_{kh} - \tau_{kf} \quad , \qquad T_{kf} = \tau_{ko} - \tau_{kg} - \tau_{kh} + \tau_{kf} \quad ,
\end{aligned} \tag{5.3}
$$

where the 16 $Z_2 \times Z_2$ characters τ_{kl} [70], combinations of products of level-one SO(2) characters, are displayed in Appendix B. These two classes of supersymmetric $Z_2 \times Z_2$ orbifolds have quite different low-energy spectra: in the first $\mathcal{N} = 2$ supergravity is coupled to 52 hyper-multiplets and 3 vector multiplets, while in the second it is coupled to 4 hyper-multiplets and 51 vector multiplets. Actually these spectra correspond to orbifold limits of Calabi-Yau manifolds with Hodge numbers $(51, 3)$ and $(3, 51)$, respectively. Finally, remembering that the two spinor representations S_2 and C_2 of $SO(2)$ are one the conjugate of the other, the model without discrete torsion corresponds to the diagonal modular invariant, while the model with discrete torsion corresponds to the charge-conjugation modular invariant. This difference in the parent modular invariants will lead to different brane configurations for the two classes of models.

Following the standard procedure for the construction of open descendants for a given bulk model, we consider the Klein bottle projection. There are actually a number of distinct choices for $\mathcal{K}$, that correspond to different choices for three signs (ϵ_1,ϵ_2,ϵ_3) that define the world-sheet projection for twisted states. In all cases these signs are related to the parameter ϵ introduced previously, since they must satisfy

$$
\epsilon_1 \; \epsilon_2 \; \epsilon_3 \; = \epsilon \quad . \tag{5.4}
$$

Thus, the models with discrete torsion require an odd number of negative signs, while the models without discrete torsion require an even number of negative signs. Let us now introduce the restrictions of the full lattice sums Λ_i to their momentum and winding sublattices. These will be denoted by P_i and W_i, where the index identifies the corresponding two-torus. For later use we also introduce sums with alternating signs in one direction of the two-tori, that we shall denote succinctly as $(-1)^m P$ or $(-1)^n W$. In terms of these quantities the Klein-bottle amplitude in the direct channel is

$$
\begin{aligned}
\mathcal{K} \;=\; & \frac{1}{8}\Big\{(P_1P_2P_3 + P_1W_2W_3 + W_1P_2W_3 + W_1W_2P_3)T_{oo} \\
& +2 \times 16[\epsilon_1(P_1 + \epsilon W_1)T_{go} + \epsilon_2(P_2 + \epsilon W_2)T_{fo} + \epsilon_3(P_3 + \epsilon W_3)T_{ho}] \left(\frac{\eta}{\theta_4}\right)^2 \Big\} \quad .
\end{aligned} \tag{5.5}
$$

Note the peculiar structure of the lattice contributions $(P_i + \epsilon W_i)$ that are crucial in order to project consistently the closed spectrum. Indeed for $\epsilon = 1$ in the torus amplitude the massless twisted characters appear in diagonal combinations and describe $N = 2$ vector multiplets. Accordingly they appear also in the Klein bottle and are symmetrized or antisymmetrized depending on the choice made for the ϵ_i. For instance, starting from the original twisted $N = 2$ vector multiplets, in the Klein-bottle projection $\epsilon_1 = 1$ selects $N = 1$ chiral multiplets, while $\epsilon_1 = -1$ selects $N = 1$ vector multiplets. When $\epsilon = -1$, the massless twisted characters are combined in an off-diagonal way and describe (chiral-linear) hypermultipets. In this case, they can not contribute to the Klein bottle amplitude and indeed the sums $(P_i - W_i)$ vanish at the origin of the lattice. The following amplitudes can be written in a more readable form introducing a compact notation which leaves the *summations* over repeated indices and the *symmetrizations* over distinct indices implicit. Using this notation, the previous amplitude can be rewritten as

$$\mathcal{K} = \frac{1}{8}\Big\{(P_1P_2P_3 + \frac{1}{2}P_kW_lW_m)T_{oo} + 2\times 16\epsilon_k(P_k + \epsilon W_k)T_{ko}\left(\frac{\eta}{\vartheta_4}\right)^2\Big\} \quad . \tag{5.6}$$

An S modular transformation turns this expression into the corresponding vacuum-channel amplitude

$$\tilde{\mathcal{K}} = \frac{2^5}{8}\Big\{(v_1v_2v_3W_1^eW_2^eW_3^e + \frac{v_k}{2v_lv_m}W_k^eP_l^eP_m^e)T_{oo} + 2\epsilon_k(v_kW_k^e + \epsilon\frac{P_k^e}{v_k})T_{ok}\left(\frac{2\eta}{\vartheta_2}\right)^2\Big\} \ , \tag{5.7}$$

where the superscript e denotes the restriction of the lattice sums to their even terms and the v_k denote the volumes of the three internal tori. At the origin of the lattices, the constraint (5.4) leads to an expression whose coefficients are perfect squares,

$$\begin{aligned}\tilde{\mathcal{K}}_0 \;=\; \frac{2^5}{8}\Big\{ &\left(\sqrt{v_1v_2v_3} + \epsilon_1\sqrt{\frac{v_1}{v_2v_3}} + \epsilon_2\sqrt{\frac{v_2}{v_1v_3}} + \epsilon_3\sqrt{\frac{v_3}{v_1v_2}}\right)^2 \tau_{oo} \\ + &\left(\sqrt{v_1v_2v_3} + \epsilon_1\sqrt{\frac{v_1}{v_2v_3}} - \epsilon_2\sqrt{\frac{v_2}{v_1v_3}} - \epsilon_3\sqrt{\frac{v_3}{v_1v_2}}\right)^2 \tau_{og} \\ + &\left(\sqrt{v_1v_2v_3} - \epsilon_1\sqrt{\frac{v_1}{v_2v_3}} + \epsilon_2\sqrt{\frac{v_2}{v_1v_3}} - \epsilon_3\sqrt{\frac{v_3}{v_1v_2}}\right)^2 \tau_{of} \\ + &\left(\sqrt{v_1v_2v_3} - \epsilon_1\sqrt{\frac{v_1}{v_2v_3}} - \epsilon_2\sqrt{\frac{v_2}{v_1v_3}} + \epsilon_3\sqrt{\frac{v_3}{v_1v_2}}\right)^2 \tau_{oh}\Big\} \quad . \end{aligned} \tag{5.8}$$

From (5.8) we can read the configuration of the orientifold planes. In particular from the crosscap coefficient of the identity τ_{oo} we can recognize the contribution of the $O9$ plane, proportional to the square root of the volume of the full T^6. The other three terms in the same crosscap coefficient give the contribution of three $O5$ planes, wrapped around one of the internal two-tori. We call $O5_i$ plane the O-plane wrapped around the torus T^i, $i = 1, 2, 3$. The freedom in the choice of the signs ϵ_i amounts to the possibility of reversing the tension and the charge of these orientifold planes. While manifestly compatible with the usual positivity requirements, this reversal clearly affects the tadpole conditions, that require the introduction of antibranes. In this respect, it should be appreciated that, according to (5.4), discrete torsion implies the reversal of at least one of the O5 charges. Therefore, taking into account the presence of the ϵ_k, one can identify four classes of models, determined by the independent choices for $(\epsilon_1, \epsilon_2, \epsilon_3)$. If $\epsilon = 1$, the choice $(+,+,+)$ gives a model with 48 chiral multiplets from the closed twisted sectors [70, 80], while the choice $(+,-,-)$ gives a model with 16 chiral multiplets and 32 vector multiplets from the twisted sectors. On the other hand, for $\epsilon = -1$ the two choices $(+,+,-)$ and $(-,-,-)$ yield the same massless twisted spectrum, namely 48 chiral multiplets.

We will describe the model $(+,+,+)$ first. In this case the tadpole conditions still allow for supersymmetric brane configurations and the determination of the open spectrum does not present any subtlety. However these class of models is a very rich laboratory for the study of freely-acting orbifolds and allow to illustrate without too many complications the general features of $Z_2 \times Z_2$ orbifolds. Since the closed partition function is not the charge-conjugation modular invariant, we expect that in this case only a subset of the characters can flow in the transverse channel. Actually the orbifold breakings are not allowed in this case: the twisted terms involve diagonal combinations of the (non-self-conjugate) characters τ_{ij}, $i \neq j$, that cannot flow in the tube channel. The annulus amplitude then reads

$$\mathcal{A} = \frac{1}{8}\Big\{(N^2\, P_1 P_2 P_3 + D_k^2\, P_k W_l W_m) T_{oo} + 2 N D_k\, P_k T_{ko} \left(\frac{\eta}{\theta_4}\right)^2 + 2 D_k D_l\, W_m T_{mo} \left(\frac{\eta}{\theta_4}\right)^2 \Big\} \quad . \tag{5.9}$$

The need for three types of branes can easily be appreciated by noticing that, relatively to the three internal two-tori, the three twists of the $Z_2 \times Z_2$ orbifold result in open string ends with boundary conditions of four types: NNN, NDD, DND and DDN. Combining these boundaries in all possible ways yields all types of open strings present in the model. Aside from the familiar NN, ND_i and D_iD_i strings, the spectrum thus includes three additional types of open strings, that are DD with respect to a torus and ND with respect to the others. An S transformation turns (5.9) into the corresponding vacuum-channel amplitude

$$\begin{aligned} \tilde{\mathcal{A}} &= \frac{2^{-5}}{8}\Big\{(N^2\; v_1 v_2 v_3 W_1 W_2 W_3 + \frac{D_i^2 v_i}{v_j v_k}\; W_i P_j P_k) T_{oo} + 2 N D_i\; v_i W_i T_{oi} \left(\frac{2\eta}{\theta_2}\right)^2 \\ &+ 2 D_j D_k\; \frac{1}{v_i} P_i T_{oi} \left(\frac{2\eta}{\theta_2}\right)^2 \Big\} \quad , \end{aligned} \tag{5.10}$$

from which we can extract the reflection coefficients for the massless states

$$\begin{aligned} \tilde{\mathcal{A}}_0 &= \frac{2^{-5}}{8}\Big\{\, (N\sqrt{v_1 v_2 v_3} + D_1\sqrt{\frac{v_1}{v_2 v_3}} + D_2\sqrt{\frac{v_2}{v_1 v_3}} + D_3\sqrt{\frac{v_3}{v_1 v_2}})^2 \tau_{oo} \\ &+ (N\sqrt{v_1 v_2 v_3} + D_1\sqrt{\frac{v_1}{v_2 v_3}} - D_2\sqrt{\frac{v_2}{v_1 v_3}} - D_3\sqrt{\frac{v_3}{v_1 v_2}})^2 \tau_{og} \end{aligned}$$

$$
\begin{aligned}
&+(N\sqrt{v_1v_2v_3} - D_1\sqrt{\frac{v_1}{v_2v_3}} + D_2\sqrt{\frac{v_2}{v_1v_3}} - D_3\sqrt{\frac{v_3}{v_1v_2}})^2\tau_{of} \\
&+(N\sqrt{v_1v_2v_3} - D_1\sqrt{\frac{v_1}{v_2v_3}} - D_2\sqrt{\frac{v_2}{v_1v_3}} + D_3\sqrt{\frac{v_3}{v_1v_2}})^2\tau_{oh} \Big\} \ . \qquad (5.11)
\end{aligned}
$$

As expected, (5.11) displays four independent squared reflection coefficients, related to the 9-branes and to the three types of 5-branes, and all terms enter this expression with signs fixed by the relation between spin and statistics for the open spectrum in $\mathcal{A}$.

These amplitudes determine by standard methods the vacuum channel of the Möbius amplitude at the origin of the lattices. One can easily reconstruct the whole $\tilde{\mathcal{M}}$ and performing a P transformation recover the direct-channel Möbius amplitude

$$
\mathcal{M} = -\frac{1}{8}\Big\{[NP_1P_2P_3 + D_iP_iW_jW_k]\hat{T}_{oo} - [(N+D_i)P_i + (D_j+D_k)W_i]\hat{T}_{oi}\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2\Big\} \quad , \qquad (5.12)
$$

where, as usual for the Möbius amplitude, we make use of the real basis of hatted characters. This model has only untwisted tadpoles and their cancellation requires

$$
N = D_1 = D_2 = D_3 = 32 \quad . \qquad (5.13)
$$

The lack of breaking terms, however, requires a rescaling by a factor two of the four charges N, D_1, D_2 and D_3, in order to grant a proper particle interpretation. The rescaling $N = 2n$ and $D_i = 2d_i$, combined with the sign of the identity in the direct Möbius amplitude, gives $USp(16)^4$ as the gauge group of the model. This rescaling by a factor of two could have been anticipated on the basis of our previous discussion of boundary states in general modular invariant. Allowed boundary states for this model must have vanishing boundary coefficients with respect to the twisted characters. They can be obtained acting with the $Z_2 \times Z_2$ orbifold group on the boundary states of the charge-conjugation model. As a consequence their boundary coefficients are rescaled by a factor of two. The massless spectrum can be easily read if we restrict $\mathcal{A}$ and $\mathcal{M}$ to the origin of the lattice

$$
\begin{aligned}
\mathcal{A}_0 &= \frac{n^2+d_1^2+d_2^2+d_3^2}{2}(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of}) + (nd_1+d_2d_3)(\tau_{go}+\tau_{gg}+\tau_{gh}+\tau_{gf}) \\
&+ (nd_2+d_1d_3)(\tau_{fo}+\tau_{fg}+\tau_{fh}+\tau_{ff}) + (nd_3+d_1d_2)(\tau_{ho}+\tau_{hg}+\tau_{hh}+\tau_{hf}) \ , \\
\mathcal{M}_0 &= \frac{1}{2}(n+d_1+d_2+d_3)(\tau_{oo}-\tau_{og}-\tau_{of}-\tau_{oh}) \quad . \qquad (5.14)
\end{aligned}
$$

These amplitudes describe the adjoint vector multiplets of the four $USp(16)$ groups, three chiral multiplets, each in the $(120,1,1,1)$, $(1,120,1,1)$, $(1,1,120,1)$ and $(1,1,1,120)$, and six chiral multiplets in the $(16,16,1,1)$ and in five additional bi-fundamental representations that differ by permutations of the factors. These spectrum is clearly not chiral, and is thus free of gauge and gravitational anomalies.

5.2 Brane supersymmetry in $Z_2 \times Z_2$ freely-acting orbifolds

We now deform this model by combining the $Z_2 \times Z_2$ orbifold operations with shifts (δ_L, δ_R) of the lattice vectors (p_L, p_R) [**shift**]. The resulting Type IIB freely-acting orbifolds correspond to singular limits of Calabi-Yau manifolds with Hodge numbers $(19, 19)$, $(11, 11)$ and $(3, 3)$ in the cases of one, two and three shifts respectively. As first noticed in [66, 81], in the construction of the open descendants we shall distinguish between symmetric *momentum* shifts $(p) = (\delta, \delta)$ and antisymmetric *winding* shifts $(w) = (\delta, -\delta)$, since the two have very different effects on the open string spectrum.

Let us begin by introducing a convenient notation to specify the orbifold action $X_i \to \sigma(X_i)$ on the complex coordinates for the three internal tori. There are several ways to combine the three operations g, f and h of eq. (5.1) with shifts in a way consistent with the $Z_2 \times Z_2$ group structure. However, up to T-dualities and corresponding redefinitions of the Ω projection, all non-trivial possibilities are captured by the matrices

$$\sigma_1(\delta_1, \delta_2, \delta_3) = \begin{pmatrix} \delta_1 & -\delta_2 & -1 \\ -1 & \delta_2 & -\delta_3 \\ -\delta_1 & -1 & \delta_3 \end{pmatrix} \quad , \tag{5.15}$$

and

$$\sigma_2(\delta_1, \delta_2, \delta_3) = \begin{pmatrix} \delta_1 & -1 & -1 \\ -1 & \delta_2 & -\delta_3 \\ -\delta_1 & -\delta_2 & \delta_3 \end{pmatrix} \quad , \tag{5.16}$$

where the three lines refer to the new operations, that we shall continue to denote by g, h and f, and where $-\delta_i$ indicates the combination of a shift in the direction i with an orbifold inversion.

T-duality transformations can be neatly discussed referring to the tables (5.15) and (5.16), if these are supplemented by an additional line corresponding to the identity operation. A T-duality along a torus simply flips the signs in the corresponding column and interchanges momentum and winding shifts. For instance, starting from σ_1 and performing a T-duality along T_{67} and T_{89} would result in the table

$$\begin{pmatrix} 1 & -1 & -1 \\ \delta_1 & \delta_2' & 1 \\ -1 & -\delta_2' & \delta_3' \\ -\delta_1 & 1 & -\delta_3' \end{pmatrix} \quad , \tag{5.17}$$

where δ' denotes a dual shift. This is equivalent to a standard table $\sigma_1(\delta_1, \delta_2', \delta_3')$, up to a redefinition of Ω, that now includes a factor $(-1)^{\delta_1+\delta_2'}$. Therefore we can confine our attention to ten different types of $Z_2 \times Z_2$ shift orbifolds listed in Table 1.

The brane content of these shift orbifolds compared to the brane content of the original $Z_2 \times Z_2$ model is altered in a simple and general way: when a line of the table contains p or $-w$, the corresponding brane is eliminated by tadpole conditions. That this is indeed the case will become clear after our discussion of the Klein bottle amplitude. Indeed it is a more complicated instance of the same phenomenon we discussed in the previous chapter, when we considered the open descendants of the free boson with a Klein bottle amplitude modified by the simple current associated to an half-radius shift.

On the other hand, our choice of referring to a conventional Ω projection will always result in leaving the $D9$ branes unaffected. Up to T-dualities and redefinitions of Ω, one can then shown that all interesting brane configurations may be captured referring to the ten different models of Table 1.

5-Branes	σ_1	σ_2
0	p_{123}	$p_1 w_2 w_3$
	w_{123}	
1		$w_1 p_2 p_3$
	$w_1 p_2 w_3$	
		$w_1 p_2$
		p_{23}
2	p_3	$w_2 p_3$
		$w_1 w_2 p_3$

Table 1. Types of shift models.

In listing these models, we have made suitable choices of axes, so that when a single $D5$ brane is present, this is always the first, $D5_1$, and when two are present, these are always $D5_1$ and $D5_2$. All these models have $N = 1$ supersymmetry in the closed sector, but present interesting instances of "brane supersymmetry" [66, 67]: additional supersymmetries are present for the massless modes, and at times also for the massive ones [82], confined to some branes. It is therefore interesting to display, for all the models of Table 1, the number of supersymmetries of the massless modes for the various branes present. The results are summarized in Table 2:

models	$D9$ susy	$D5_1$ susy	$D5_2$ susy
p_3	N=1	N=2	N=2
w_2p_3	N=2	N=2	N=4
$w_1w_2p_3$	N=4	N=4	N=4
p_{23}	N=1	N=2	–
w_1p_2	N=2	N=4	–
$w_1p_2p_3$	N=2	N=4	–
$w_1p_2w_3$	N=4	N=4	–
p_{123}	N=1	–	–
$p_1w_2w_3$	N=2	–	–
$w_1w_2w_3$	N=4	–	–

Table 2. Brane supersymmetry for the various models.

In all models with one or two shifts, the supersymmetry enhancement for the massless modes on the branes may be traced directly to the phenomenon described in [66, 81]: projections accompanied by momentum shifts orthogonal to a brane do not reduce the supersymmetry of the massless modes. As an example, let us consider the w_1p_2 model that contains D_9 and $D5_1$ branes. After performing a T-duality along the first and the third tori, the $D9$ branes turn into $D5_2$ branes and the $D5_1$ branes turn into $D5_3$ branes, while the resulting model has momentum shifts in the first two directions. These are both orthogonal to the $D5_3$ branes, and the corresponding massless modes have indeed $N = 4$ supersymmetry. On the other hand, one of these shifts is parallel to the $D5_2$ branes and breaks the corresponding supersymmetry to $N = 2$. Models with three shifts are more subtle, due to the $Z_2 \times Z_2$ group structure of the transformations, but may be understood along similar lines.

From Table 2 one can conclude that the amount of supersymmetry for the massless modes of a brane is enhanced when the Scherk-Schwarz breaking is induced by momentum shifts orthogonal to it, or in all cases that may be linked to this by suitable T-dualities.

Let us now justify the previous considerations describing the vacuum amplitudes for the open descendants of $Z_2 \times Z_2$ shift-orbifold models. In all cases the starting point is a deformation of the torus amplitude (5.2), where momentum or winding shifts induce corresponding modifications of the twisted contributions. For instance, for the case of two momentum shifts the torus amplitude is

$$\mathcal{T} = \frac{1}{4}\Big\{|T_{oo}|^2\Lambda_1\Lambda_2\Lambda_3 + |T_{og}|^2\Lambda_1|\frac{4\eta^2}{\theta_2^2}|^2 + |T_{of}|^2(-1)^{m_2}\Lambda_2|\frac{4\eta^2}{\theta_2^2}|^2 + |T_{oh}|^2(-1)^{m_3}\Lambda_3|\frac{4\eta^2}{\theta_2^2}|^2$$
$$+ \quad |T_{go}|^2\Lambda_1|\frac{4\eta^2}{\theta_4^2}|^2 + |T_{gg}|^2\Lambda_1|\frac{4\eta^2}{\theta_3^2}|^2 + |T_{fo}|^2\Lambda_2^{n_2+1/2}|\frac{4\eta^2}{\theta_4^2}|^2 \qquad (5.18)$$

$$+ \quad |T_{ff}|^2(-1)^{m_2}\Lambda_2^{n_2+1/2}|\frac{4\eta^2}{\theta_3^2}|^2 + |T_{ho}|^2\Lambda_3^{n_3+1/2}|\frac{4\eta^2}{\theta_4^2}|^2 + |T_{hh}|^2(-1)^{m_3}\Lambda_3^{n_3+1/2}|\frac{4\eta^2}{\theta_3^2}|^2\Big\} \quad .$$

In all these deformed models, the independent orbit related to the discrete torsion is absent. This is due to the fact that the presence of a freely-acting operation allows for the existence of fixed two-tori but no for the existence of fixed points. On the other hand the independent orbit in the torus amplitude (5.2) takes into account precisely the contribution of the 64 orbifold fixed points, as it is confirmed by the absence of internal volume factors. The corresponding open descendants are essentially determined by the direct-channel Klein-bottle amplitude $\mathcal{K}$ and by the transverse-channel annulus amplitude $\tilde{\mathcal{A}}$. Moreover, the former, to which we now turn, is fully specified by the corresponding table (σ_1 or σ_2) and by our choice of working with a conventional Ω projection.

The Klein-bottle amplitudes are generically affected by the shifts, that can make the transverse-channel contributions massive. The direct-channel amplitude, $\mathcal{K}$, includes four distinct contributions, corresponding to $P_1P_2P_3$, $P_1W_2W_3$, $W_1P_2W_3$ and $W_1W_2P_3$ reflecting the presence of $O9$ and $O5_i$ planes. However, when the lattice sum corresponding to the $O5_i$ plane is accompanied in $\mathcal{K}$ by phases induced by the shifts, the corresponding vacuum-channel contribution is lifted in mass. As a consequence, the $O5_i$ plane does not induce tadpoles for the massless fields and there is no need for branes of type $D5_i$. The end result is that, aside from twisted contributions, the Klein bottle projections for the two cases in (5.15) and (5.16) may be written in the form

$$\mathcal{K}_1 = \frac{1}{8}\left\{P_1P_2P_3 + (-1)^{\delta_1+\delta_2}P_1W_2W_3 + (-1)^{\delta_2+\delta_3}W_1P_2W_3 + (-1)^{\delta_1+\delta_3}W_1W_2P_3\right\}T_{oo} \quad , \tag{5.19}$$

and

$$\mathcal{K}_2 = \frac{1}{8}\left\{P_1P_2P_3 + (-1)^{\delta_1}P_1W_2W_3 + (-1)^{\delta_2+\delta_3}W_1P_2W_3 + (-1)^{\delta_1+\delta_2+\delta_3}W_1W_2P_3\right\}T_{oo} \quad . \tag{5.20}$$

In order to describe the brane content of a given model, one can then specialize these equations to the corresponding choice of (momentum or winding) shifts. Whenever a phase has a nontrivial effect on a restricted lattice sum, the corresponding transverse-channel contribution is lifted in mass, and the tadpole conditions eliminate the corresponding brane. For instance, for the $P_1W_2W_3$ sum this is the case for a momentum shift along the first torus, and/or for winding shifts along the other two tori. In order to discuss a concrete example, let us consider the p_{23} model. In this case the only phases present are $\delta_2 = m_2$ and $\delta_3 = m_3$, while the corresponding multiplication table is σ_2. Then, aside from twisted contributions

$$\mathcal{K} = \frac{1}{8}\left\{P_1P_2P_3 + P_1W_2W_3 + (-1)^{m_2}W_1P_2W_3 + (-1)^{m_3}W_1W_2P_3\right\}T_{oo} \quad , \tag{5.21}$$

and consequently one is left only with $D9$ and $D5_1$ branes.

Let us now turn to the transverse-channel annulus amplitude $\tilde{\mathcal{A}}$. This presents some subtleties, since the shift orbifold restricts the allowed untwisted modes. These constraints have a simple geometrical origin: in general, the fixed tori of one operation are paired by the others and form multiplets, whose members must accommodate identical brane sets. As a result, branes are accompanied by a number of images, and the restrictions on the form of $\tilde{\mathcal{A}}$ determine their minimal configurations. This is reminiscent of what happens in ordinary orbifolds when, by a continuous deformation, a brane is moved off a fixed point, but in shift orbifolds the presence of image branes is in general unavoidable. In order to prepare the grounds for the discussion of the shift orbifolds, it is instructive to see how the displacement of branes works in the T^4/Z_2 orbifold. This phenomenon, first discussed in [83], may be given a complete description in conformal field theory starting from the torus amplitude

$$\mathcal{T} = \frac{1}{2}\Big\{\Lambda^4|Q_o+Q_v|^2+|Q_o-Q_v|^2|\frac{2\eta}{\theta_2}|^4+|Q_s+Q_c|^2|\frac{2\eta}{\theta_4}|^4+|Q_s-Q_c|^2|\frac{2\eta}{\theta_3}|^4\Big\} , \quad (5.22)$$

where Λ denotes a one-dimensional toroidal lattice sum and, as usual, we have introduced the four supersymmetric combinations of $SO(4)$ characters

$$\begin{aligned} Q_o &= V_4O_4 - C_4C_4 \quad , \qquad Q_v = O_4V_4 - S_4S_4 \ , \\ Q_s &= O_4C_4 - S_4O_4 \quad , \qquad Q_c = V_4S_4 - C_4V_4 \ . \end{aligned} \qquad (5.23)$$

The Klein-bottle amplitudes are then

$$\begin{aligned} \mathcal{K} &= \frac{1}{4}\Big\{(Q_o+Q_v)(P^4+W^4) + 2\times 16(Q_s+Q_c)(\frac{\eta}{\theta_4})^2\Big\} \quad , \\ \tilde{\mathcal{K}} &= \frac{2^5}{4}\Big\{(Q_o+Q_v)(v(W^e)^4+\frac{1}{v}(P^e)^4) + 2(Q_o-Q_v)(\frac{2\eta}{\theta_2})^2\Big\} \quad , \end{aligned} \qquad (5.24)$$

where P^4 and W^4 denote the restrictions of the lattice sum Λ^4 to zero windings and to zero momenta, respectively. The simplest configuration, where all $D5$ branes are at the same fixed point and no Wilson lines are introduced, results in the gauge group $U(16)_9 \times U(16)_5$. The familiar spectrum [84, 83] comprises two pairs of hypermultiplets in the $(120, 1)$ and $(1, 120)$ from the 99 and 55 sectors, and an additional 59 hypermultiplet in the $(16, 16)$.

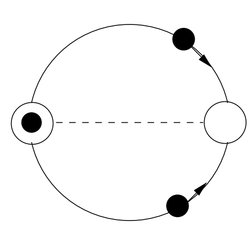

Figure 1. Pairs of image branes moved away from a fixed point.

If some pairs of $D5$ branes are moved a distance $2\pi\alpha R$ away from the fixed point along the last circle, the other transverse-channel amplitudes become

$$\tilde{\mathcal{A}} = \frac{2^{-5}}{4}\Big\{(Q_o+Q_v)\Big[N^2 v W^4 + \frac{1}{v}\sum_m (D+\frac{\delta}{2}e^{2i\pi\alpha m}+\frac{\delta}{2}e^{-2i\pi\alpha m})^2 P^3 P_m\Big]$$

$$+\ 2N(D+\delta)(Q_o-Q_v)(\frac{2\eta}{\theta_2})^2 + 4(Q_s+Q_c)\ (R_N^2+R_D^2)\ (\frac{2\eta}{\theta_4})^2 - 2R_N R_D\ (Q_s-Q_c)(\frac{2\eta}{\theta_3})^2\Big\}\ ,$$

$$\begin{aligned}
\tilde{\mathcal{M}} &= -\frac{1}{2}\Big\{(\hat{Q}_o+\hat{Q}_v)\Big[Nv(W^e)^4 + \frac{1}{v}\sum_m (D+\frac{\delta}{2}e^{4i\pi\alpha m}+\frac{\delta}{2}e^{-4i\pi\alpha m})(P^e)^3 P_{2m}\Big] \\
&+ (N+D+\delta)(\hat{Q}_o-\hat{Q}_v)\left(\frac{\hat{\eta}}{\hat{\theta}_2}\right)^2\Big\}\ ,
\end{aligned} \qquad (5.25)$$

and thus the corresponding direct-channel amplitudes are

$$\begin{aligned}
\mathcal{A} &= \frac{1}{4}\Big\{(Q_o+Q_v)\Big[N^2P^4+(D^2+\frac{\delta^2}{2})W^4+\frac{\delta^2}{4}W^3(W_{2\alpha}+W_{-2\alpha})+D\delta W^3(W_\alpha+W_{-\alpha})\Big] \\
&+ 2N(D+\delta)(Q_s+Q_c)(\frac{\eta}{\theta_4})^2 + (Q_o-Q_v)(R_N^2+R_D^2)(\frac{2\eta}{\theta_2})^2 + 2R_N R_D(Q_s-Q_c)(\frac{\eta}{\theta_3})^2\Big\}\ , \\
\mathcal{M} &= -\frac{1}{4}\Big\{(\hat{Q}_o+\hat{Q}_v)\Big[N\ P^4 + D\ W^4 + \frac{\delta}{2}W^3(W_{2\alpha}+W_{-2\alpha})\Big] \\
&- (\hat{Q}_o-\hat{Q}_v)(N+D+\delta)\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2\Big\}\ .
\end{aligned} \qquad (5.26)$$

Here N denotes the total number of $D9$ branes, D denotes the total number of $D5$ branes left at the origin, δ counts the number of displaced branes, and R_N and R_D account for the effect of the orbifold breaking on the Chan-Paton charges at the fixed points. If $\alpha \neq 1/2$, the moved branes are also away from the other fixed point on the circle, and the corresponding breaking terms R_δ are absent. Since the consistency of the conformal theory in both channels allows only breaking terms for branes sitting at fixed points, one is forced to let $\delta = 2d$, thus effectively splitting the charge between image branes. This is somewhat reminiscent of the splitting induced by a quantized NS B_{ab} [85], and indeed the rank of the gauge group is correspondingly reduced. These features reflect the geometry of the brane configuration, that includes pairs of images interchanged by orbifold operations. The structure of $\mathcal{M}$ implies that in this case the gauge groups carried by the displaced branes are symplectic and have a reduced rank [83]. We would like to stress that, while in the absence of breaking terms the δ contributions appear to have enhanced supersymmetry, this is actually reduced by the Möbius projection, as was the case for the supersymmetric $Z_2 \times Z_2$ model without discrete torsion previously discussed. On the other hand, in shift orbifolds the enhancement of supersymmetry will be respected by $\mathcal{M}$ leading to the phenomenon of *brane supersymmetry*, as we will see later in an explicit example.

Summarizing, for $\alpha \neq 1/2$ the $D5$ gauge group generically breaks from $U(16)$ to $U(16-2d)\times USp(2d)$. On the other hand, for $\alpha = 1/2$ the pairs of images meet at the other fixed point on the circle. The orbifold breakings are now allowed for all $D5$ branes, and the rank of the $D5$ gauge group is correspondingly enhanced. In the transverse channel, the modified breaking terms

$$\frac{2^{-5}}{4}\Big\{16(Q_s+Q_c)\ (R_N^2+R_D^2+R_\delta^2)\ (\frac{\eta}{\theta_4})^2 - 8R_N(R_D+R_\delta)\ (Q_s-Q_c)(\frac{\eta}{\theta_3})^2\Big\} \qquad (5.27)$$

reflect the position of the $D5$ branes, that now occupy both fixed points along the circle, and indeed at the origin eq. (5.27) becomes

$$\frac{1}{4}\Big[(R_N-4R_D)^2+(R_N-4R_\delta)^2+14R_N^2\Big] \quad . \qquad (5.28)$$

Let us now apply these results to the case of freely-acting orbifolds. If some orbifold operation were to interchange the two fixed points where we have placed the $D5$ branes, one would be forced to set $D=\delta$ and $R_D=R_\delta$, and as a result for $\alpha=1/2$ $\tilde{\mathcal{A}}$ would include the projector

$$\Pi=\frac{1}{2}\left(1+(-1)^m\right) \quad . \qquad (5.29)$$

Summarizing, the presence of brane multiplets in the vacuum configuration reflects itself in the presence of corresponding projection operators in $\tilde{\mathcal{A}}$. The projection operators for the two classes of models corresponding to σ_1 and σ_2 are

$$\Pi_1 \sim 1+(-1)^{\delta_1+\delta_2}+(-1)^{\delta_2+\delta_3}+(-1)^{\delta_1+\delta_3} \quad , \qquad (5.30)$$

$$\Pi_2 \sim 1+(-1)^{\delta_1}+(-1)^{\delta_2+\delta_3}+(-1)^{\delta_1+\delta_2+\delta_3} \quad . \qquad (5.31)$$

In order to give a concrete example, let us refer again to the p_{23} model, that as we have seen has $D9$ and $D5_1$ branes. In this case

$$\Pi=\frac{1}{2}\left(1+(-1)^{m_2+m_3}\right) \quad , \qquad (5.32)$$

and all states flowing in the transverse channel of the annulus must have correlated momenta, both even or both odd in the corresponding two circles of the last two tori. Clearly, Π has no effect on the $D9$ contribution, that in the transverse-channel annulus amplitude only involves windings. However, it does play a crucial role for the sector associated to the $D5_1$ brane. In this case, the transverse-channel contribution corresponds to $W_1P_2P_3$, and the projection affects the momenta in the last two sums. Since the Poisson transform of the restricted sum is proportional to $P_1(W_2W_3+W_2^{n+1/2}W_3^{n+1/2})$, where the $W_i^{n+1/2}$ in the last two factors denote shifted sums, in the direct channel Π gives rise to a doublet of branes. As we have seen, this is precisely the type of configuration required by the structure of the shift orbifold.

We now describe in some detail one of the models in Table 2, the p_{23} model, and then we will comment briefly on the other cases. This model is obtained introducing in the table σ_2 of eq. (5.16) momentum shifts p_2 and p_3 in two lattice directions. In this case $N=2$ supersymmetry is spontaneously broken to $N=1$. The previous analysis reveals that, in addition to $D9$-branes, they also contain $D5_1$ branes. They have the rather interesting feature of displaying various numbers of supersymmetries in the bulk and on the branes in a relatively simple setting. Indeed we expect $N=1$ supersymmetry for the $D9$ branes, since the two shifts are parallel to their world volume, and $N=2$ supersymmetry for the massless modes of the $D5_1$ branes, since they are acted upon by the g projection, while the two shifts are orthogonal to their world volume.

The direct and transverse Klein bottle amplitudes for this model

$$\mathcal{K} = \frac{1}{8}\Big\{T_{oo}[P_1P_2P_3+P_1W_2W_3+W_1(-1)^{m_2}P_2W_3+W_1W_2(-1)^{m_3}P_3]+2\times 16T_{go}\ P_1\left(\frac{\eta}{\theta_4}\right)^2\Big\},$$

$$\begin{aligned}\tilde{\mathcal{K}} &= \frac{2^5}{8}\Big\{T_{oo}[v_1v_2v_3W_1^eW_2^eW_3^e+\frac{v_1}{v_2v_3}W_1^eP_2^eP_3^e+\frac{v_2}{v_1v_3}P_1^eW_2^oP_3^e+\frac{v_3}{v_1v_2}P_1^eP_2^eW_3^o] \\ &+ 2T_{og}v_1W_1^e\left(\frac{2\eta}{\theta_2}\right)^2\Big\} \quad ,\end{aligned} \tag{5.33}$$

are determined by the general rules we explained in the previous paragraphs. In particular, the twisted contribution is fixed by the behavior of $\tilde{\mathcal{K}}$ at the origin of the lattices, and ensures that the various independent sectors of the spectrum have reflection coefficients that are perfect squares. Indeed:

$$\tilde{\mathcal{K}}_0 = \frac{2^5}{8}\Big\{(\tau_{oo}+\tau_{og})\Big(\sqrt{v_1v_2v_3}+\sqrt{\frac{v_1}{v_2v_3}}\Big)^2+(\tau_{of}+\tau_{oh})\Big(\sqrt{v_1v_2v_3}-\sqrt{\frac{v_1}{v_2v_3}}\Big)^2\Big\}\ . \tag{5.34}$$

The vacuum channel annulus amplitude

$$\begin{aligned}\tilde{\mathcal{A}} &= \frac{2^{-5}}{8}\Big\{T_{oo}\Big[N^2v_1v_2v_3W_1W_2W_3+D_1^2\frac{v_1}{v_2v_3}W_1(P_2^eP_3^e+P_2^oP_3^o)\Big] \\ &+ 4(G^2+2G_1^2)T_{go}W_1v_1(\frac{2\eta}{\theta_4})^2+4F^2T_{fo}W_2^{n+1/2}v_2(\frac{2\eta}{\theta_4})^2+4H^2T_{ho}W_3^{n+1/2}v_3(\frac{2\eta}{\theta_4})^2 \\ &+ 2ND_1T_{og}W_1v_1(\frac{2\eta}{\theta_2})^2+4GG_1T_{gg}W_1v_1(\frac{2\eta}{\theta_3})^2\Big\}\end{aligned} \tag{5.35}$$

inherits the projection operator

$$\Pi = \frac{1}{2}\ [1+(-1)^{m_2+m_3}] \tag{5.36}$$

that restricts the 55 contribution to the even-even (ee) and odd-odd (oo) subsets of momentum eigenvalues in the last two tori.

Leaving a discussion of the breaking terms momentarily aside, by an S transformation we can turn this expression into the direct-channel annulus amplitude

$$\mathcal{A} = \frac{1}{8}\Big\{T_{oo}\Big[N^2P_1P_2P_3+\frac{D_1^2}{2}P_1(W_2W_3+W_2^{n+1/2}W_3^{n+1/2})\Big]+(G^2+2G_1^2)T_{og}P_1(\frac{2\eta}{\theta_2})^2$$

$$+ \quad F^2\, T_{of}(-1)^{m_2} P_2(\frac{2\eta}{\theta_2})^2 + H^2\, T_{oh}(-1)^{m_3} P_3(\frac{2\eta}{\theta_2})^2 + 2ND_1 T_{go} P_1(\frac{\eta}{\theta_4})^2$$
$$+ \quad 4GG_1 T_{gg} P_1(\frac{\eta}{\theta_3})^2 \Big\} \quad . \tag{5.37}$$

Whereas the 99 strings are rather conventional, and have the usual three projection terms, the configuration of the 55 strings is more peculiar, and admits a nice geometrical interpretation: it corresponds to a doublet of branes, associated to a pair of tori fixed by g and interchanged by f and h. Only the g projection is present, since in this sector the physical states are combinations of pairs localized on the image branes. In this case the *full* $D5_1$ spectrum, not only its massless modes, has enhanced supersymmetry. We have already stressed that multiplets of branes are a generic feature of these shift orbifolds. Even if one attempts to insert all branes at a fixed point, the other operations typically move them, and give rise to multiple images. This is summarized for this class of models in figure 2, where the three axes refer to the three two-tori T_{45}, T_{67} and T_{89}. As usual, the massive untwisted excitations of these orbifold models have enhanced supersymmetry, although they have only a fraction of the lattice modes present in the toroidal case. More precisely, the g projection forbids this extension in the first torus, while in the second and third tori the projected combinations of lattice states have indeed $N = 2$ supersymmetry in the 99 sector and $N = 4$ supersymmetry in the 55 sector.

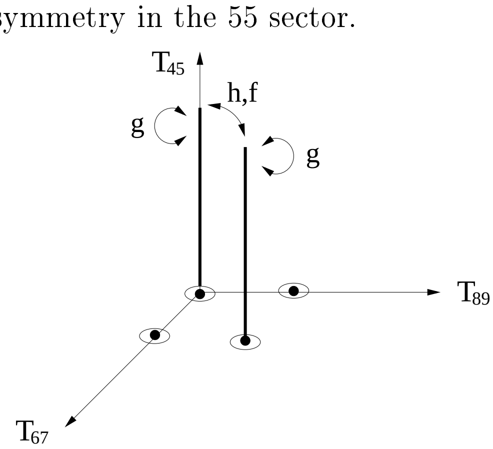

Figure 2. $D5_1$ brane configuration for the p_{23} model

The previous considerations determine the structure of the breaking terms in the transverse channel: as we have anticipated, the $D5_1$ branes occupy a pair of fixed tori, while all others are empty. Then, according to the rule discussed in the previous Section, in the transverse channel the breaking terms are to group into the structures

$$\frac{2}{4}(G \mp 4G_1)^2 + \frac{14}{4}G^2 \ , \tag{5.38}$$

precisely as demanded by the direct-channel amplitude $\mathcal{A}$. The relative factor between G and G_1 in the first term counts the number of occupied fixed points; more precisely in all these models it is equal to

$$\sqrt{\frac{\#\text{ of fixed points}}{\#\text{ of occupied fixed points}}} \quad . \tag{5.39}$$

At the origin of the lattices, the untwisted contribution to the transverse-channel annulus amplitude is

$$\tilde{\mathcal{A}}_0 = \frac{2^{-5}}{8}\Big\{(\tau_{oo}+\tau_{og})\Big(N\sqrt{v_1v_2v_3}+D_1\sqrt{\frac{v_1}{v_2v_3}}\Big)^2+(\tau_{of}+\tau_{oh})\Big(N\sqrt{v_1v_2v_3}-D_1\sqrt{\frac{v_1}{v_2v_3}}\Big)^2\Big\} \ , \tag{5.40}$$

and, together with $\tilde{\mathcal{K}}_0$ of eq. (5.34), determines the transverse-channel Möbius amplitude

$$\begin{aligned}\tilde{\mathcal{M}} \ &= \ -\frac{1}{4}\Big\{\hat{T}_{oo}\Big[Nv_1v_2v_3W_1^eW_2^eW_3^e + D_1\frac{v_1}{v_2v_3}W_1^eP_2^eP_3^e\Big] + \hat{T}_{og}(N+D_1)W_1^ev_1\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2 \\ &+ \ N\hat{T}_{of}W_2^ov_2\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2 - N\hat{T}_{oh}W_3^ov_3\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2\Big\} \quad .\end{aligned} \tag{5.41}$$

Finally, a P transformation determines the direct-channel Möbius projection

$$\begin{aligned}\mathcal{M} \ &= \ -\frac{1}{8}\Big\{\hat{T}_{oo}\Big[NP_1P_2P_3+D_1P_1W_2W_3\Big]-\hat{T}_{og}(D_1+N)P_1\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2-N\hat{T}_{of}(-1)^{m_2}P_2\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2 \\ &+ \ N\hat{T}_{oh}(-1)^{m_3}P_3\left(\frac{2\hat{\eta}}{\hat{\theta}_2}\right)^2\Big\} \quad ,\end{aligned} \tag{5.42}$$

whose contributions at the origin of the lattices imply that both types of gauge groups are unitary. The untwisted tadpole conditions

$$N = D_1 = 32 \tag{5.43}$$

fix the (maximal) size of the two gauge groups, while the twisted tadpole conditions

$$G = G_1 = 0 \tag{5.44}$$

are identically satisfied if one parameterizes the charges according to

$$\begin{aligned} N \ &= \ o+g+\bar{o}+\bar{g} \ , \qquad & G = i(o+g-\bar{o}-\bar{g}) \ , \\ H \ &= \ (o-g+\bar{o}-\bar{g}) \ , \qquad & F = i(o-g-\bar{o}+\bar{g}) \ , \\ D_1 \ &= \ 2(d+\bar{d}) \ , \qquad & G_1 = i(d-\bar{d}) \ , \end{aligned} \tag{5.45}$$

provided $o+g=16$ and $d=8$. The massless modes from the open sector are then summarized in

$$\begin{aligned}\mathcal{A}_0 \ &+ \ \mathcal{M}_0 = \ (o\bar{o}+g\bar{g})\tau_{oo}+(o\bar{g}+g\bar{o})\tau_{og}+(og+\bar{o}\bar{g})\tau_{of} \\ &+ \ \Big[\frac{o(o-1)}{2}+\frac{g(g-1)}{2}+\frac{\bar{o}(\bar{o}-1)}{2}+\frac{\bar{g}(\bar{g}-1)}{2}\Big]\tau_{oh} \\ &+ \ d\bar{d}(\tau_{oo}+\tau_{og})+\Big[\frac{d(d-1)}{2}+\frac{\bar{d}(\bar{d}-1)}{2}\Big](\tau_{of}+\tau_{oh}) \\ &+ \ [(o+g)d+(\bar{o}+\bar{g})\bar{d}](\tau_{gf}+\tau_{gh}) \quad ,\end{aligned} \tag{5.46}$$

while the gauge groups allowed by the tadpole conditions in this class of models are

$$[U(o) \times U(g)]_9 \times U(8)_{5_1} \ . \tag{5.47}$$

The tree-level spectrum of the $D9$ branes has $N = 1$ supersymmetry, while the whole spectrum of the $D5_1$ branes, orthogonal to the directions used to induce supersymmetry breaking, has indeed $N = 2$ supersymmetry. This is one more instance of the phenomenon discussed in [66, 81], and extended to the whole spectrum in [82], as suggested in [67]. The massless spectrum is not chiral, and includes 99 chiral multiplets in the representations $(o(o-1)/2, 1, 1)$, $(1, g(g-1)/2, 1)$, $(o, \bar{g}, 1)$, $(o, g, 1)$ and their hermitian conjugates, one pair of 55 hypermultiplets in the representation $(1, 1, d(d-1)/2)$, and 59 hypermultiplets in the representations $(o, 1, d)$ and $(1, g, d)$.

The other models in Table 2 can be constructed in a similar way. The resulting brane configurations are very rich and we can illustrate very briefly the most interesting cases with the aid of some figure. Let us consider the three classes of models containing $D9$, $D5_1$ and $D5_2$ branes, the first being the model p_3 obtained by introducing a single momentum shift in the table σ_1 of eq. (5.15). The corresponding brane configuration is shown in Figure 3, where the $D5_1$ branes are parallel to the T_{45} axis and the $D5_2$ branes are parallel to the T_{67} axis.

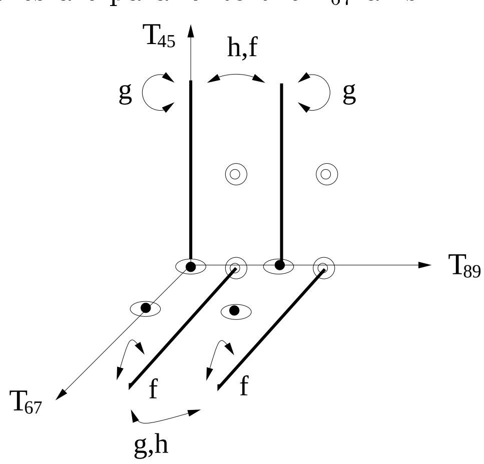

Figure 3. $D5_1$ and $D5_2$ brane configuration for the p_3 model

The gauge group is

$$[U(o) \times U(g)]_9 \times [U(8)]_{5_1} \times [U(8)]_{5_2} \ . \tag{5.48}$$

with $o + g = 16$. The massless spectrum is again not chiral and has $N = 1$ supersymmetry on the $D9$ branes and $N = 2$ supersymmetry on the two types of $D5$ branes, that are orthogonal to the direction used for the breaking. The rank reduction for the $D5$ brane gauge group is again due to the formation of brane multiplets containing brane exchanged by two of the three orbifold operations. In the decompactification limit the two sets of $D5$ branes and the corresponding $U(8) \times U(8)$ gauge group can be given a dual M-theory interpretation as Horava-Witten walls [86], as in the simpler examples discussed in [66, 81].

The second model is $w_2 p_3$ and corresponds to introducing a winding shift w_2 and a momentum shift p_3 in the table σ_2 of eq. (5.16). The brane configuration, after a T-duality along T_{45} and T_{67}, that turns the $D9$ branes into $D5'_3$ branes (dashed), the $D5_1$ branes into $D5'_2$ branes (horizontal) and the $D5_2$ branes into $D5'_1$ branes (vertical), is shown in Figure 4

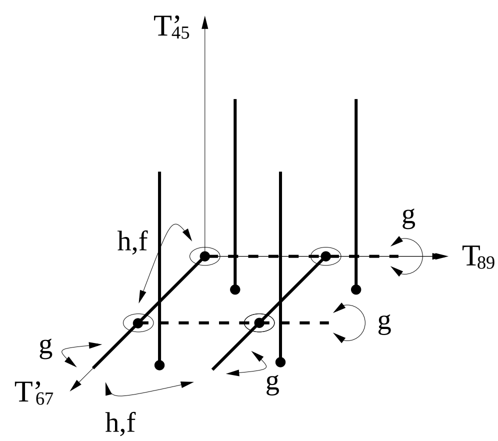

Figure 4. D_9 (dashed), $D5_1$ and $D5_2$ branes for the $w_2 p_3$ model.

The gauge group is

$$U(8)_9 \times U(8)_{5_1} \times SO(8)_{5_2} \quad . \tag{5.49}$$

The massless spectrum is again not chiral and has $N = 2$ supersymmetry on the $D9$ and $D5_1$ branes, and $N = 4$ supersymmetry on the $D5_2$ branes. The further rank reduction for the gauge group on the $D5_2$ branes and the parallel enhancement of supersymmetry are due to the fact that these branes are moved by all the orbifold operations and therefore are naturally arranged in quadruplets.

Finally we can introduce a momentum shift p_3 along the third torus, and winding shifts w_1 and w_2 along the others in the table σ_2 of eq. (5.16). The resulting model is the $w_1 w_2 p_3$ model and Figure 5 shows the brane configuration, after a T-duality along T_{45} and T_{67}, that turns the $D9$ branes into $D5'_3$ branes (dashed), the $D5_1$ branes into $D5'_2$ branes (horizontal) and the $D5_2$ branes into $D5'_1$ branes (vertical).

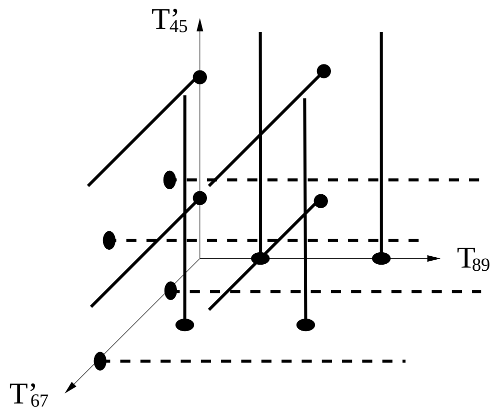

Figure 5. D_9 (dashed), $D5_1$ and $D5_2$ branes for the $w_1 w_2 p_3$ model.

All the branes are now organized in quadruplets and therefore their massless spectrum has $N = 4$ supersymmetry and gauge group

$$SO(8)_9 \times SO(8)_{5_1} \times SO(8)_{5_2} \quad . \tag{5.50}$$

The other models in Table 2 can be considered as truncations of the previous ones in the sense that some set of branes is removed since it is no more required by tadpole cancellation. We limit ourselves to give a list of the resulting maximal gauge groups compatible with local tadpole cancellation. For the models $w_1 p_2$, $w_1 p_2 p_3$ and $w_1 p_2 w_3$, which contain two set of branes, the $D9$ and the $D5_1$, we have respectively $U(8)_9 \times SO(8)_{5_1}$, $U(8)_9 \times SO(8)_{5_1}$ and $SO(8)_9 \times SO(8)_{5_1}$. For the models $p_1 p_2 p_3$, $p_1 w_2 w_3$ and $w_1 w_2 w_3$, which contain only the $D9$, we obtain the following gauge groups: $U(8)_9$ and $SO(8)_9$. The supersymmetries preserved by the various branes are as reported in Table 1.

5.3 Brane supersymmetry breaking

We conclude our discussion of supersymmetry breaking in open string models presenting two four-dimensional orientifolds whose tadpole conditions require the simultaneous presence in the vacuum of branes and anti-branes. We emphasize that in these models the types and the total numbers of D-branes present in the open descendants is determined by the structure of the closed sector and therefore the breaking of supersymmetry on the branes is directly enforced by the consistency of the model. Before presenting the four-dimensional models, let us discuss a simpler system of branes and anti-branes in ten-dimensions. The perspective is however slightly different, since in this case we start from a supersymmetric open sector and deform it allowing for the simultaneous presence of branes and antibranes of the same type. Indeed the tadpole conditions only fix the total RR charge, and although the option of saturating it by a single type of D-brane stands out as the only one compatible with space-time supersymmetry, there are no evident obstructions to considering vacuum configurations where branes and antibranes with a fixed total RR charge are simultaneously present.

We start from the type IIB torus amplitude

$$\mathcal{T} = |V_8 - S_8|^2 \quad , \tag{5.51}$$

and the usual Klein bottle projection

$$\mathcal{K} = \frac{1}{2}(V_8 - S_8) \quad . \tag{5.52}$$

In the transverse channel, the latter becomes

$$\tilde{\mathcal{K}} = \frac{2^5}{2}(V_8 - S_8) \quad , \tag{5.53}$$

and requires an open sector with a *net* number of 32 branes in order to cancel the resulting RR tadpole. Actually, both V_8 and S_8 develop tadpoles in this case, that in the usual type I model are related by supersymmetry, but are conceptually quite different. While NS-NS tadpoles result in redefinitions of the vacuum configuration, RR tadpoles signal in general genuine inconsistencies, and their presence is a symptom of the emergence of serious pathologies [14].

As stated before, one can conceive a more general construction in this case [73], allowing in the transverse-channel annulus different reflection coefficients for the V_8 and S_8 characters, so that

$$\tilde{\mathcal{A}} = \frac{2^{-5}}{2}\left[(n_+ + n_-)^2\, V_8 - (n_+ - n_-)^2\, S_8\right] \,, \tag{5.54}$$

where n_+ and n_- actually count the total numbers of D9 and D$\bar{9}$ branes. It should be appreciated how their relative minus sign in the coefficient of S_8 accounts neatly for their opposite RR charges, while they have clearly identical couplings to the graviton, consistently with the coefficient of V_8. The direct-channel annulus amplitude

$$\mathcal{A} = \frac{n_+^2 + n_-^2}{2}\,(V_8 - S_8) + n_+ n_-\,(O_8 - C_8) \tag{5.55}$$

reflects the opposite GSO projections for open strings stretched between two D-branes of the same type (99 or $\bar{9}\bar{9}$) and of different types ($9\bar{9}$ or $\bar{9}9$) [77]. While the former yields the supersymmetric type I spectrum, the latter eliminates the vector and its spinorial superpartners, while retaining the tachyon and the spinor of opposite chirality. As a result, supersymmetry is broken and an instability, signaled by the presence of the tachyonic ground state, emerges.

The Möbius amplitude

$$\mathcal{M} = \pm\frac{1}{2}(n_+ + n_-)\,\hat{V}_8 + \frac{1}{2}(n_+ - n_-)\,\hat{S}_8 \tag{5.56}$$

now involves naturally an undetermined sign for V_8, whose tadpole is generally incompatible with the one of S_8, and is to be relaxed. Together with $\mathcal{A}$, the two signs lead to symplectic or orthogonal gauge groups with S fermions in (anti)symmetric representations and tachyons and C fermions in bi-fundamentals.

In these ten-dimensional models, the only way to eliminate the tachyon consists in introducing only D9-branes. Depending on the signs in the Möbius amplitude, one thus recovers either the SO(32) superstring or the USp(32) model of [73]. On the other hand, more can be done if one compactifies the theory on some internal manifold. In this case, one can introduce Wilson lines (or, equivalently, separate the branes) in such a way that in the open strings stretched between separate 9 and $\bar{9}$ branes the tachyon actually becomes massive. It is instructive to analyze in some detail the simple case of circle compactification. As in the previous example, the Klein bottle amplitude is not affected, and is given by

$$\mathcal{K} = \frac{1}{2}\,(V_8 - S_8)\,P\ , \tag{5.57}$$

where P denotes the sum over momentum states. However, the Wilson line affects the annulus amplitude, that in the transverse-channel now reads

$$\tilde{\mathcal{A}} = \frac{2^{-5}}{2}\,\left[(n_+ + (-1)^n n_-)^2\,V_8 - (n_+ - (-1)^n n_-)^2\,S_8\right]\,W \quad , \tag{5.58}$$

where $(-1)^n W$ denotes an oscillating winding sum. As a result, in the direct channel amplitude

$$\mathcal{A} = \frac{n_+^2 + n_-^2}{2}\,(V_8 - S_8)\,P + n_+ n_-\,(O_8 - C_8)\,P^{(1/2)}\ , \tag{5.59}$$

where $P^{(1/2)}$ denotes a sum over $\frac{1}{2}$-shifted momentum states, both the tachyon and the C spinor are lifted. The open sector is completed by the Möbius amplitude

$$\mathcal{M} = \frac{1}{2}\,\left[\pm(n_+ + n_-)\,\hat{V}_8 + (n_+ - n_-)\,\hat{S}_8\right]\,P\ , \tag{5.60}$$

and at the massless level comprises gauge bosons in the adjoint of $\mathrm{SO}(n_+)\times\mathrm{SO}(n_-)$ (or $\mathrm{USp}(n_+)\times\mathrm{USp}(n_-)$, depending on the sign of $\hat{V}_8$ in $\mathcal{M}$) and S spinors in (anti)symmetric representations.

Having illustrated the main features of vacua containing both branes and anti-branes, we now turn to the discussion of the open descendants of the $T^6/Z_2\times Z_2$ [**bsb**] for the other possible choices of signs in the Klein bottle projection (5.6). In order to describe the annulus amplitude, it is convenient to introduce a compact notation.

If $T^{\rm NS}_{kl}$ ($T^{\rm R}_{kl}$) denote the NS (R) parts of the usual combinations of supersymmetric $Z_2 \times Z_2$ characters (Appendix B), we thus define

$$\tilde{T}^{(\varepsilon)}_{kl} = T^{\rm NS}_{kl} - \varepsilon T^{\rm R}_{kl} \quad , \tag{5.61}$$

where $\varepsilon = \pm 1$. Whereas $\tilde{T}^{(+)}_{kl} (= T^{(+)}_{kl})$, that in the following we simply denote by T_{kl} for the sake of brevity, form a closed set under S modular transformations, the additional combinations $\tilde{T}^{(-)}_{kl}$ with reversed RR charges, associated to the interactions between branes and antibranes, do not. As a result, the corresponding terms in $\mathcal{A}$, describing open strings stretched between branes and antibranes, contain new combinations $T^{(-)}_{kl}$, obtained from the $T^{(+)}_{kl}$ interchanging O_2 with V_2 and S_2 with C_2 in the last three factors, as explained in Appendix B. In the transverse-channel annulus amplitude, as expected, the RR part of every term describing the interaction between a brane and an antibrane has a reversed sign. Moreover, as for the shift orbifolds, the breaking terms reflect rather neatly the geometry of the brane and antibrane configuration. Indeed, the coefficient that multiplies a given twisted character is a sum of squares associated to the fixed tori of the various twisted sectors, and each square contains the breaking terms for the branes present in the fixed tori, with factors $\sqrt{v}$ if they are wrapped around them or $1/\sqrt{v}$ if they are localized on them. The relative coefficients of these terms are also directly linked to the brane geometry, and are given by (5.39). Thus, for a given twisted sector, the numerator counts the fixed tori, while the denominator counts the fixed tori where branes are actually present. Moreover, the R portions of the characters describing brane-antibrane exchanges have reverted signs also in these twisted contributions, as expected. Let us consider as an explicit example the g-twisted sector of the $(++-)$ model, that contains $\mathrm{D}\bar{5}_3$ branes. The reflection coefficients for the massless modes in τ_{gh} are

$$\frac{2^{-5}}{8}\Big[\left(N_g\sqrt{v_1} - 4D_{g;g}\sqrt{v_1} - 2D_{f;g}\frac{1}{\sqrt{v_1}} + 2D_{h;g}\frac{1}{\sqrt{v_1}}\right)^2 + 3\left(N_g\sqrt{v_1} - 2D_{f;g}\frac{1}{\sqrt{v_1}}\right)^2 + 3\left(N_g\sqrt{v_1} + 2D_{h;g}\frac{1}{\sqrt{v_1}}\right)^2 + 9N_g^2 v_1\Big] \tag{5.62}$$

for the NS portion, and

$$\frac{2^{-5}}{8}\Big[\left(N_g\sqrt{v_1} - 4D_{g;g}\sqrt{v_1} - 2D_{f;g}\frac{1}{\sqrt{v_1}} - 2D_{h;g}\frac{1}{\sqrt{v_1}}\right)^2 + 3\left(N_g\sqrt{v_1} - 2D_{f;g}\frac{1}{\sqrt{v_1}}\right)^2 + 3\left(N_g\sqrt{v_1} - 2D_{h;g}\frac{1}{\sqrt{v_1}}\right)^2 + 9N_g^2 v_1\Big] \tag{5.63}$$

for the R portion. In the previous expressions the two indices carried by $D_{i;j}$ label respectively the type of $D5$-brane and the type of orbifold projection. According to (5.39), the coefficient of N_g is $\sqrt{v_1}$, since the D9 are wrapped around all fixed tori, the coefficient of $D_{g;g}$ is $4\sqrt{v_1}$, since the $\mathrm{D}5_1$ are only wrapped around one fixed torus, while the coefficients of $D_{f;g}$ and $D_{h;g}$ are $2/\sqrt{v_1}$, since the D5_2 and D$\bar{5}_3$ are confined to four of the fixed tori. Finally, out of the 16 g-fixed tori, one sees all the branes, three see only the D9 and the D5_2, three see only the D9 and the D$\bar{5}_3$ and, finally, nine see only the D9.

Let us discuss briefly the spectra of these models. The models where only one ϵ_k is negative have discrete torsion and contain one set of D$\bar{5}$. For the D9 and the two sets of D5 branes, the gauge groups are U(8) $\times$ U(8), with $\mathcal{N}=1$ supersymmetry, while for the D$\bar{5}$ branes the gauge group is USp(8)^4, with $\mathcal{N}=0$. Moreover, the 59 and $5_k 5_l$ strings are supersymmetric, while the $9\bar{5}$ and $5_k\bar{5}$ strings are not. Also the model $(\epsilon_1, \epsilon_2, \epsilon_3) = (- - -)$ has discrete torsion but in this case there are D9 branes and three sets of D$\bar{5}$ branes. The D9 branes have $\mathcal{N}=1$ supersymmetry, with gauge group SO(8)^4 while each antibrane gives a non-supersymmetric spectrum, with gauge group U(8) $\times$ U(8). These models have chiral spectra and are free of non-abelian anomalies. Models without discrete torsion are instead not chiral, as we have already seen in details for the case $(\epsilon_1, \epsilon_2, \epsilon_3) = (+++)$. Another model without discrete torsion but with two sets of D$\bar{5}$ branes, can be obtained letting two of the ϵ_k be negative. The D9 and D5 branes give orthogonal gauge groups SO(16) with $\mathcal{N}=1$ supersymmetry, while the two sets of D$\bar{5}$ branes give symplectic gauge groups USp(16) with no supersymmetry.

Another model whose tadpole conditions do not allow for supersymmetric solutions but which can be nonetheless constructed if one introduces anti-branes is the T^6/Z_4 orbifold. Before the Ω projection, the massless sector describes $\mathcal{N}=2$ supergravity coupled to 7 vector multiplets and 32 hypermultiplets, and can thus be associated with the singular limit of a Calabi-Yau manifold with Hodge numbers $(h_{11}, h_{21}) = (31, 7)$. For this model we limit ourselves to show what is the difficulty in solving the tadpole conditions. Let us consider the Klein bottle amplitude

$$\begin{aligned} \mathcal{K} &= \frac{1}{8}\Big\{\psi_{00}\Big(P_1P_2P_3 + W_1W_2P_3\Big) + 2\,\psi_{02}\left(\frac{2\eta}{\vartheta_2}\right)^2 W_3 \\ &+ \epsilon\Big[2\times 16\,\psi_{20}\left(\frac{\eta}{\vartheta_4}\right)^2 P_3 + 2\times 4\,\psi_{22}\left(\frac{\eta}{\vartheta_3}\right)^2 W_3\Big]\Big\}\,, \end{aligned} \qquad (5.64)$$

where the ψ_{ij} are suitable characters for this model. As in the previous cases, we have inserted a sign $\epsilon = \pm 1$ in front of the twisted contributions. The choice $\epsilon = 1$ gives the usual Ω projection, while the choice $\epsilon = -1$ inverts the charge of the O5 plane and, as in the previous examples, requires the introduction of antibranes. In the transverse channel, this amplitude turns into

$$\begin{aligned} \tilde{\mathcal{K}} &= \frac{2^5}{8}\Big\{\psi_{00}\Big(v_1v_2v_3W_1^eW_2^eW_3^e + \frac{v_3}{v_1v_2}P_1^eP_2^eW_3^e\Big) + 2\,\psi_{20}\left(\frac{\eta}{\vartheta_4}\right)^2\frac{P_3^e}{v_3} \\ &+ 2\,\epsilon\,\psi_{02}\left(\frac{2\eta}{\vartheta_2}\right)^2 W_3^e v_3 - 2\,\epsilon\,\psi_{22}\left(\frac{\eta}{\vartheta_3}\right)^2\frac{P_3^e}{v_3}\Big\}\,, \end{aligned} \qquad (5.65)$$

and a closer look at the contributions at the origin of the internal lattices

$$\begin{aligned}\tilde{\mathcal{K}}_0 &= \frac{2^5}{8}\Big\{\Big(\sqrt{v_1v_2v_3}+\epsilon\sqrt{\frac{v_3}{v_1v_2}}\Big)^2(\rho_{00}+\rho_{02})+\Big(\sqrt{v_1v_2v_3}-\epsilon\sqrt{\frac{v_3}{v_1v_2}}\Big)^2(\rho_{01}+\rho_{03})\\&+ \frac{4}{v_3}\frac{1}{2}\left[(1-\epsilon)(\rho_{20}+\rho_{22})+(1+\epsilon)(\rho_{21}+\rho_{23})\right]\Big\}\end{aligned} \tag{5.66}$$

reveals the presence of a term proportional to $1/v_3$. For $\epsilon = 1$, this gives rise to a massless tadpole that can not be canceled by the annulus and Möbius contributions. For $\epsilon = -1$, however, this term becomes massive, and one can complete the construction of the open descendants without any further difficulties [**bsb**]. In this case, the projected massless closed spectrum is $\mathcal{N} = 1$ supergravity coupled to 6 vector multiplets and 33 chiral multiplets. The open sector contains $D9$ with $\mathcal{N} = 1$ supersymmetry and non-supersymmetric $\bar{D}5$ branes with gauge group $\mathrm{SO}(8)_9 \times \mathrm{SO}(8)_9 \times \mathrm{U}(8)_9 \times \mathrm{USp}(8)_{\bar{5}} \times \mathrm{USp}(8)_{\bar{5}} \times \mathrm{U}(8)_{\bar{5}}$. The spectrum is chiral, but is free of irreducible gauge anomalies.

Appendix A

Characters of the T-O-I models

A.0.1 Octahedron

Untwisted sector

$$\begin{aligned}
u_+ &= \tfrac{1}{24}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \tfrac{3}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix} \ , && h = 0 \ , \\
u_- &= \tfrac{1}{24}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} - \tfrac{1}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix} \ , && h = 9 \ , \\
u_f &= \tfrac{1}{12}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] - \tfrac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix} \ , && h = 4 \ , \\
j_+ &= \tfrac{1}{8}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix} \ , && h = 1 \ , \\
j_- &= \tfrac{1}{8}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] - \tfrac{3}{8}\theta\begin{bmatrix}0\\1/2\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}0\\1/4\end{bmatrix} \ , && h = 4 \ , \\
\phi_+ &= \tfrac{1}{12}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \tfrac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}1/2\\1/4\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}1/2\\3/4\end{bmatrix} \ , && h = \tfrac{1}{4} \ , \\
\phi_- &= \tfrac{1}{12}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \tfrac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}1/2\\1/4\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}1/2\\3/4\end{bmatrix} \ , && h = \tfrac{25}{4} \ , \\
\phi_f &= \tfrac{1}{6}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \tfrac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} \ , && h = \tfrac{9}{4}
\end{aligned} \tag{A.1}$$

$\mathbb{Z}_2$-twisted sector

$$\begin{aligned}
\mu_0 &= \tfrac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \tfrac{1}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} \ , && h = \tfrac{1}{16} \ , \\
\mu_1 &= \tfrac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \tfrac{1}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} \ , && h = \tfrac{9}{16} \ ,
\end{aligned} \tag{A.2}$$

$\mathbb{Z}_3$-twisted sector $(i = 0, 1, 2)$

$$\begin{aligned}
\omega_i &= \tfrac{1}{3}\theta\begin{bmatrix}1/3\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/3\\2/3\end{bmatrix} \ , && h = \tfrac{1}{9}, \tfrac{4}{9}, \tfrac{16}{9} \ , \\
\theta_i &= \tfrac{1}{3}\theta\begin{bmatrix}1/6\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/6\\2/3\end{bmatrix} \ , && h = \tfrac{1}{36}, \tfrac{25}{36}, \tfrac{49}{36} \ ,
\end{aligned} \tag{A.3}$$

$\mathbb{Z}_4$-twisted sector $(k = 0, 1, 2, 3)$

$$\begin{aligned}
\alpha_k &= \tfrac{1}{4}\theta\begin{bmatrix}1/8\\0\end{bmatrix} + \tfrac{i^k}{4}\theta\begin{bmatrix}1/8\\1/4\end{bmatrix} + \tfrac{(-1)^k}{4}\theta\begin{bmatrix}1/8\\1/2\end{bmatrix} + \tfrac{(-i)^k}{4}\theta\begin{bmatrix}1/8\\3/4\end{bmatrix} \ , && h = \tfrac{1}{64}, \tfrac{49}{64}, \tfrac{225}{64}, \tfrac{81}{64} \ , \\
\beta_k &= \tfrac{1}{4}\theta\begin{bmatrix}3/8\\0\end{bmatrix} + \tfrac{i^k}{4}\theta\begin{bmatrix}3/8\\1/4\end{bmatrix} + \tfrac{(-1)^k}{4}\theta\begin{bmatrix}3/8\\1/2\end{bmatrix} + \tfrac{(-i)^k}{4}\theta\begin{bmatrix}3/8\\3/4\end{bmatrix} \ , && h = \tfrac{9}{64}, \tfrac{65}{64}, \tfrac{169}{64}, \tfrac{121}{64} \ , \\
\sigma_+ &= \tfrac{1}{4}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}1/4\\3/4\end{bmatrix} \ , && h = \tfrac{1}{16} \ , \\
\sigma_- &= \tfrac{1}{4}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}1/4\\1/4\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}1/4\\3/4\end{bmatrix} \ , && h = \tfrac{1}{16} \ , \\
\tau_+ &= \tfrac{1}{4}\theta\begin{bmatrix}1/4\\0\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + \tfrac{i}{4}\theta\begin{bmatrix}1/4\\1/4\end{bmatrix} - \tfrac{i}{4}\theta\begin{bmatrix}1/4\\3/4\end{bmatrix} \ , && h = \tfrac{1}{16} \ , \\
\tau_- &= \tfrac{1}{4}\theta\begin{bmatrix}1/4\\0\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} - \tfrac{i}{4}\theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + \tfrac{i}{4}\theta\begin{bmatrix}1/4\\3/4\end{bmatrix} \ , && h = \tfrac{1}{16} \ ,
\end{aligned} \tag{A.4}$$

A.0.2 Icosahedron

Untwisted sector

$$\begin{aligned}
u_0 &= \tfrac{1}{60}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \tfrac{1}{5}\theta\begin{bmatrix}0\\1/5\end{bmatrix} + \tfrac{1}{5}\theta\begin{bmatrix}0\\2/5\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix} \ , && h = 0 \ , \\
u_1 &= \tfrac{1}{20}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1+\sqrt{5}}{10}\theta\begin{bmatrix}0\\1/5\end{bmatrix} + \tfrac{1-\sqrt{5}}{10}\theta\begin{bmatrix}0\\2/5\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix} \ , && h = 1 \ , \\
u_2 &= \tfrac{1}{20}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1-\sqrt{5}}{10}\theta\begin{bmatrix}0\\1/5\end{bmatrix} + \tfrac{1+\sqrt{5}}{10}\theta\begin{bmatrix}0\\2/5\end{bmatrix} - \tfrac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix} && h = 9 \ , \\
u_3 &= \tfrac{1}{15}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \tfrac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} - \tfrac{1}{5}\theta\begin{bmatrix}0\\1/5\end{bmatrix} - \tfrac{1}{5}\theta\begin{bmatrix}0\\2/5\end{bmatrix} \ , && h = 9 \ , \\
u_4 &= \tfrac{1}{15}\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] - \tfrac{1}{3}\theta\begin{bmatrix}0\\1/3\end{bmatrix} + \tfrac{1}{4}\theta\begin{bmatrix}0\\1/2\end{bmatrix} \ , && h = 4 \ , \\
\phi_1 &= \tfrac{1}{30}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \tfrac{1+\sqrt{5}}{10}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} - \tfrac{1-\sqrt{5}}{10}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} - \tfrac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} \ , && h = \tfrac{1}{4} \ , \\
\phi_2 &= \tfrac{1}{30}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \tfrac{1-\sqrt{5}}{10}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} - \tfrac{1+\sqrt{5}}{10}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} - \tfrac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} \ , && h = \tfrac{49}{4} \ , \\
\phi_3 &= \tfrac{1}{15}\theta\begin{bmatrix}1/2\\0\end{bmatrix} - \tfrac{1}{5}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} - \tfrac{1}{5}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} + \tfrac{\overline{\omega}}{3}\theta\begin{bmatrix}1/2\\1/3\end{bmatrix} \ , && h = \tfrac{9}{4} \ , \\
\phi_4 &= \tfrac{1}{10}\theta\begin{bmatrix}1/2\\0\end{bmatrix} + \tfrac{1}{5}\overline{\zeta}^2\theta\begin{bmatrix}1/2\\1/5\end{bmatrix} + \tfrac{1}{5}\zeta\theta\begin{bmatrix}1/2\\2/5\end{bmatrix} \ , && h = \tfrac{25}{4} \ ,
\end{aligned} \tag{A.5}$$

$\mathbb{Z}_2$-twisted sector

$$\begin{aligned}
\sigma &= \tfrac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \tfrac{1}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} \ , && h = \tfrac{1}{16} \ , \\
\tau &= \tfrac{1}{2}\theta\begin{bmatrix}1/4\\0\end{bmatrix} + \tfrac{1}{2}\theta\begin{bmatrix}1/4\\1/2\end{bmatrix} \ , && h = \tfrac{9}{16} \ ,
\end{aligned} \tag{A.6}$$

$\mathbb{Z}_3$-twisted sector $(i = 0, 1, 2)$

$$\begin{aligned}
\omega_i &= \tfrac{1}{3}\theta\begin{bmatrix}1/3\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/3\\2/3\end{bmatrix} \ , && h = \tfrac{1}{9}, \tfrac{4}{9}, \tfrac{16}{9} \ , \\
\theta_i &= \tfrac{1}{3}\theta\begin{bmatrix}1/6\\0\end{bmatrix} + \tfrac{\omega^i}{3}\theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \tfrac{\overline{\omega}^i}{3}\theta\begin{bmatrix}1/6\\2/3\end{bmatrix} \ , && h = \tfrac{1}{36}, \tfrac{25}{36}, \tfrac{49}{36} \ ,
\end{aligned} \tag{A.7}$$

$\mathbb{Z}_5$-twisted sector $(k = 0, 1, 2, 3, 4)$

$$\begin{aligned}
\pi_k &= \tfrac{1}{5}\theta\begin{bmatrix}1/5\\0\end{bmatrix} + \tfrac{\zeta^k}{5}\theta\begin{bmatrix}1/5\\1/5\end{bmatrix} + \tfrac{\zeta^{2k}}{5}\theta\begin{bmatrix}1/5\\2/5\end{bmatrix} + \tfrac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}1/5\\3/5\end{bmatrix} + \tfrac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}1/5\\4/5\end{bmatrix} \ , && h = \tfrac{1}{25}, \tfrac{16}{25}, \tfrac{81}{25}, \tfrac{121}{25}, \tfrac{36}{25} \ , \\
\rho_k &= \tfrac{1}{5}\theta\begin{bmatrix}1/10\\0\end{bmatrix} + \tfrac{\zeta^k}{5}\theta\begin{bmatrix}1/10\\1/5\end{bmatrix} + \tfrac{\zeta^{2k}}{5}\theta\begin{bmatrix}1/10\\2/5\end{bmatrix} + \tfrac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}1/10\\3/5\end{bmatrix} + \tfrac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}1/10\\4/5\end{bmatrix} \ , && h = \tfrac{1}{100}, \tfrac{81}{100}, \tfrac{361}{100}, \tfrac{441}{100}, \tfrac{121}{100} \ , \\
\lambda_k &= \tfrac{1}{5}\theta\begin{bmatrix}2/5\\0\end{bmatrix} + \tfrac{\zeta^k}{5}\theta\begin{bmatrix}2/5\\1/5\end{bmatrix} + \tfrac{\zeta^{2k}}{5}\theta\begin{bmatrix}2/5\\2/5\end{bmatrix} + \tfrac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}2/5\\3/5\end{bmatrix} + \tfrac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}2/5\\4/5\end{bmatrix} \ , && h = \tfrac{4}{25}, \tfrac{9}{25}, \tfrac{64}{25}, \tfrac{144}{25}, \tfrac{49}{25} \ , \\
\xi_k &= \tfrac{1}{5}\theta\begin{bmatrix}3/10\\0\end{bmatrix} + \tfrac{\zeta^k}{5}\theta\begin{bmatrix}3/10\\1/5\end{bmatrix} + \tfrac{\zeta^{2k}}{5}\theta\begin{bmatrix}3/10\\2/5\end{bmatrix} + \tfrac{\overline{\zeta}^{2k}}{5}\theta\begin{bmatrix}3/10\\3/5\end{bmatrix} + \tfrac{\overline{\zeta}^k}{5}\theta\begin{bmatrix}3/10\\4/5\end{bmatrix} \ , && h = \tfrac{9}{100}, \tfrac{49}{100}, \tfrac{289}{100}, \tfrac{529}{100}, \tfrac{169}{100} \ ,
\end{aligned}$$

where $\zeta = \exp(2i\pi/5)$.

The S matrix in this basis is reported in Table A.1

	u_0	u_1	u_2	u_3	u_4	ϕ_1	ϕ_2	ϕ_3	ϕ_4	σ	τ	ω_j	θ_j	π_l	ρ_l	λ_l	ξ_l
u_0	1	3	3	4	5	2	2	4	6	30	30	20	20	12	12	12	12
u_1	3	9	9	12	15	6	6	12	18	-30	-30	0	0	$\widetilde{g}$	g	g	$\widetilde{g}$
u_2	3	9	9	12	15	6	6	12	18	-30	-30	0	0	g	$\widetilde{g}$	$\widetilde{g}$	g
u_3	4	12	12	16	20	8	8	16	24	0	0	20	20	-12	-12	-12	-12
u_4	5	15	15	20	25	10	10	20	30	30	30	-20	-20	0	0	0	0
ϕ_1	2	6	6	8	10	-4	-4	-8	-12	0	0	-20	20	$-g$	$\widetilde{g}$	$-\widetilde{g}$	g
ϕ_2	2	6	6	8	10	-4	-4	-8	-12	0	0	-20	20	$-\widetilde{g}$	g	$-g$	$\widetilde{g}$
ϕ_3	4	12	12	16	20	-8	-8	-16	-24	0	0	20	-20	-12	12	-12	12
ϕ_4	6	18	18	24	30	-12	-12	-24	-36	0	0	0	0	12	-12	12	-12
σ	30	-30	-30	0	30	0	0	0	0	$30\sqrt{2}$	$-30\sqrt{2}$	0	0	0	0	0	0
τ	30	-30	-30	0	30	0	0	0	0	$-30\sqrt{2}$	$30\sqrt{2}$	0	0	0	0	0	0
ω_i	20	0	0	20	-20	-20	-20	20	0	0	0	a_{ij}	b_{ij}	0	0	0	0
θ_i	20	0	0	20	-20	-20	-20	20	0	0	0	b_{ji}	d_{ij}	0	0	0	0
π_k	12	$\widetilde{g}$	g	-12	0	$-g$	$-\widetilde{g}$	-12	12	0	0	0	0	pp_{kl}	pr_{kl}	pl_{kl}	px_{kl}
ρ_k	12	g	$\widetilde{g}$	-12	0	$\widetilde{g}$	g	12	-12	0	0	0	0	pr_{lk}	rr_{kl}	rl_{kl}	rx_{kl}
λ_k	12	g	$\widetilde{g}$	-12	0	$-\widetilde{g}$	$-g$	-12	12	0	0	0	0	pl_{lk}	rl_{lk}	ll_{kl}	lx_{kl}
ξ_k	12	$\widetilde{g}$	g	-12	0	g	$\widetilde{g}$	12	-12	0	0	0	0	px_{lk}	rx_{lk}	lx_{lk}	xx_{kl}

The submatrices are defined as follows:

$$pp_{kl} = \mathrm{Re}\,\left(e^{-\frac{4i\pi}{25}}\zeta^{2(k+l)}\right)\,, \quad pr_{kl} = \mathrm{Re}\,\left(e^{-\frac{2i\pi}{25}}\zeta^{k+2l}\right)\,, \quad pl_{kl} = \mathrm{Re}\,\left(e^{-\frac{8i\pi}{25}}\zeta^{2l-k}\right)\,,$$
$$px_{kl} = \mathrm{Re}\,\left(e^{-\frac{6i\pi}{25}}\zeta^{2(l-k)}\right)\,, \quad rr_{kl} = \mathrm{Re}\,\left(e^{-\frac{i\pi}{25}}\zeta^{k+l}\right)\,, \quad rl_{kl} = \mathrm{Re}\,\left(e^{-\frac{4i\pi}{25}}\zeta^{2(k+l)}\right)\,,$$
$$rx_{kl} = \mathrm{Re}\,\left(e^{-\frac{4i\pi}{25}}\zeta^{2(k+l)}\right)\,, \quad ll_{kl} = \mathrm{Re}\,\left(e^{-\frac{4i\pi}{25}}\zeta^{2(k+l)}\right)\,, \quad lx_{kl} = \mathrm{Re}\,\left(e^{-\frac{4i\pi}{25}}\zeta^{2(k+l)}\right)\,,$$
$$xx_{kl} = \mathrm{Re}\,\left(e^{-\frac{4i\pi}{25}}\zeta^{2(k+l)}\right)\,.$$

Table A.1: Icosahedron S matrix ($\times 60\sqrt{2}$)

A.0.3 Super-Tetrahedron

The characters of this orbifold model can be expressed in terms of the Θ functions (4.14) and of the three Ising characters o, v and s. They can be divided into classes as follows.

Untwisted NS sector $(i = 0, 1, 2)$

$$\begin{aligned}
\chi_i &= \tfrac{1}{3}ooo + \tfrac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\2/3\end{bmatrix}\right)(o+v) + \tfrac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\1/6\end{bmatrix}\right)(o-v)\ ,\\
h &= 0, 2, 2\ ,\\
\chi_- &= \tfrac{1}{3}(ovv + vov + vvo)\ , \qquad h = 1\ ,\\
\xi_i &= \tfrac{1}{3}vvv + \tfrac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\2/3\end{bmatrix}\right)(o+v) - \tfrac{1}{6\eta}\left(\omega^i\Theta\begin{bmatrix}0\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}0\\1/6\end{bmatrix}\right)(o-v)\\
h &= \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{5}{2}\ ,\\
\xi_- &= \tfrac{1}{3}(voo + ovo + oov)\ , \qquad = \tfrac{1}{2}\ ,
\end{aligned} \tag{A.8}$$

Untwisted R sector $(i = 0, 1, 2)$

$$\rho_i = \tfrac{1}{3}sss - \tfrac{1}{3\eta}\left(\omega^i\Theta\begin{bmatrix}1/2\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/2\\2/3\end{bmatrix}\right)s\ , \qquad h = \tfrac{19}{16}, \tfrac{19}{16}, \tfrac{3}{16}\ , \tag{A.9}$$

$\mathbb{Z}_2$-twisted NS sector: $\lambda s = \{\sigma_s, \tau_s\}$

$$\begin{aligned}
\sigma_s &= \tfrac{1}{3}(oss + sos + sso)\ , \qquad h = \tfrac{1}{8}\ ,\\
\tau_s &= \tfrac{1}{3}(vss + svs + ssv)\ , \qquad h = \tfrac{5}{8}\ ,
\end{aligned} \tag{A.10}$$

$\mathbb{Z}_2$-twisted R sector $\{\sigma_I, \tau_I | I = o, v\}$,

$$\begin{aligned}
\sigma_o &= \tfrac{1}{3}(soo + oso + oos)\ , \qquad h = \tfrac{1}{16}\ ,\\
\sigma_v &= \tfrac{1}{3}(osv + svo + vos)\ , \qquad h = \tfrac{9}{16}\ ,\\
\tau_o &= \tfrac{1}{3}(vso + sov + ovs)\ , \qquad h = \tfrac{9}{16}\ ,\\
\tau_v &= \tfrac{1}{3}(svv + vsv + vvs)\ , \qquad h = \tfrac{17}{16}\ ,
\end{aligned} \tag{A.11}$$

$\mathbb{Z}_3$-twisted NS sector $(i = 0, 1, 2)$

$$\begin{aligned}
\omega_i^{\pm} &= \tfrac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \overline{w}^i\Theta\begin{bmatrix}1/3\\2/3\end{bmatrix}\right)(o+v) \pm \tfrac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\1/2\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\1/6\end{bmatrix}\right)(o-v)\ ,\\
h &= \tfrac{1}{18}, \tfrac{13}{18}, \tfrac{25}{18}, \tfrac{10}{18}, \tfrac{4}{18}, \tfrac{16}{18}\ ,\\
\pi_i^{\pm} &= \tfrac{1}{6\eta}\left(\Theta\begin{bmatrix}2/3\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}2/3\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}2/3\\2/3\end{bmatrix}\right)(o+v) \pm \tfrac{1}{6\eta}\left(\Theta\begin{bmatrix}2/3\\1/2\end{bmatrix} + \omega^i\Theta\begin{bmatrix}2/3\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}2/3\\1/6\end{bmatrix}\right)(o-v)\ ,\\
h &= \tfrac{4}{18}, \tfrac{16}{18}, \tfrac{10}{18}, \tfrac{13}{18}, \tfrac{25}{18}, \tfrac{1}{18}\ ,
\end{aligned} \tag{A.12}$$

$\mathbb{Z}_3$-twisted R sector $(i = 0, 1, 2)$

$$\begin{aligned}
\lambda_i &= \tfrac{1}{3\eta}\left(\Theta\begin{bmatrix}1/6\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/6\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/6\\2/3\end{bmatrix}\right)s\ ,\ , \qquad h = \tfrac{11}{144}, \tfrac{107}{144}, \tfrac{59}{144}\ ,\\
\psi_i &= \tfrac{1}{3\eta}\left(\Theta\begin{bmatrix}5/6\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}5/6\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}5/6\\2/3\end{bmatrix}\right)s\ , \qquad h = \tfrac{59}{144}, \tfrac{11}{144}, \tfrac{107}{144}\ ,
\end{aligned} \tag{A.13}$$

The S matrix in this basis is reported in Table A.2.

	χ_j	χ_-	ξ_j	ξ_-	ρ_j	μs	σ_J	τ_J	$\omega_l^{\pm}$	$\pi_l^{\pm}$	λ_j	ψ_j
χ_i	1	3	1	3	$2\sqrt{2}$	6	$3\sqrt{2}$	$3\sqrt{2}$	$4\omega^i$	$4\overline{\omega}^i$	$4\sqrt{2}\omega^i$	$4\sqrt{2}\overline{\omega}^i$
χ_-	3	9	3	9	$6\sqrt{2}$	-6	$-3\sqrt{2}$	$-3\sqrt{2}$	0	0	0	0
ξ_i	1	3	1	3	$-2\sqrt{2}$	6	$-3\sqrt{2}$	$-3\sqrt{2}$	$4\omega^i$	$4\overline{\omega}^i$	$-4\sqrt{2}\omega^i$	$-4\sqrt{2}\overline{\omega}^i$
ξ_-	3	9	3	9	$-6\sqrt{2}$	-6	$3\sqrt{2}$	$3\sqrt{2}$	0	0	0	0
ρ_i	$2\sqrt{2}$	$6\sqrt{2}$	$-2\sqrt{2}$	$-6\sqrt{2}$	0	0	0	0	$\pm 4\sqrt{2}\omega^{i+1}$	$\mp 4\sqrt{2}\overline{\omega}^{i+i}$	0	0
λs	6	-6	6	-6	0	0	$-6\sqrt{2}\epsilon_{\lambda J}$	$6\sqrt{2}\epsilon_{\lambda J}$	0	0	0	0
σ_I	$3\sqrt{2}$	$-3\sqrt{2}$	$-3\sqrt{2}$	$3\sqrt{2}$	0	$-6\sqrt{2}\epsilon_{I\mu}$	$12\delta_{IJ}$	$12(1-\delta_{IJ})$	0	0	0	0
τ_I	$3\sqrt{2}$	$-3\sqrt{2}$	$-3\sqrt{2}$	$3\sqrt{2}$	0	$6\sqrt{2}\epsilon_{I\mu}$	$12(1-\delta_{IJ})$	$12\delta_{IJ}$	0	0	0	0
$\omega_i^{\pm}$	$4\omega^j$	0	$4\omega^j$	0	$\pm 4\sqrt{2}\omega^{j+1}$	0	0	0	$4\overline{\alpha}\omega^{i+j}$	$4\overline{\alpha}^2\overline{w}^{i+j}$	$\pm 4\sqrt{2}\beta w^{i+j-1}$	$\pm 4\sqrt{2\beta}\overline{w}^{i+j}$
$\pi_i^{\pm}$	$4\overline{\omega}^j$	0	$4\omega^j$	0	$\mp 4\sqrt{2}\overline{\omega}^{j+1}$	0	0	0	$4\overline{\alpha}^2\overline{w}^{i+j}$	$4\overline{\alpha}^4\omega^{i+j}$	$\pm 4\sqrt{2}\beta^8\overline{w}^{i+j}$	$\pm 4\sqrt{2\beta}^2\omega^{i+j}$
λ_i	$4\sqrt{2}\omega^j$	0	$-4\sqrt{2}\omega^j$	0	0	0	0	0	$\pm 4\sqrt{2}\beta w^{i+j-1}$	$\pm 4\sqrt{2}\beta^8\overline{w}^{i+j}$	0	0
ψ_i	$4\sqrt{2}\omega^j$	0	$-4\sqrt{2}\overline{\omega}^j$	0	0	0	0	0	$\pm 4\sqrt{2\beta}\overline{w}^{i+j}$	$\pm 4\sqrt{2\beta}^2\omega^{i+j}$	0	0

Table A.2: Super-tetrahedron S matrix ($\times 24$): $\epsilon_{ij} = (-1)^{i+j}$, $\omega = \exp(2i\pi/3)$, $\alpha = \exp(2i\pi/9)$ and $\beta = \exp(i\pi/9)$.

A.0.4 Super-Octahedron

Untwisted NS sector

$$\begin{aligned}
u_\pm &= \tfrac{1}{6}ooo \pm \tfrac{1}{2}o(2\tau)o + \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) + \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\ , && h=0,5\ ,\\
u &= \tfrac{1}{3}ooo - \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) - \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\ , && h=2\ ,\\
j_\pm &= \tfrac{1}{6}(ovv+vov+vvo) \pm \tfrac{1}{2}v(2\tau)o\ , && h=1,2\ ,\\
v_\pm &= \tfrac{1}{6}vvv \pm \tfrac{1}{2}v(2\tau)v + \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) - \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\ , && h=\tfrac{3}{2},\tfrac{9}{2}\ ,\\
v &= \tfrac{1}{3}vvv - \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\1/3\end{bmatrix}(o+v) + \tfrac{1}{6\eta}\Theta\begin{bmatrix}0\\5/6\end{bmatrix}(o-v)\ , && h=\tfrac{5}{2}\ ,\\
h_\pm &= \tfrac{1}{6}(voo+ovo+oov) \pm \tfrac{1}{2}o(2\tau)v\ , && h=\tfrac{1}{2},\tfrac{5}{2}\ ,
\end{aligned} \tag{A.14}$$

Untwisted R sector

$$\begin{aligned}
\rho_\pm &= \tfrac{1}{6}sss \pm \tfrac{1}{2}s(2\tau)s - \tfrac{\bar\omega}{3\eta}\Theta\begin{bmatrix}1/2\\1/3\end{bmatrix}s\ , && h=\tfrac{3}{16},\tfrac{51}{16}\ ,\\
\rho &= \tfrac{1}{3}sss + \tfrac{\bar\omega}{3\eta}\Theta\begin{bmatrix}1/2\\1/3\end{bmatrix}s\ , && h=\tfrac{19}{16},
\end{aligned} \tag{A.15}$$

$\mathbb{Z}_3$-twisted NS sector $(i=0,1,2)$

$$\begin{aligned}
\omega_i^\pm &= \tfrac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\1/3\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\2/3\end{bmatrix}\right)(o+v) \pm \tfrac{1}{6\eta}\left(\Theta\begin{bmatrix}1/3\\1/2\end{bmatrix} + \omega^i\Theta\begin{bmatrix}1/3\\5/6\end{bmatrix} + \overline{\omega}^i\Theta\begin{bmatrix}1/3\\1/6\end{bmatrix}\right)(o-v)\ ,\\
h &= \tfrac{1}{18},\tfrac{13}{18},\tfrac{25}{18},\tfrac{10}{18},\tfrac{4}{18},\tfrac{16}{18}\ ,
\end{aligned} \tag{A.16}$$

$\mathbb{Z}_3$-twisted R sector $(i=0,1,2)$

$$\lambda_i = \tfrac{1}{3\eta}\left(\Theta\begin{bmatrix}5/6\\0\end{bmatrix} + \omega^i\Theta\begin{bmatrix}5/6\\1/3\end{bmatrix} + \overline{w}^i\Theta\begin{bmatrix}5/6\\2/3\end{bmatrix}\right)s\ , \qquad h=\tfrac{59}{144},\tfrac{11}{144},\tfrac{107}{144}\ , \tag{A.17}$$

$\mathbb{Z}_4$-twisted NS sector: $\lambda_{s\pm}=\{\sigma_{s\pm},\tau_{s\pm}\}$ $(k=0,1,2,3)$

$$\begin{aligned}
\sigma_{s\pm} &= \tfrac{1}{2}(sso \pm s(2\tau)o)\ , && h=\tfrac{1}{8},\tfrac{9}{8}\ ,\\
\tau_{s\pm} &= \tfrac{1}{2}(ssv \pm s(2\tau)v)\ , && h=\tfrac{5}{8},\tfrac{13}{8}\ ,\\
\alpha_k &= \tfrac{1}{4\eta}\left(\Theta\begin{bmatrix}1/4\\0\end{bmatrix} + i^k\Theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + (-1)^k\Theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + (-i)^k\Theta\begin{bmatrix}1/4\\3/4\end{bmatrix}\right)o\ , && h=\tfrac{1}{32},\tfrac{9}{32},\tfrac{49}{32},\tfrac{25}{32}\ ,\\
\beta_k &= \tfrac{1}{4\eta}\left(\Theta\begin{bmatrix}1/4\\0\end{bmatrix} + i^k\Theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + (-1)^k\Theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + (-i)^k\Theta\begin{bmatrix}1/4\\3/4\end{bmatrix}\right)v\ , && h=\tfrac{17}{32},\tfrac{25}{32},\tfrac{65}{32},\tfrac{41}{32}\ ,
\end{aligned} \tag{A.18}$$

$\mathbb{Z}_4$-twisted R sector $(I=o,v;\ k=0,1,2,3)$

$$\begin{aligned}
\sigma_{o\pm} &= \tfrac{1}{2}(soo \pm o(2\tau)s)\ , && h=\tfrac{1}{16},\tfrac{33}{16}\ ,\\
\tau_{v\pm} &= \tfrac{1}{2}(svv \pm v(2\tau)s)\ , && h=\tfrac{17}{16},\tfrac{33}{16}\ ,\\
osv &= osv\ , && h=\tfrac{9}{16}\ ,\\
\gamma_k &= \tfrac{1}{4\eta}\left(\Theta\begin{bmatrix}1/4\\0\end{bmatrix} + i^k\Theta\begin{bmatrix}1/4\\1/4\end{bmatrix} + (-1)^k\Theta\begin{bmatrix}1/4\\1/2\end{bmatrix} + (-i)^k\Theta\begin{bmatrix}1/4\\3/4\end{bmatrix}\right)s\ , && h=\tfrac{3}{32},\tfrac{11}{32},\tfrac{51}{32},\tfrac{27}{32}\ ,
\end{aligned} \tag{A.19}$$

$\mathbb{Z}_2$-twisted NS sector $(i=0,1)$

$$\begin{aligned}
\mu_{0s} &= \tfrac{1}{3}(oss+sos+sso)\ , && h=\tfrac{1}{8}\ ,\\
\mu_{1s} &= \tfrac{1}{3}(vss+svs+ssv)\ , && h=\tfrac{5}{8}\ ,
\end{aligned} \tag{A.20}$$

$\mathbb{Z}_2$-twisted R sector ($I = o, v$, $i = 0, 1$)

$$\begin{aligned}
\mu_{0o} &= \tfrac{1}{3}(soo + oso + oos) \ , \qquad & h &= \tfrac{1}{16} \ , \\
\mu_{0v} &= \tfrac{1}{3}(osv + svo + vos) \ , \qquad & h &= \tfrac{9}{16} \ , \\
\mu_{1o} &= \tfrac{1}{3}(vso + sov + ovs) \ , \qquad & h &= \tfrac{9}{16} \ , \\
\mu_{1v} &= \tfrac{1}{3}(svv + vsv + vvs) \ , \qquad & h &= \tfrac{17}{16} \ ,
\end{aligned} \tag{A.21}$$

The S matrix in this basis is reported in Table A.3.

	$u_\pm$	u	$j_\pm$	$v_\pm$	v	$h_\pm$	$\rho_\pm$	ρ	$\sigma_{s\pm}$	$\tau_{s\pm}$	$\sigma_{I\pm}$	osv	ω_j^β	λ_j	α_l	β_l	γ_l	μ_{js}	μ_{jJ}
$u_\pm$	1	2	3	1	2	3	$2\sqrt{2}$	$4\sqrt{2}$	6	6	$3\sqrt{2}$	$6\sqrt{2}$	8	$8\sqrt{2}$	± 6	± 6	$\pm 6\sqrt{2}$	± 12	$\pm 6\sqrt{2}$
u	2	4	6	2	4	6	$4\sqrt{2}$	$8\sqrt{2}$	12	12	$6\sqrt{2}$	$12\sqrt{2}$	-8	$-8\sqrt{2}$	0	0	0	0	0
$j_\pm$	3	6	9	3	6	9	$6\sqrt{2}$	$12\sqrt{2}$	-6	-6	$-3\sqrt{2}$	$-6\sqrt{2}$	0	0	± 6	± 6	$\pm 6\sqrt{2}$	∓ 12	$\mp 6\sqrt{2}$
$v_\pm$	1	2	3	1	2	3	$-2\sqrt{2}$	$-4\sqrt{2}$	6	6	$-3\sqrt{2}$	$-6\sqrt{2}$	8	$-8\sqrt{2}$	± 6	± 6	$\mp 6\sqrt{2}$	± 12	$\mp 6\sqrt{2}$
v	2	4	6	2	4	6	$-4\sqrt{2}$	$-8\sqrt{2}$	12	12	$-6\sqrt{2}$	$-12\sqrt{2}$	-8	$8\sqrt{2}$	0	0	0	0	0
$h_\pm$	3	6	9	3	6	9	$-6\sqrt{2}$	$-12\sqrt{2}$	-6	-6	$3\sqrt{2}$	$6\sqrt{2}$	0	0	± 6	± 6	$\mp 6\sqrt{2}$	∓ 12	$\pm 6\sqrt{2}$
$\rho_\pm$	$2\sqrt{2}$	$4\sqrt{2}$	$6\sqrt{2}$	$-2\sqrt{2}$	$-4\sqrt{2}$	$-6\sqrt{2}$	0	0	0	0	0	0	$(-1)^\beta 8\sqrt{2}$	0	$\mp(-1)^l 12$	$\pm(-1)^l 12$	0	0	0
ρ	$4\sqrt{2}$	$8\sqrt{2}$	$12\sqrt{2}$	$-4\sqrt{2}$	$-8\sqrt{2}$	$-12\sqrt{2}$	0	0	0	0	0	0	$-(-1)^\beta 8\sqrt{2}$	0	0	0	0	0	0
$\sigma_{s\pm}$	6	12	-6	6	12	-6	0	0	0	0	$6\sqrt{2}$	$-12\sqrt{2}$	0	0	$\mp(-1)^l 6\sqrt{2}$	$\mp(-1)^l 6\sqrt{2}$	$\mp(-1)^l 12$	0	0
$\tau_{s\pm}$	6	12	-6	6	12	-6	0	0	0	0	$-6\sqrt{2}$	$12\sqrt{2}$	0	0	$\mp(-1)^l 6\sqrt{2}$	$\mp(-1)^l 6\sqrt{2}$	$\pm(-1)^l 12$	0	0
$\sigma_{I\pm}$	$3\sqrt{2}$	$6\sqrt{2}$	$-3\sqrt{2}$	$-3\sqrt{2}$	$-6\sqrt{2}$	$3\sqrt{2}$	0	0	$6\sqrt{2}$	$-6\sqrt{2}$	$(-1)^{I+J} 12$	0	0	0	$\pm 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	0	$\pm(-1)^{I+J}$
osv	$6\sqrt{2}$	$12\sqrt{2}$	$-6\sqrt{2}$	$-6\sqrt{2}$	$-12\sqrt{2}$	$6\sqrt{2}$	0	0	$-12\sqrt{2}$	$12\sqrt{2}$	0	0	0	0	0	0	0	0	0
ω_i^α	8	-8	0	8	-8	0	$(-1)^\alpha 8\sqrt{2}$	$-(-1)^\alpha 8\sqrt{2}$	0	0	0	0	ww_{ij}	$(-1)^\alpha wl_{ij}$	0	0	0	0	0
λ_i	$8\sqrt{2}$	$-8\sqrt{2}$	0	$-8\sqrt{2}$	$8\sqrt{2}$	0	0	0	0	0	0	0	$(-1)^\beta wl_{ij}$	0	0	0	0	0	0
α_k	± 6	0	± 6	± 6	0	± 6	$\mp(-1)^k 12$	0	$\mp(-1)^k 6\sqrt{2}$	$\mp(-1)^k 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	0	0	aa_{kl}	0	0	0	0
β_k	± 6	0	± 6	± 6	0	± 6	$\mp(-1)^k 12$	0	$\mp(-1)^k 6\sqrt{2}$	$\mp(-1)^k 6\sqrt{2}$	$\pm 6\sqrt{2}$	0	0	0	aa_{kl}	0	0	0	0
γ_k	$\pm 6\sqrt{2}$	0	$\pm 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	$\mp 6\sqrt{2}$	0	0	$\mp(-1)^k 12$	$\mp(-1)^k 12$	0	0	0	0	$\sqrt{2} aa_{kl}$	0	0	0	0
μ_{is}	± 12	0	∓ 12	∓ 12	0	∓ 12	0	0	0	0	0	0	0	0	0	0	0	0	$(-1)^{i+j+J} 12\sqrt{2}$
μ_{iI}	$\pm 6\sqrt{2}$	0	$\mp 6\sqrt{2}$	$\mp 6\sqrt{2}$	0	$\pm 6\sqrt{2}$	0	0	0	0	$\pm(-1)^{I+J}$	0	0	0	0	0	0	$(-1)^{i+I+j} 12\sqrt{2}$	$(-1)^{i+j} 12$

The sub-matrices are defined as follows

$$ww_{ij} = 16\mathrm{Re}\ \left(\epsilon^{-2i\pi/9}\omega^{i+j}\right)\ ,\quad wl_{ij} = 16\sqrt{2}\mathrm{Re}\ \left(\epsilon^{-5i\pi/9}\omega^{i+j}\right)\ ,\quad aa_{kl} = 12\mathrm{Re}\ \left(\epsilon^{-i\pi/8}i^{k+l}\right)\ ,$$

Table A.3: Super-octahedron S matrix ($\times 48$)

Appendix B

Characters for the $T^6/Z_2 \times Z_2$ orbifolds

In this Appendix we list the characters of the $T^6/Z_2 \times Z_2$ orbifold. They may be expressed as ordered products of the four SO(2) level-one characters, O_2, V_2, S_2 and C_2, where the ordering of the four factors refers to the eight transverse dimensions of space time and, in particular, the first factor is associated to the two transverse space-time directions.

$$
\begin{aligned}
\tau_{oo} &= V_2O_2O_2O_2 + O_2V_2V_2V_2 - S_2S_2S_2S_2 - C_2C_2C_2C_2 \ , \\
\tau_{og} &= O_2V_2O_2O_2 + V_2O_2V_2V_2 - C_2C_2S_2S_2 - S_2S_2C_2C_2 \ , \\
\tau_{oh} &= O_2O_2O_2V_2 + V_2V_2V_2O_2 - C_2S_2S_2C_2 - S_2C_2C_2S_2 \ , \\
\tau_{of} &= O_2O_2V_2O_2 + V_2V_2O_2V_2 - C_2S_2C_2S_2 - S_2C_2S_2C_2 \ , \\
\tau_{go} &= V_2O_2S_2C_2 + O_2V_2C_2S_2 - S_2S_2V_2O_2 - C_2C_2O_2V_2 \ , \\
\tau_{gg} &= O_2V_2S_2C_2 + V_2O_2C_2S_2 - S_2S_2O_2V_2 - C_2C_2V_2O_2 \ , \\
\tau_{gh} &= O_2O_2S_2S_2 + V_2V_2C_2C_2 - C_2S_2V_2V_2 - S_2C_2O_2O_2 \ , \\
\tau_{gf} &= O_2O_2C_2C_2 + V_2V_2S_2S_2 - S_2C_2V_2V_2 - C_2S_2O_2O_2 \ , \\
\tau_{ho} &= V_2S_2C_2O_2 + O_2C_2S_2V_2 - C_2O_2V_2C_2 - S_2V_2O_2S_2 \ , \\
\tau_{hg} &= O_2C_2C_2O_2 + V_2S_2S_2V_2 - C_2O_2O_2S_2 - S_2V_2V_2C_2 \ , \\
\tau_{hh} &= O_2S_2C_2V_2 + V_2C_2S_2O_2 - S_2O_2V_2S_2 - C_2V_2O_2C_2 \ , \\
\tau_{hf} &= O_2S_2S_2O_2 + V_2C_2C_2V_2 - C_2V_2V_2S_2 - S_2O_2O_2C_2 \ , \\
\tau_{fo} &= V_2S_2O_2C_2 + O_2C_2V_2S_2 - S_2V_2S_2O_2 - C_2O_2C_2V_2 \ , \\
\tau_{fg} &= O_2C_2O_2C_2 + V_2S_2V_2S_2 - C_2O_2S_2O_2 - S_2V_2C_2V_2 \ , \\
\tau_{fh} &= O_2S_2O_2S_2 + V_2C_2V_2C_2 - C_2V_2S_2V_2 - S_2O_2C_2O_2 \ , \\
\tau_{ff} &= O_2S_2V_2C_2 + V_2C_2O_2S_2 - C_2V_2C_2O_2 - S_2O_2S_2V_2 \ .
\end{aligned}
\tag{B.1}
$$

While these are sufficient to describe all supersymmetric $Z_2 \times Z_2$ amplitudes, when brane supersymmetry breaking is present additional characters are needed to describe the open strings stretched between branes and antibranes. These new characters, that we denote by $\tau_{kl}^{(-)}$, are obtained from the others in eq. (B.1) interchanging O_2 with V_2 and S_2 with C_2 in the last three factors, that correspond to the three internal tori:

$$\begin{aligned}
\tau_{oo}^{(-)} &= O_2O_2O_2O_2 + V_2V_2V_2V_2 - C_2S_2S_2S_2 - S_2C_2C_2C_2 \ , \\
\tau_{og}^{(-)} &= V_2V_2O_2O_2 + O_2O_2V_2V_2 - S_2C_2S_2S_2 - C_2S_2C_2C_2 \ , \\
\tau_{oh}^{(-)} &= V_2O_2O_2V_2 + O_2V_2V_2O_2 - S_2S_2S_2C_2 - C_2C_2C_2S_2 \ , \\
\tau_{of}^{(-)} &= V_2O_2V_2O_2 + O_2V_2O_2V_2 - S_2S_2C_2S_2 - C_2C_2S_2C_2 \ , \\
\tau_{go}^{(-)} &= O_2O_2S_2C_2 + V_2V_2C_2S_2 - C_2S_2V_2O_2 - S_2C_2O_2V_2 \ , \\
\tau_{gg}^{(-)} &= V_2V_2S_2C_2 + O_2O_2C_2S_2 - C_2S_2O_2V_2 - S_2C_2V_2O_2 \ , \\
\tau_{gh}^{(-)} &= V_2O_2S_2S_2 + O_2V_2C_2C_2 - S_2S_2V_2V_2 - C_2C_2O_2O_2 \ , \\
\tau_{gf}^{(-)} &= V_2O_2C_2C_2 + O_2V_2S_2S_2 - C_2C_2V_2V_2 - S_2S_2O_2O_2 \ , \\
\tau_{ho}^{(-)} &= O_2S_2C_2O_2 + V_2C_2S_2V_2 - S_2O_2V_2C_2 - C_2V_2O_2S_2 \ , \\
\tau_{hg}^{(-)} &= V_2C_2C_2O_2 + O_2S_2S_2V_2 - S_2O_2O_2S_2 - C_2V_2V_2C_2 \ , \\
\tau_{hh}^{(-)} &= V_2S_2C_2V_2 + O_2C_2S_2O_2 - C_2O_2V_2S_2 - S_2V_2O_2C_2 \ , \\
\tau_{hf}^{(-)} &= V_2S_2S_2O_2 + O_2C_2C_2V_2 - S_2V_2V_2S_2 - C_2O_2O_2C_2 \ , \\
\tau_{fo}^{(-)} &= O_2S_2O_2C_2 + V_2C_2V_2S_2 - C_2V_2S_2O_2 - S_2O_2C_2V_2 \ , \\
\tau_{fg}^{(-)} &= V_2C_2O_2C_2 + O_2S_2V_2S_2 - S_2O_2S_2O_2 - C_2V_2C_2V_2 \ , \\
\tau_{fh}^{(-)} &= V_2S_2O_2S_2 + O_2C_2V_2C_2 - S_2V_2S_2V_2 - C_2O_2C_2O_2 \ , \\
\tau_{ff}^{(-)} &= V_2S_2V_2C_2 + O_2C_2O_2S_2 - S_2V_2C_2O_2 - C_2O_2S_2V_2 \ .
\end{aligned} \tag{B.2}$$

List of Publications

[shift] I. Antoniadis, G. D'Appollonio, E. Dudas and A. Sagnotti, *Open descendants of Z(2) x Z(2) freely-acting orbifolds*, Nucl. Phys. B **565** (2000) 123, hep-th/9907184.

[bsb] C. Angelantonj, I. Antoniadis, G. D'Appollonio, E. Dudas and A. Sagnotti, *"Type I vacua with brane supersymmetry breaking*, Nucl. Phys. B **572** (2000) 36, hep-th/9911081.

[ct1] A. Cappelli and G. D'Appollonio, *On the trace anomaly as a measure of degrees of freedom*, Phys. Lett. B **487** (2000) 87, hep-th/0005115.

[ct2] A. Cappelli, G. D'Appollonio, R. Guida and N. Magnoli, *On the c-theorem in more than two dimensions*, hep-th/0009119.

[toi] A. Cappelli and G. D'Appollonio, *Boundary states for c=1 and 3/2 Rational Conformal Field Theories*, hep-th/yymmnnn, DFF 378/12/2001, PAR-LPTHE 01-79.

Bibliography

[1] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl. Phys. B **426** (1994) 19 [Erratum-ibid. B **430** (1994) 485]; K. A. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality*, Nucl. Phys. Proc. Suppl. **45BC** (1996) 1.

[2] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nucl. Phys. B **438** (1995) 109; E. Witten, *String theory dynamics in various dimensions*, Nucl. Phys. B **443** (1995) 85.

[3] For a general introduction see M. Green, J. Schwarz and E. Witten, *Superstring Theory vol.1-2*, Cambridge Univ. Pr. (1987) and J. Polchinski, *String Theory vol.1-2*, Cambridge Univ. Pr. (1998).

[4] D. Friedan, E. J. Martinec and S. H. Shenker, *Conformal Invariance, Supersymmetry And String Theory* Nucl. Phys. B **271** (1986) 93.

[5] N. Seiberg and E. Witten, *Spin Structures In String Theory* Nucl. Phys. B **276** (1986) 272.

[6] D. J. Gross, J. A. Harvey, E. J. Martinec and R. Rohm, *Heterotic String Theory. 1. The Free Heterotic String* Nucl. Phys. B **256** (1985) 253; *Heterotic String Theory. 2. The Interacting Heterotic String* Nucl. Phys. B **267** (1986) 75.

[7] A. Sagnotti, in: Cargèse '87, Non-Perturbative Quantum Field Theory, eds. G. Mack et al. (Pergamon Press, Oxford, 1988) p. 521; G. Pradisi and A. Sagnotti, *Phys. Lett.* **B216** (89) 59; M. Bianchi and A. Sagnotti, *Phys. Lett.* **B247** (90) 517, *Nucl. Phys.* **B361** (91) 519.

[8] M. B. Green and J. H. Schwarz, *Anomaly Cancellation In Supersymmetric D=10 Gauge Theory And Superstring Theory* Phys. Lett. B **149** (1984) 117.

[9] There are several reviews on this subject as, for instance, M. J. Duff, R. R. Khuri and J. X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213; K. S. Stelle, *BPS branes in supergravity.*

[10] J. Polchinski, *Dirichlet-Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724

[11] J. Dai, R. G. Leigh and J. Polchinski, *New Connections Between String Theories*, Mod. Phys. Lett. A **4** (1989) 2073.

[12] E. Witten, *Bound States Of Strings And p-Branes*, Nucl. Phys. B **460** (1996) 335.

[13] E. S. Fradkin and A. A. Tseytlin, *Nonlinear Electrodynamics From Quantized Strings*, Phys. Lett. B **163** (1985) 123; M. R. Douglas, *Branes within branes*, hep-th/9512077; Y. K. Cheung and Z. Yin, *Anomalies, branes, and currents*, Nucl. Phys. B **517** (1998) 69; R. Minasian and G. W. Moore, *K-theory and Ramond-Ramond charge*, JHEP **9711** (1997) 002.

[14] J. Polchinski and Y. Cai, *Consistency Of Open Superstring Theories*, Nucl. Phys. B **296** (1988) 91; M. Bianchi and J. F. Morales, *Anomalies and tadpoles*, JHEP **0003** (2000) 030.

[15] A. Strominger and C. Vafa, *Microscopic Origin of the Bekenstein-Hawking Entropy*, Phys. Lett. B **379** (1996) 99.

[16] For a general review see A. Giveon and D. Kutasov, *Brane dynamics and gauge theory*, Rev. Mod. Phys. **71** (1999) 983.

[17] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231, Int. J. Theor. Phys. **38** (1998) 1113. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B **428** (1998) 105; E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253.

[18] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, *New York, USA: Springer (1997).*

[19] J. L. Cardy, *Conformal Invariance And Surface Critical Behavior*, Nucl. Phys. B **240** (1984) 514.

[20] N. Ishibashi, *The Boundary And Crosscap States In Conformal Field Theories*, Mod. Phys. Lett. A **4** (1989) 251.

[21] H. Sonoda, *Sewing Conformal Field Theories*, Nucl. Phys. B **311** (1988) 401; H. Sonoda, *Sewing Conformal Field Theories. 2*, Nucl. Phys. B **311** (1988) 417.

[22] E. Verlinde, *Fusion Rules And Modular Transformations In 2-D Conformal Field Theory*, Nucl. Phys. B **300** (1988) 360.

[23] J. L. Cardy and D. C. Lewellen, *Bulk And Boundary Operators In Conformal Field Theory*, Phys. Lett. B **259** (1991) 274; D. C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, Nucl. Phys. B **372** (1992) 654.

[24] J. L. Cardy, *Boundary Conditions, Fusion Rules And The Verlinde Formula* Nucl. Phys. B **324** (1989) 581.

[25] A. N. Schellekens and Y. S. Stanev, *Trace formulas for annuli*, hep-th/0108035.

[26] A. Sagnotti and Y. S. Stanev, *Open descendants in conformal field theory*, Fortsch. Phys. **44** (1996) 585, [Nucl. Phys. Proc. Suppl. **55B** (1996) 200].

[27] G. Pradisi, A. Sagnotti and Y. S. Stanev, *Completeness Conditions for Boundary Operators in 2D Conformal Field Theory*, Phys. Lett. B **381** (1996) 97.

[28] G. Pradisi, A. Sagnotti and Y. S. Stanev, *Planar duality in SU(2) WZW models*, Phys. Lett. B **354** (1995) 279.

[29] T. Gannon, *Integers in the open string*, Phys. Lett. B **473** (2000) 80.

[30] P. Bantay, *Frobenius-Schur indicators, the Klein-bottle amplitude, and the principle of orbifold covariance*, Phys. Lett. B **488** (2000) 207.

[31] L. R. Huiszoon, A. N. Schellekens and N. Sousa, *Klein bottles and simple currents*, Phys. Lett. B **470** (1999) 95; L. R. Huiszoon, A. N. Schellekens and N. Sousa, *Open descendants of non-diagonal invariants*, Nucl. Phys. B **575** (2000) 401; L. R. Huiszoon and A. N. Schellekens, *Crosscaps, boundaries and T-duality*, Nucl. Phys. B **584** (2000) 705.

[32] L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *Strings On Orbifolds*, Nucl. Phys. B **261** (1985) 678; L. J. Dixon, J. A. Harvey, C. Vafa and E. Witten, *Strings On Orbifolds. 2*, Nucl. Phys. B **274** (1986) 285.

[33] C. Vafa, *Modular Invariance And Discrete Torsion On Orbifolds*, Nucl. Phys. B **273** (1986) 592.

[34] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, *The Operator Algebra Of Orbifold Models,* Commun. Math. Phys. **123** (1989) 485.

[35] A. N. Schellekens and S. Yankielowicz, *Simple Currents, Modular Invariants And Fixed Points*, Int. J. Mod. Phys. A **5** (1990) 2903; K. A. Intriligator, *Bonus Symmetry In Conformal Field Theory*, Nucl. Phys. B **332** (1990) 541; A. N. Schellekens and S. Yankielowicz, *Modular Invariants From Simple Currents: An Explicit Proof*, Phys. Lett. B **227** (1989) 387; A. N. Schellekens and S. Yankielowicz, *Extended Chiral Algebras And Modular Invariant Partition Functions*, Nucl. Phys. B **327** (1989) 673.

[36] G. W. Moore and N. Seiberg, *Naturality In Conformal Field Theory*, Nucl. Phys. B **313** (1989) 16.

[37] E. Witten, *Nonabelian Bosonization In Two Dimensions*, Commun. Math. Phys. **92** (1984) 455.

[38] A. Cappelli, C. Itzykson and J. B. Zuber, *The Ade Classification Of Minimal And A1(1) Conformal Invariant Theories*, Commun. Math. Phys. **113** (1987) 1.

[39] M. Kreuzer and A. N. Schellekens, *Simple Currents Versus Orbifolds With Discrete Torsion: A Complete Classification*, Nucl. Phys. B **411** (1994) 97.

[40] J. Fuchs, A. N. Schellekens and C. Schweigert, *A matrix S for all simple current extensions*, Nucl. Phys. B **473** (1996) 323.

[41] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. I: General theory*, Nucl. Phys. B **558** (1999) 419; J. Fuchs and C. Schweigert, *Symmetry breaking boundaries. II: More structures, examples*, Nucl. Phys. B **568** (2000) 543; L. Birke, J. Fuchs and C. Schweigert, *Symmetry breaking boundary conditions and WZW orbifolds*, Adv. Theor. Math. Phys. **3** (1999) 671.

[42] J. Fuchs and C. Schweigert, *A classifying algebra for boundary conditions*, Phys. Lett. B **414** (1997) 251; J. Fuchs and C. Schweigert, *Branes: From free fields to general backgrounds*, Nucl. Phys. B **530** (1998) 99.

[43] J. Fuchs, L. R. Huiszoon, A. N. Schellekens, C. Schweigert and J. Walcher, *Boundaries, crosscaps and simple currents*, Phys. Lett. B **495** (2000) 427.

[44] A. Y. Alekseev and V. Schomerus, *D-branes in the WZW model*, Phys. Rev. D **60** (1999) 061901.

[45] G. Felder, J. Frohlich, J. Fuchs and C. Schweigert, The geometry of WZW branes, J. Geom. Phys. **34** (2000) 162.

[46] C. Bachas, M. R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, JHEP **0005** (2000) 048.

[47] P. Bordalo, S. Ribault and C. Schweigert, *Flux stabilization in compact groups*, JHEP **0110** (2001) 036.

[48] I. Brunner, *On orientifolds of WZW models and their relation to geometry*, hep-th/0110219; C. Bachas, N. Couchoud and P. Windey, *Orientifolds of the 3-sphere*, JHEP **0112** (2001) 003.

[49] L. R. Huiszoon, K. Schalm and A. N. Schellekens, *Geometry of WZW orientifolds*, hep-th/0110267.

[50] G. Pradisi, A. Sagnotti and Y. S. Stanev, *The Open descendants of nondiagonal SU(2) WZW models*, Phys. Lett. B **356** (1995) 230.

[51] R. E. Behrend, P. A. Pearce, V. B. Petkova and J. B. Zuber, *Boundary conditions in rational conformal field theories*, Nucl. Phys. B **579** (2000) 707.

[52] D. Friedan, Z. a. Qiu and S. H. Shenker, *Superconformal Invariance In Two-Dimensions And The Tricritical Ising Model*, Phys. Lett. B **151** (1985) 37.

[53] E. Witten, *Dynamical Breaking Of Supersymmetry*, Nucl. Phys. B **188** (1981) 513.

[54] J. Maldacena, G. W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, JHEP **0107** (2001) 046.

[55] P. Ginsparg, *Curiosities at $c = 1$*, Nucl. Phys. B **295** (1988) 153.

[56] L. J. Dixon, P. Ginsparg and J. A. Harvey, *$c = 1$ Superconformal Field Theory,* Nucl. Phys. B **306** (1988) 470.

[57] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Infinite Conformal Symmetry In Two-Dimensional Quantum Field Theory*, Nucl. Phys. B **241** (1984) 333.

[58] R. Dijkgraaf, E. Verlinde and H. Verlinde, *$C = 1$ Conformal Field Theories On Riemann Surfaces* Commun. Math. Phys. **115** (1988) 649.

[59] E. B. Kiritsis, *Proof Of The Completeness Of The Classification Of Rational Conformal Theories With $C = 1$*, Phys. Lett. B **217** (1989) 427.

[60] E. G. Gimon and C. V. Johnson, *K3 Orientifolds*, Nucl. Phys. B **477** (1996) 715; A. Dabholkar and J. Park, *Strings on Orientifolds*, Nucl. Phys. B **477** (1996) 701; C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y. S. Stanev, *Comments on Gepner models and type I vacua in string theory*, Phys. Lett. B **387** (1996) 743.

[61] P. Bantay, *Characters and modular properties of permutation orbifolds*, Phys. Lett. B **419** (1998) 175.

[62] M. Dine, R. Rohm, N. Seiberg and E. Witten, *Gluino Condensation In Superstring Models*, Phys. Lett. B **156** (1985) 55.

[63] J. Scherk and J.H. Schwarz, *Nucl. Phys.* **B153** (79) 61, *Phys. Lett.* **B82** (79) 60;
R. Rohm, *Nucl. Phys.* **B237** (84) 553;
P. Fayet, *Phys. Lett.* **B159** (85) 121, *Nucl. Phys.* **B263** (86) 649.

[64] C. Kounnas and M. Porrati, *Nucl. Phys.* **B310** (88) 355;
S. Ferrara, C. Kounnas, M. Porrati and F. Zwirner, *Nucl. Phys.* **B318** (89) 75;
C. Kounnas and B. Rostand, *Nucl. Phys.* **B341** (90) 641;
I. Antoniadis, *Phys. Lett.* **B246** (90) 377;
I. Antoniadis and C. Kounnas, *Phys. Lett.* **B261** (91) 369.

[65] E. Kiritsis and C. Kounnas, *Nucl. Phys.* **B503** (97) 117.

[66] I. Antoniadis, E. Dudas and A. Sagnotti, *Nucl. Phys.* **B544** (99) 469.

[67] Z. Kakushadze and S.-H.H. Tye, *Nucl. Phys.* **B548** (99) 180.

[68] J. D. Lykken, *Weak Scale Superstrings*, Phys. Rev. D **54** (1996) 3693; E. Witten, *Strong Coupling Expansion Of Calabi-Yau Compactification*, Nucl. Phys. B **471** (1996) 135. Phenomenological applications of D-branes are reviewed in I. Antoniadis, *String and D-brane physics at low energy*, hep-th/0102202.

[69] I. Antoniadis, *A Possible New Dimension At A Few Tev*, Phys. Lett. B **246** (1990) 377; N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *The hierarchy problem and new dimensions at a millimeter*, Phys. Lett. B **429** (1998) 263; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, *New dimensions at a millimeter to a Fermi and superstrings at a TeV*, Phys. Lett. B **436** (1998) 257.

[70] M. Bianchi, Ph.D. Thesis, preprint ROM2F-92/13; A. Sagnotti, hep-th/9302099.

[71] G. Zwart, *Nucl. Phys.* **B526** (98) 378; Z. Kakushadze, G. Shiu and S.H.H. Tye, *Nucl. Phys.* **B533** (98) 25; G. Aldazabal, A. Font, L.E. Ibáñez and G. Violero, *Nucl. Phys.* **B536** (98) 29.

[72] I. Antoniadis, E. Dudas and A. Sagnotti, *Phys. Lett.* **B464** (99) 38.

[73] S. Sugimoto, *Anomaly cancellations in type I D9-D9-bar system and the USp(32) string theory*, Prog. Theor. Phys. **102** (1999) 685.

[74] G. Aldazabal and A. M. Uranga, *Tachyon-free non-supersymmetric type IIB orientifolds via brane-antibrane systems*, JHEP **9910** (1999) 024.

[75] E. Dudas and J. Mourad, *Brane solutions in strings with broken supersymmetry and dilaton tadpoles*, Phys. Lett. B **486** (2000) 172.

[76] E. Dudas and J. Mourad, *Consistent gravitino couplings in non-supersymmetric strings*, Phys. Lett. B **514** (2001) 173; G. Pradisi and F. Riccioni, *Geometric couplings and brane supersymmetry breaking*, Nucl. Phys. B **615** (2001) 33.

[77] A. Sen, *Stable non-BPS states in string theory*, JHEP **9806** (1998) 007; *Stable non-BPS bound states of BPS D-branes*, JHEP **9808** (1998) 010; *Tachyon condensation on the brane antibrane system*, JHEP **9808** (1998) 012. For a review of non-BPS states in string theory see: A. Sen, *Stable Non-Bps States In String Theory, Prepared for 29th International Conference on High-Energy Physics (ICHEP 98), Vancouver, British Columbia, Canada, 23-29 Jul 1998*; A. Lerda and R. Russo, *Stable non-BPS states in string theory: A pedagogical review*, Int. J. Mod. Phys. A **15** (2000) 771.

[78] A. Adams, J. Polchinski and E. Silverstein, *Don't panic! Closed string tachyons in ALE space-times*, JHEP **0110** (2001) 029.

[79] C. Bachas, *A Way to break supersymmetry*,

[80] M. Berkooz and R.G. Leigh, *Nucl. Phys.* **B483** (97) 187.

[81] I. Antoniadis, G. D'Appollonio, E. Dudas and A. Sagnotti, *Partial breaking of supersymmetry, open strings and M-theory*, Nucl. Phys. B **553** (1999) 133.

[82] R. Blumenhagen and L. Gorlich, *Orientifolds of non-supersymmetric, asymmetric orbifolds*, Nucl. Phys. B **551** (1999) 601; C. Angelantonj, I. Antoniadis and K. Forger, *Non-supersymmetric type I strings with zero vacuum energy*, Nucl. Phys. B **555** (1999) 116.

[83] E. G. Gimon and J. Polchinski, *Consistency Conditions for Orientifolds and D-Manifolds*, Phys. Rev. D **54** (1996) 1667.

[84] M. Bianchi and A. Sagnotti, *On The Systematics Of Open String Theories*, Phys. Lett. B **247** (1990) 517.

[85] M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, Nucl. Phys. B **376** (1992) 365.

[86] P. Horava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. B **460** (1996) 506.