Università degli Studi di Firenze
Dipartimento di Fisica
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Tesi di Dottorato

Noncommutative Geometry and the Quantum Hall effect

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Contents

Chapter 1

Introduction

1.1 Forewords

Last century witnessed the birth and growth of a new way of thinking about the physical world. Our aim is not to make a history of modern physics: the present section is meant solely to motivate the reader to go through the work presented, while expressing a few more personal considerations. The argumentation presented are inspired by the reviews [49, 48], among the others.

The route the physical thought has followed in said period of time, has been influenced in a very peculiar way by the theoretical novelties the physics community has discovered during this period. All of those conceptual steps the physical knowledge of fundamental processes has made, have been went along with corresponding steps in mathematics. It is not occasional, nowadays, that some novel concept in theoretical physics triggers a new investigation or discovery in mathematics, or vice versa; anyway this conceptual coupling is not a prerogative of modern thought. Indeed geometry in its early days was clearly oriented to describe the space in which natural events take place. After the very early works by Einstein upon General Relativity, the scope of the previous sentence has enlarged in an almost dramatic fashion.

In the same period, it became clear that physics on a smaller scale was different from what it appeared to be in everyday life (at human-size scale). It took several years for most of the physical community to accept the new-birth Quantum Mechanics. The appearance of uncertainty in physics puzzled most of those physicists who did not promptly accept it. It compelled to shift the traditional view on physical phenomena, to a more indirect one: on a quantum system, there are several questions that cannot be asked any longer. This has been accepted long ago understanding that the nature of physical phenomena is such, and we must bear it. Also, this shift in the attention, has become a virtue, in physics, being it more abstract and hence allowing for further reaching work.

So in modern physics one just gets used to several abstract concepts, being always able to trace them back to their very sources, by mean of the physical meaning of each of them. This needs to be the case for the not-so-recent Noncommutative Geometry. Already in the early times of Quantum Mechanics and Quantum Field Theory, [48], the introduction, as coordinates, of objects which did not commute was considered as a resource in order to cure the infinite self-energies that plagued Quantum Field Theory, before the Renormalization had become a well-established matter. This has already been noticed by Heisenberg in the 30's, and analysed thoroughly by Snyder in 1947 [46].

More recently, mathematicians have studied this new geometry in several ways: we will use mainly the point of view established by Alain Connes (see [7, 8] and references therein), adhering in this way to the choice of a fairly large part of the physical community (see [28], [30] and references therein).

1.2 Quantum Mechanics

The simplest physical instance of noncommutative geometry is that of the phase space of a mechanical system after quantization. The theory with which we start is described by (regular) functions on the $2n$-dimensional phase space Γ of the system. The space Γ is endowed with a closed nondegenerate 2-form ω that defines the Poisson brackets, which can be seen as a bilinear antisymmetric functional defined on the algebra of observables (regular functions) $\mathcal{A} \doteq C^r(\Gamma)$. One can choose local coordinates in which the canonical expression of the Poisson bracket between the two observables $f(q,p)$ and $g(q,p)$ is

$$\{f,g\} \doteq \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \ . \tag{1.1}$$

The standard quantization procedure requires to introduce the Hilbert space $\mathcal{H}$ of the physical states of the system, and map the algebra of observables $\mathcal{A}$ in an algebra $\widehat{\mathcal{A}}$ of operators acting on $\mathcal{H}$. The latter correspondence is defined in such a way that the Poisson brackets (1.1) of any pair of observables $f(q,p)$ and $g(q,p)$ mapped into the commutator of the corresponding operators $\hat{f}$ and $\hat{g}$:

$$i\hbar\,\widehat{\{f,g\}} = \hat{f}\cdot\hat{g} - \hat{g}\cdot\hat{f}$$

where the hat stands for the quantization map $\mathcal{A} \ni f \longmapsto \hat{f} \in \widehat{\mathcal{A}}$.[1]
In particular the position and momentum observables q_i and p_i are mapped respectively to the operators $\hat{q}_i$ and $\hat{p}_i$ which have the canonical commutation relation

$$[q_i, p_j] = i\hbar\,\delta_{ij}\ .$$

It is well known that from this relation, which is an obstruction to find simultaneous eigenvectors of both the position and the momentum operators, there arise the Heisenberg relations of uncertainty on the measurements of the position and momentum

$$(\Delta q_i)^2(\Delta p_i)^2 \gtrsim \frac{1}{4}\hbar^2\ , \tag{1.2}$$

which express quantitatively the loss of localization of points in the phase space Γ. From the uncertainty relations (1.2) one see that a point in the phase space cannot be resolved in an area smaller that that of a Planck cell. This fact causes the loss of the very notion of "point" in a quantized phase space[2]. In Quantum Mechanics physical properties are worked out, generically speaking, by algebraic relations among operators, since this allows to work in a more abstract context. Therefore for a physicist it is more profitable to refer to the traditional physical lore of Quantum Mechanics when considering the introduction of a noncommutative geometry in a problem.

1.3 Quantization of Geometry

Since the foundation of General Relativity by Einstein on 1916, the paradigm of the physical theory of Gravity has been to identify the Gravity with the Geometry

[1] To define this quantization map properly, one need also to define the quantization of any symmetric product of observables of $\mathcal{A}$. We are not entering into any detail here, because it would be beyond the scope of this chapter.

[2] Hence J. von Neumann happened to call that of a quantized phase space a "pointless" geometry.

of the Space-Time. Hence, it is expected on a general basis that quantization of Gravity will lead to a noncommutative Space-Time geometry. Starting by the direct application of Heisenberg's principle to the Einstein's Gravity, one can obtain by a semi-classical evaluation the uncertainty relations for the coordinates in absence of a strong external field [11]:

$$\sum_{i<j} \Delta x_i \, \Delta x_j \ \gtrsim \lambda_P^2 \,, \tag{1.3}$$

whereas $\lambda_P \sim 10^{-33}$cm is the *Planck length*, and Δx_i are the uncertainties on the measurement of the coordinates. The argument to find the above goes as follows [11]. To perform a measurement of the localization of an event, we give to our test particles en energy of order $\frac{h}{a}$ where a is the minimum among the uncertainties Δx_i; at the locations of the test particles the density of energy is $\frac{h}{a}$. We must ensure that this energy density does not exceed the threshold for the formation of a black hole, because otherwise the horizon will take the region around the event away from the observation.[3] In [11] there can be found more details on how to introduce an algebra of operators from whose commutation relations one can obtain the uncertainty relations (1.3). For our purposes, we need only to notice that the relations (1.3) require that the coordinate of the almost-Minkowski Space-Time are "promoted" to noncommuting operators

$$[x_\mu, x_\nu] = iQ_{\mu\nu} \qquad \text{with } \lambda_P \sim \sqrt{|Q|} \ .$$

Therefore, very general arguments based on Quantum Mechanics and General Relativity lead, at a semi-classical level, to noncommuting coordinates in the Space-Time.

1.4 Strings and Branes

The arguments on the quantization of Space-Time above can be improved by the analysis of scattering amplitudes of strings at high energy (see section 3.1 in [49] and references therein). The out-coming Heisenberg relations between the uncertainty on position and momentum of the string, get a term due to the finite spatial extent of the string itself:

$$\Delta x \gtrsim \frac{\hbar}{2}\left(\frac{1}{(\Delta p)} + \ell_s^2(\Delta p)\right) \ .$$

[3] We can also restrict to stationary solutions of Einstein equation, when the uncertainty on time localization is very large, obtaining that a must be smaller of the Schwarzschild radius relative to the energy $\frac{h}{a}$.

This in turn implies that the uncertainty on the measurement of spatial distances is bounded from below by

$$\Delta x \gtrsim \ell_s \ ,$$

i.e. from the finite length of the string. Therefore strings even at high energies cannot probe space-time at distances lower than the length of the strings themselves. This is an effect of the intrinsic non-locality of String Theory. To reach lower length scales it is necessary to use D-brane as probes [49, 48].

1.4.1 Strings in a Magnetic NS Background

From bosonic open string theory we can extract a simple example of how noncommutativity of coordinates arise in a fundamental context. This should display some of the motivations behind the excitement Noncommutative Geometry causes in the Theoretical Physics community. Consider the action of an open bosonic string moving in euclidean flat Space-Time, in presence of a background 2-form antisymmetric B field [49]

$$S = \frac{1}{4\pi\ell_s^2}\int_\Sigma d^2z\ (g_{\mu\nu}\partial^a X^\mu \partial_a X^\nu - 2\pi i \ell_s^2 \epsilon^{ab} B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)$$

The ends of the open string are attached on D-branes. The antisymmetric field $B_{\mu\nu}$ plays the role of a magnetic field on the D-branes. Let us restrict ourselves to the constant $B_{\mu\nu}$ case. Moreover we take the so called Seiberg-Witten limit[45]

$$g_{\mu\nu} \sim \ell_s^4 \sim \varepsilon \longrightarrow 0 \quad \text{while } B \text{is fixed,}$$

in which the massive modes decouple and the bulk dynamics disappear, the theory becoming topological; only the boundary theory survives[4]

$$S_B = -\frac{i}{2}\int_{\partial\Sigma} dt\ B_{\mu\nu} Y^\mu \frac{d}{dt} Y^\nu \ , \tag{1.4}$$

where $Y^\mu \doteq X^\mu|_{\partial\Sigma}$ is the restriction to the boundary of the string maps X^μ. This is just the theory of a charged particle in a strong uniform magnetic field B, therefore projected on the lowest Landau level. The canonical Poisson brackets obtained by the action (1.4) are

$$\{Y^\mu, Y^\nu\} = i\Theta^{\mu\nu} \qquad \text{where } \Theta^{\mu\nu} \doteq -\frac{1}{2}(B^{-1})^{\mu\nu} \ .$$

[4]Let us notice that since $\ell_s \longrightarrow 0$ the Seiberg-Witten limit is also a point-particle limit of the open string.

Upon quantization, these brackets become the commutators defining the usual non-commutative $\mathbb{R}^n$ euclidean space. Therefore we obtain a quantum effective theory describing strings in the low energy Seiberg-Witten limit, which is a Quantum Field Theory on a Noncommutative space.

1.5 Quantum Hall effect

The famous Peierls' substitution [39] was introduced for the first time in the problem of the motion of a electrons on a plane, in a uniform magnetic field B. As already mentioned, the total action of this system in the limit of strong magnetic field $B \longrightarrow \infty$ (or small mass $m \longrightarrow 0$) is

$$S = \int dt \frac{eB}{2} \epsilon^{ab} x_a(t) \dot{x}_b(t) \ .$$

The canonical quantization leads us to the commutation relations

$$[x_1, x_2] = i\frac{\hbar}{eB} \tag{1.5}$$

Also a coordinate depending potential $V(x_a)$ could be added, without changing the canonical commutation relations. This is maybe the easiest physical instance of noncommutative plane, and will be analysed in detail in chapter 3. We only notice here that the non-vanishing commutator (1.5) implies that the electron cannot be localized with infinite precision in the strong B limit.

1.5.1 Susskind's proposal

Inspired to the analogies between the physics of electrons in a strong magnetic field and the properties of D-Branes in String Theory, Susskind [47] proposed a model to describe Laughlin incompressible fluid. He derived a Noncommutative Chern-Simons Field Theory starting from the Lagrange description of the incompressible fluid, and constructing a noncommutative extension of it: the key feature to make this extension was that in the limit of high density the noncommutative theory reproduced the equations of motion of the Lagrange incompressible fluid (see chapter 4 of the present thesis for the detailed analysis).

The original proposal in [47] described the incompressible fluid in its thermodynamic limit, i.e. it described the infinite fluid. To describe a finite sample, and to avoid the problems of proper regularisation the infinite fluids theory presented, Polychronakos [42] proposed a truncation of the model, introducing the so called *Chern-Simons Matrix Quantum Mechanics*. The theory exposed also the boundary excitations, which have always had a great importance in the study of the Quantum Hall effect (for a review see [51]).

Polychronakos model is a model of $N \times N$ (hermitian) matrices. The truncation has been carried on through the introduction of a N-dimensional auxiliary time dependent vector, which corresponds to the boundary fields of low energy edge excitations [51]. He showed that this model possesses a $U(N)$ gauge invariance, and reduced it to a Calogero model of one-dimensional non-relativistic fermions with a repulsive interaction: the coordinates and momenta of these 1-dimensional fermions where the eigenvalues of the matrices of the original theory.
This model shares many features with Laughlin theory of Quantum Hall fluid, but the two models are not equivalent to each other. More precisely, while the states of the two models are isomorphic, the correspondence is not isometric: the measure of integration of Calogero model is real and one-dimensional, but the one of Laughlin quantum Hall fluid is complex and two-dimensional.
Anyhow the classical solutions of the matrix model presented the expected feature of the Hall fluid and the fractional charge vortex excitations as well. In [23, 22] the expected Hall conductivity has been derived from the noncommutative theory.

Karabali and Sakita [27, 26] analysed the reduction of the matrix theory to complex eigenvalues using the coherent states of electrons in the lowest Landau level (Bargmann-Fock space). Though they could not disentangle the electron coordinates from the auxiliary variables of the boundary fields, they performed some explicit calculations at low N. They found that the overlaps of states contain, along with the Laughlin wavefunction, a nontrivial measure factor which modifies the short distance properties of the fluid.
Hence the two authors concluded that either the matrix model did not describe the physics of Laughlin fluid, or the correspondence happened in an unknown set of coordinates.

1.6 Plan of the Thesis

In chapter 2 it will be presented a concise review of the Noncommutative Geometry in the Connes' paradigm. That chapter is not intended as a substitution of more classical text, but as a handy review of the subject; the whole chapter is mainly based on the book [28], but there are many other books on the subject, as well as reviews written by both mathematicians and physicists (e.g. [7, 9, 8, 21, 50, 18] and many others).

In chapter 3, we will first review several features of the problem of the electron in the Landau levels, stressing in particular the important role of W_∞ algebra of area preserving diffeomorphisms [13] in the mathematical description of the conditions of incompressibility [6], and in the characterisation of the quantum Hall fluids and their excitations. The W_∞ algebra plays an important role in the matrix model as well, since the Hilbert space of states of the system holds a representation of the this algebra.
Also the topic of the projection to the first n Landau levels is addressed, and it is shown what is the result of this projection on the algebra of observables of the system (see also [32, 31]).
Moreover it will be analysed a deformation of the algebra defining the Landau levels inspired by a paper by Nair and Polychronakos [38]: the device of Weyl quantization map will be used to define in the more abstract way the expectation values of products observables of the theory. The whole machinery will be employed to compute the density expectation value and the density-density correlation function on the ground state of a droplet of Quantum Hall fluid. The result will show that the fluid after the deformation of the algebra keeps its characteristic feature of (almost) uniform density and of incompressibility. Also it will be provided a simple computation which will make explicit a physical effect of noncommutativity, in terms of an effective repulsion appearing when a two-body attractive potential is switched on.

Chapter 4 will be devoted to the concise presentation of the work of Susskind [47]: the Lagrange description of the the incompressible fluid will be thoroughly presented along with its extension to the noncommutative theory, following [47] and [25]. The resulting theory will be a theory with a constraint, the Gauss' law, which ensures the noncommutativity of coordinates. The following chapter 5, will contain the statement of the truncation to finite N of the noncommutative theory by the introduction of the auxiliary time-depending complex vector Ψ. The quantization of the model will employ the path integral, and the Faddeev-Popov procedure will be used to fix the $U(N)$ gauge symmetry, as customary in field theory. As a result, together with the level quantization, we will obtain the reduction of the problem to the one-dimensional Calogero model of non-relativistic fermions, with a repulsive potential generated by the noncommutativity of coordinates. Also the scalar product for the quantum theory will be written, in terms of the coherent states of the matrix model, and the change of statistics induced by the integration measure in the scalar product.

The last part of the thesis, chapter 6, will present the Holomorphic quantization of the Chern-Simons Matrix Quantum Mechanics [5]. Complex (matrix) coordinates $X, X^\dagger$ will be introduced. A canonical transformation will be used to solve the Gauss' law constraint in terms of the eigenvalues of X: the path integral will be reduced to that of the electrons in the lowest Landau level, the electrons coordinates and momenta being the complex eigenvalues of X and their canonically conjugated variables.

In Schrödinger representation, while the coordinates will have the obvious diagonal form, the canonical momenta will get a term which geometrically is interpreted as a nontrivial affine connection; the appearance of this connection has an analogous in the appearance of the statistical interaction induced by the ordinary Chern-Simons interaction solved in terms of the sources (for a review see [53]).

The incompressibility will be defined in terms of the matrix extension of the generators of W_∞ algebra. In chapter 6 it is also performed the analysis of the realisation of W_∞ algebra in the Matrix Model, and the highest weight conditions defining incompressibility [6] are proved to hold for the latter; also the finite-size corrections arising from the finiteness of dimensionality of the matrices are taken into account and included into the set of generators of the W_∞ algebra.

It is argued as well that from the W_∞ symmetry of the model it is possible to compute all the scalar products of the states of the Chern-Simons Matrix Model. The problem of deciding whether the Chern-Simons Matrix Model describes the Laughlin theory of Quantum Hall fluid has been reduced to the proof that W_∞ symmetry holds for the Matrix theory. The latter has been done for its expression in general gauge, but is still not complete for the gauge fixed theory [5].

Chapter 2

Brief introduction to Noncommutative Geometry

This chapter is a concise review of the mathematical setting of Noncommutative geometry, mostly based on the book [28], both for the logical order of the arguments, and for the terminology used therein. Other sources [21, 50, 7, 8, 9] have been used as well to get a more complete view of this subject.

2.1 A technical preamble

We are going to review some general definitions, needed to understand the mathematical language of Noncommutative Geometry. We start by defining here the basic objects.

Definition 2.1 (Banach spaces) *A* **vector space** $\mathcal{V}$*, of arbitrary dimension, over the field of complex numbers* $\mathbb{C}$*,* [1] *equipped with a* **norm***, i.e. an application*

$$\|\cdot\| : \mathcal{V} \longrightarrow \mathbb{R}$$

which is ($\forall a \in \mathbb{C}, v, w \in \mathcal{V}$)

- $\|a\ v\| = |a|\ \|v\|$
- $\|v\| \geq 0 \quad , \quad \|v\| = 0 \iff v = 0$
- $\|v + w\| \leq \|v\| + \|w\|$

[1] In this thesis, we will consider only vector spaces over $\mathbb{C}$.

With respect to this norm, the space is required to be **complete**, *i.e. any Cauchy sequence is a convergent one, to some point of the space.*

Definition 2.2 (Banach Algebra) *A* **Banach space** $\mathcal{A}$ *endowed with an internal composition law* $\cdot$

$$\cdot : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$$

such that it is **distributive** *with respect to the vector space addition. Moreover it is required that*

$$\forall v, w \in \mathcal{A} \qquad \|v \cdot w\| \leq \|v\|\|w\|$$

A Banach algebra is said **unital** *if it is endowed of a multiplicative unit* $\mathbb{I}$*,* $\forall a \in \mathcal{A} \quad a \cdot \mathbb{I} = \mathbb{I} \cdot a = a$ *.*

Definition 2.3 (C^*-algebra) *A* **Banach algebra** *equipped with an antilinear involution* $*$ *leaving the norm invariant* $a^{**} = a, \quad \|a^*\| = \|a\|$*, and such that*

$$\|a^* a\| = \|a\|^2$$

Notice that there is no requirement here about the commutativity or the associativity of the algebra product.

Definition 2.4 (Ideal of a Banach algebra $\mathcal{A}$) *A subspace* $\mathcal{I} \subset \mathcal{A}$*, with the property that either*

$$\forall a \in \mathcal{A}, g \in \mathcal{I}, \quad a \cdot g \in \mathcal{I}$$

for a **left ideal***, or*

$$\forall a \in \mathcal{A}, g \in \mathcal{I}, \quad g \cdot a \in \mathcal{I}$$

for a **right ideal***. If both of the above are satisfied, then we deal with a* **two-sided ideal**.[2]
And ideal $\mathcal{I}$ *is* **maximal** *if there is no proper ideal* $\mathcal{I}'$ *such that* $\mathcal{I} \subset \mathcal{I}' \subsetneq \mathcal{A}$*.*

If $\mathcal{A}$ is a C^*-algebra , and $\mathcal{I} \subset \mathcal{A}$ a two-sided closed $*$-ideal (i.e. it has an involution induced by that of $\mathcal{A}$), then the quotient $\mathcal{A}/\mathcal{I}$ is a C^*-algebra . A **simple** C^*-algebra has no nontrivial two-sided ideals.

[2]If $\mathcal{A}$ has an involution (e.g. if it is a C^*-algebra) then its ideals are all two-sided.

Definition 2.5 (Resolvent set) *Given a unital C^*-algebra $\mathcal{A}$, and $a \in \mathcal{A}$, the* **resolvent set of** *a $r(a)$ is the subset of $\mathbb{C}$*

$$r(a) \doteq \{z \in \mathbb{C} \mid (a - z\,\mathbb{I}) \text{ is invertible}\}$$

For $z \in r(a)$, the operator $(a - z\,\mathbb{I})^{-1}$ is the **resolvent of** *a* **at** *z.*
The set $\sigma(a) = \mathbb{C}\backslash r(a)$ is the **spectrum of** *a.*

For a C^*-algebra $\mathcal{A}$, the spectrum of any $a \in \mathcal{A}$ is nonempty and compact.

Definition 2.6 (Spectral radius of $a \in \mathcal{A}$) *It is*

$$\rho(a) \doteq \sup\{|z|, z \in \sigma(a)\}$$

Now, if $\mathcal{A}$ is a C^*-algebra , then it holds the following

$$\forall a \in \mathcal{A}, \quad \|a\|^2 = \rho(a^* a)$$

So we see that for a C^*-algebra , the norm is unique and fixed by the algebraic structure.

Definition 2.7 (Morphism of C^*-algebra) *A $\mathbb{C}$-linear application $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ such that*

$$\phi(a_1 \cdot a_2) = \phi(a_1) \cdot \phi(a_2)$$

When bijective it is a $*$-isomorphism

A morphism $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is continuous and such that

$$\|a\|_{\mathcal{A}} \geq \|\phi(a)\|_{\mathcal{B}}$$

Moreover it maps a C^*-algebra in a C^*-algebra .

Definition 2.8 (Representation of a C^*-algebra $\mathcal{A}$) *A pair $(\mathcal{H}, \pi)$, with $\mathcal{H}$ an Hilbert space, such that*

$$\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$$

is a $*$-morphism in the space of bounded operators on $\mathcal{H}$. [3]
It is a **faithful representation** *if* **Ker** $\pi = \{0\}$*, or equivalently if $\forall a \in \mathcal{A}$, $\|\pi(a)\| = \|a\|$.*
It is an **irreducible representation** *if there are no nontrivial closed subspaces of*

[3] Actually the latter turns out to be a C^*-algebra as well

$\mathcal{H}$ which are invariant under the action of $\pi(\mathcal{A})$, or equivalently if the **center**[4] *of $\mathcal{A}$ satisfies $\mathbf{Z}(\mathcal{A}) = \{z\ \mathbb{I} | z \in \mathbb{C}\}$.*
Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are said **unitary equivalent representations** *if there exists an unitary operator $\mathbf{U} : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ such that $\mathbf{U}\pi_1 \equiv \pi_2 \mathbf{U}$.*

Definition 2.9 (Primitive ideal) *A subspace $\mathcal{I}$ of the C^*-algebra $\mathcal{A}$ such that $\mathcal{I} =$ **Ker** π for some irreducible $(\mathcal{H}, \pi)$ representation of $\mathcal{A}$. It is obviously a two-sided ideal.*
The space of **primitive ideals** *of a C^*-algebra $\mathcal{A}$ is called $Prim(\mathcal{A})$*

Definition 2.10 (Compact operator) *An operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ on a Hilbert space mapping weakly convergent sequences of $\mathcal{H}$ in strongly convergent ones. Equivalently a* **compact operator** *is an operator which is approximable in norm by a sequence $\{T_n\}$ of operators for which the orthogonal complement of each of the kernels* **Ker** *T_n is finite dimensional.*
The space of all compact operators on an Hilbert space $\mathcal{H}$ is usually called $\mathcal{K}(\mathcal{H})$.

Now a few properties of compact operators follow:

Proposition 2.1 (Polar decomposition) *The spectrum of a compact operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ is discrete and has no limit point in the complex plane, eventually except the origin. Any nonzero eigenvalue has finite multiplicity. Moreover, it may be written*

$$T = \sum_m \gamma_m(T) \psi_m \circ \phi_m \quad , \quad \mathbb{R}_+ \ni \gamma_m(T) \searrow 0$$

with $\{\psi_m\}$ and $\{\phi_m\}$ two orthonormal sets.

Proposition 2.2 *If $T : \mathcal{H} \longrightarrow \mathcal{H}$ is compact and self-adjoint, then there exists an orthonormal basis $\{\psi_m\}$ of $\mathcal{H}$ such that $T\psi_m = \lambda_n \psi_m$ with $\lim_{m \to \infty} \lambda_m = 0$.*

Definition 2.11 (Infinitesimal) *An infinitesimal of order $\mu \in \mathbb{R}_+$ is a $T \in \mathcal{K}(\mathcal{H})$ such that for $m \sim \infty$, $\gamma_m(T) = \mathcal{O}(1/m^\mu)$*

It turns out that $\mathcal{K}(\mathcal{H})$ is the largest norm closed two-sided ideal of the space of limited operators $\mathcal{B}(\mathcal{H})$. It is also a (non-unital) C^*-algebra having only one class of irreducible representations. Another important property of $\mathcal{K}(\mathcal{H})$ is that if a C^*-algebra $\mathcal{A}$ acts irreducibly on an Hilbert space, and contains some compact operator, then it contains all of them: $\mathcal{K}(\mathcal{H}) \subseteq \mathcal{A}$.

[4] It is the subspace of elements of $\mathcal{A}$ commuting with *all* the elements of $\mathcal{A}$

Definition 2.12 (Liminal C^*-algebra) *A C^*-algebra $\mathcal{A}$ for which the image of any irreducible representation $(\mathcal{H}, \pi)$ is coincident with $\mathcal{K}(\mathcal{H})$.*
Equivalently (see the above properties of compact operators) $\mathcal{A}$ is a **liminal C^*-algebra** *iff $\pi(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{H})$.*

Definition 2.13 (Postliminal C^*-algebra) *A C^*-algebra $\mathcal{A}$ for which the image of any irreducible representation $(\mathcal{H}, \pi)$ is contained in $\mathcal{K}(\mathcal{H})$.*
Equivalently (see above) $\mathcal{A}$ is a **postliminal C^*-algebra** *iff $\pi(\mathcal{A}) \cap \mathcal{K}(\mathcal{H}) \neq \emptyset$.*

For a postliminal C^*-algebra the classes of irreducible representations are uniquely characterised by their kernels.

Now we will need a new point of view about known things, suitable for extending the theory of (ordinary) Geometry.

2.2 Commutative Spaces

Firstly, let us consider a commutative C^*-algebra $\mathcal{A}$. From the commutativity it follows that its irreducible representations are all (unitary equivalent to) one-dimensional representations. So, every irreducible representation is a functional $\phi : \mathcal{A} \longrightarrow \mathbb{C}$, which preserves the algebra product (being it a $*$-morphism). It is customary to use the symbol $\widehat{\mathcal{A}}$ for the space of all such functionals, i.e. for the space of all the equivalence classes of irreducible representations of $\mathcal{A}$, the so called **structure space**.[5] The space $\widehat{\mathcal{A}}$ can be endowed with the weak topology (the *Gel'fand Topology*) σ_w, defined on the sequences as follows

$$\{\phi_n\} \subset \widehat{\mathcal{A}}, \qquad \phi_n \longrightarrow 0 \iff \forall a \in \mathcal{A}, \quad \phi_n(a) \longrightarrow 0$$

It can be shown that with this topology $\widehat{\mathcal{A}}$ is a $\mathbf{T}_2$ topological locally compact space[6]. This is true if $\mathcal{A}$ is only a Banach commutative $*$-algebra as well.

Definition 2.14 (Gel'fand Transform) *It is this correspondence between a C^*-algebra $\mathcal{A}$ and the space of complex functions $\widehat{\mathcal{A}} \longrightarrow C$*

$$\begin{array}{rcl} \hat{\cdot} \ : \quad \mathcal{A} & \longrightarrow & \widehat{\mathcal{A}}' \\ a & \longmapsto & \hat{a} \\ \hat{a}(\phi) & \doteq & \phi(a), \quad \forall \phi \in \widehat{\mathcal{A}}' \end{array}$$

[5]Such a space, for commutative algebras, is the space of all the characters of $\mathcal{A}$

[6]In the case $\mathcal{A}$ is an unital algebra, $(\widehat{\mathcal{A}}, \sigma_w)$ is a compact topological space.

An important case of this mapping is given by the algebra of measurable functions $\mathbf{L}^1(\mathbb{R})$ endowed with its natural norm

$$\|f\|_1 \doteq \int dx\ |f(x)|$$

and with the product of convolution as algebra product

$$\forall f, g \in \mathbf{L}^1(\mathbb{R}) \quad f \star g(x) \doteq \int dy\ f(x-y)g(y)$$

It is a Banach $*$-algebra with the complex conjugation as involution, as it can be easily shown with the standard machinery of Banach algebra theory[7]. Moreover, any irreducible representation of $\mathbf{L}^1(\mathbb{R})$ is continuous and can be written in integral form in the following fashion

$$\phi(a) = \int dx\ \underline{\phi}(x)a(x)$$

with $\underline{\phi}$ a suitable function in $\mathbf{L}^\infty(\mathbb{R}) \simeq \mathbf{L}^1(\mathbb{R})'$, the dual of our algebra. The convolution product is mapped to the point-wise product. The fact ϕ is a representation, hence a $*$-morphism, i.e. $\phi(a \star b) = \phi(a)\phi(b)$, implies (given $\underline{\phi} \in \mathbf{L}^\infty(\mathbb{R})$ and $a, b \in \mathbf{L}^1(\mathbb{R})$)

$$\phi(a \star b) \doteq \int dx\ \underline{\phi}(x) \int dy\ a(x-y)b(y) = \int\int dxdy\ \underline{\phi}(x+y)a(x)b(y)$$

$$\phi(a) \cdot \phi(b) \doteq \int dx\ \underline{\phi}(x)a(x) \int dy\ \underline{\phi}(y)b(y)$$

so that $\underline{\phi}(x+y) = \underline{\phi}(x)\underline{\phi}(y)$. This qualifies $\underline{\phi}(\cdot)$ as the exponential map. From this and from the limitedness it follows $\underline{\phi}(x) = \exp(ikx)$, $k \in \mathbb{R}$. So each representation is identified with a real number. Putting everything together, we find that the Gel'fand transform of an element $a \in \mathbf{L}^1(\mathbb{R})$ evaluated on a representation ϕ is

$$\hat{a}(\phi) \doteq \phi(a) = \int dx\ a(x)e^{ik_\phi x}$$

i.e. it is the Fourier transform of $a \in \mathbf{L}^1(\mathbb{R})$ at the frequency k_ϕ. Notice that the Gel'fand transform of a measurable function is a continuous function of the real line.

One can prove in general, that any Banach $*$-algebra $\mathcal{A}$ is mapped to the space of continuous functions on the structure space $\widehat{\mathcal{A}}$; if the latter is only locally compact (i.e. if $\mathcal{A}$ has no unit), it will be the space $\mathbf{C}_0(\widehat{\mathcal{A}})$ of continuous functions vanishing at infinity.

For a C^*-algebra there is the following stronger statement

[7]Let us notice that this algebra is a simple example of non-unital algebra, since the unit of convolution product is the Dirac δ *function*, which is, of course, not a function but a distribution, so it does not belong to $\mathbf{L}^1(\mathbb{R})$.

Theorem 2.1 (Gel'fand-Naimark) *Given a commutative unital C^*-algebra $\mathcal{A}$, there exists a compact Hausdorff space X such that the Gel'fand transform is an isometric $*$-isomorphism between $\mathcal{A}$ and $\mathbf{C}(X)$. This correspondence is fixed up to homeomorphisms.*

If the C^*-algebra $\mathcal{A}$ is non-unital then the space X will be only locally compact.[8] So for the Gel'fand-Naimark theorem each commutative C^*-algebra "is" the space of functions of a Hausdorff space. Hence, in principle, any result of classical geometry can be translated in this algebraic framework, provided that we can write all the data, such as differential forms an so on. This requires more work, but in the meanwhile we can render more material the correspondence between algebra and geometry.
So suppose we have the space $\mathbf{C}(X)$ of continuous functions on the compact $\mathbf{T}_2$ space X. Let us introduce the *evaluation map*

$$\forall p \in X \quad \psi_p : \mathbf{C}(X) \longrightarrow \mathbb{C} \quad \psi_p(f) \doteq f(p)$$

So any point of X define, roughly speaking, a morphism of $\mathbf{C}(X)$ to $\mathbb{C}$. Using the Gel'fand topology and the definitions for commutative Banach algebras, one can prove that these maps define an homeomorphism $\psi_\cdot$ between X and $\widehat{\mathbf{C}(X)}$, and that any maximal ideal of $\mathbf{C}(X)$ is the kernel of some map ψ_p, which in turn is by the definitions identified with a point $p \in X$.
In example we can associate to each point $p \in X$ the ideal of continuous functions vanishing on that point. The latter ideal is a maximal ideal of $C(X)$, and is the kernel of an irreducible representation of the algebra $\mathcal{A} \simeq C(X)$ itself. Thus we see that in the commutative case the manifold X can be identified with the maximal ideals of the algebra of functions defined on itself, and moreover given a generic commutative C^*-algebra $\mathcal{A}$ we can find a space whose points are ideals (indeed, primitive ideals) of $\mathcal{A}$ itself. This fact is a useful generalisation, which allows us to generalise the definition of space itself, as we shall see in the next section.

2.3 Noncommutative Spaces

The above discussion is not adequate when we go on considering what happens when one considers noncommutative C^*-algebra $\mathcal{A}$. Indeed, in this more general case, it is

[8] There is a nice correspondence between the one-point compactification of X and the unitalization of $\mathcal{A}$, in that the one point compactification of the structure space $\widehat{\mathcal{A}}$ of a Banach $*$-algebra is the structure space of the algebra $\mathcal{A} + \{z\mathbb{I} \mid z \in \mathbb{C}\}$ (the unitalization of $\mathcal{A}$).

no more true that the irreducible representations are characterised by their kernel. Now we introduce a topology on both $Prim\mathcal{A}$ and $\widehat{\mathcal{A}}$.
Let us start with the former: a subset $W \subset Prim\mathcal{A}$ is closed if and only if[9]

$$\forall \mathcal{I} \in W \mathcal{I} \subseteq \mathcal{J} \Longrightarrow \mathcal{J} \in W$$

With this topology the space $Prim\mathcal{A}$ is $\mathbf{T}_0$.[10] It can be proven that if $\mathcal{A}$ is a liminal C^*-algebra , then $Prim\mathcal{A}$ is $\mathbf{T}_1$[11].
We can now pass to the structure space $\widehat{\mathcal{A}}$ by the canonical surjection $\pi \longmapsto \mathbf{Ker}\ \pi$. We endow $\widehat{\mathcal{A}}$ with the coarsest topology which renders this surjection continuous, i.e. the quotient topology. In this topology the two objects $\widehat{\mathcal{A}}$ and $Prim\mathcal{A}$ are homeomorphic if and only if $\widehat{\mathcal{A}}$ is $\mathbf{T}_0$ as well as $Prim\mathcal{A}$. This is e.g. the case if the C^*-algebra $\mathcal{A}$ is postliminal.
Also in the noncommutative case it is true that the structure space $\widehat{\mathcal{A}}$ of a C^*-algebra is locally compact (compact if it has a unit), and $Prim\mathcal{A}$ share this property.
Noncommutative Geometry is based on the extension of the classical and familiar concepts of geometry, and what we just saw is a basic example of how this is usually done in this branch of mathematics. Starting from an ordinary (commutative) space, we pass to describe it in terms of the algebra of (continuous, smooth, etc.) functions defined on it, knowing we can recover the ordinary quantities of geometry in a formal way. Now it has been made possible to generalise this structure, without altering the relation it has with the geometric concepts we may be interested in (i.e. points, vectors and so on, as we shall see in the next sections). [8]
In this more abstract terms, there are two proposals for the identification of points: we can identify them with the primitive ideals of the C^*-algebra $\mathcal{A}$, or with the equivalence classes of irreducible representations of $\mathcal{A}$, i.e. elements of $\widehat{\mathcal{A}}$, the structure space of $\mathcal{A}$. We will restrict the analyses only to the cases in which these two notions are the same. As we saw this is the case when e.g. the C^*-algebra $\mathcal{A}$ is postliminal. The treatment of more general cases is left to the literature (see [28] and references

[9]Such a topology is equivalent to the so called *Jacobson topology*, more usual in this context. We are introducing this one instead for the sake of simplicity.

[10]A $\mathbf{T}_0$ space is such when for any pair of points there is an open neighbourhood of *one* of them which does not contain the other. This leads to a "lack of localisation in a general $\mathbf{T}_0$ space, because there are points that "stick" to some other point. Also, not all points in a $\mathbf{T}_0$ space are closed. This feature emerges when one studies the so called *Noncommutative lattices*.

[11]In $\mathbf{T}_1$ space for any pair of points, *any* of them has an open neighbourhood not containing the other one.

therein). In the commutative case it was also the case, and we saw how to identify points by the ideals of functions vanishing at those points.
The interested reader may find in reference [28] maybe the simplest example of a noncommutative space, the *two-points space*; it turns out to be important for the formulation made by Connes and Lott of the standard model of electro-weak interactions.

2.4 Modules

Till now we have dealt with algebraic structures representing the geometrical objects, roughly speaking, for themselves. Now we own only the basic tools to treat the topology of a space. In what follows we are going to introduce a kind of structure generalising the concepts of vector bundle geometry. Let us start with the basic definitions

Definition 2.15 (Module) *A nonempty set $\mathcal{E}$, endowed with an abelian composition law $+ : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{E}$, rendering it an abelian group, and an external composition law on a given ring $\mathcal{R}$, $\cdot : \mathcal{R} \times \mathcal{E} \longrightarrow \mathcal{E}$, the latter having the following (associativity and distributivity) properties*

$$\forall a, b \in \mathcal{R}; \eta, \xi \in \mathcal{E}$$

$$\begin{aligned} (a +_{\mathcal{R}} b) \cdot \eta &= a \cdot \eta +_{\mathcal{E}} b \cdot \eta \\ a \cdot (\eta +_{\mathcal{E}} \xi) &= a \cdot \eta +_{\mathcal{E}} a \cdot \xi \\ (ab) \cdot \eta &= a \cdot (b \cdot \eta) \end{aligned}$$

In this case $\mathcal{E}$ is called a **left module over** $\mathcal{R}$*. When the external composition law is in the form $\cdot : \mathcal{E} \times \mathcal{R} \longrightarrow \mathcal{E}$ then $\mathcal{E}$ is called* **right module**.[12]

It is clearly apparent that this is just a generalisation of the usual concept of *vector space*. The usual notion is clearly restored when $\mathcal{R} \simeq \mathbb{C}$. We will make use of modules over algebras, instead of rings. In this case we explicitly require that the module be a $\mathbb{C}$-vector space as well. This is automatic, of course, when the algebra is unital.
Of course the distinction between left and right structure for a module is totally

[12] We are being overnice here, in order to make clear, e.g. which sum are we talking about in writing a "+". Of course we will abandon this clumsy notation, just because it is usually superfluous outside formal definitions.

immaterial when only one kind of structure is chosen. Infact it is enough to consider, for any left (right) $\mathcal{A}$-module $\mathcal{E}$, the opposite algebra $\mathcal{A}^o$ defined by the relation $(ab)^o \doteq b^o a^o$, and so use the isomorphic right (left) $\mathcal{A}^o$-module structure over $\mathcal{E}$. This is not this the case, instead, when dealing with *bimodules*, i.e. modules with both a left and a right structure. In this case we of course could exchange the left and right structure, but we must require the compatibility between the two of them.

Definition 2.16 (Bimodule over an algebra) *A* left *and* right *module $\mathcal{E}$ over an algebra $\mathcal{A}$ for which it is satisfied the relation*

$$\forall \eta \in \mathcal{E}, \forall a, b \in \mathcal{A} \qquad (a\eta)b = a(\eta b)$$

i.e. the left and right structures can be supported in a compatible way.
Moreover a bimodule $\mathcal{E}$ over a $*$-algebra $\mathcal{A}$ is a **$*$-bimodule** *if there is an involution $* : \mathcal{E} \longrightarrow \mathcal{E}$ such that $\forall a, b \in \mathcal{A}; \eta \in \mathcal{E}$ there is the identity $(a\eta b)^* = b^* \eta^* a^*$.*

Definition 2.17 (Modules morphism) *Let $\mathcal{A}$ be an algebra, $\mathcal{E}$, $\mathcal{F}$ two left (right) $\mathcal{A}$-modules. Then $\phi : \mathcal{E} \longrightarrow \mathcal{E}$ is a* **module morphism** *iff it is $\mathbb{C}$-linear and $\mathcal{A}$-linear, i.e. it satisfies*

$$\forall \eta \in \mathcal{E}, a \in \mathcal{A} z \in \mathbb{C} \qquad \begin{cases} \phi(a\eta) &= a \cdot \phi(\eta) \\ \phi(z\eta) &= z\phi(\eta) \end{cases}$$

Definition 2.18 (Dual module) *Given a left (right) module $\mathcal{E}$ over the algebra $\mathcal{A}$, its* **dual** *$\mathcal{E}'$ is*

$$\mathcal{E}' \doteq Hom(\mathcal{E}, \mathcal{A}) = \{\phi : \mathcal{E} \longrightarrow \mathcal{A} \mid \phi \text{ is a morphism}\}$$

It is also a right (left) $\mathcal{A}$-module, defined by

$$\forall a \in \mathcal{A}, \phi \in \mathcal{E}' \qquad \phi(\diamond) \cdot_{\mathcal{A}\times\mathcal{E}'} a \doteq (\phi(\diamond))a$$

2.4.1 Modules from inside

Being the modules objects more general than the vector spaces, there are many subtleties about them. Now we begin to describe the most elementary ones.

Definition 2.19 (Generating family) *Given $\mathcal{E}$ a left (right) module, a generating family is a net $\{e_n\} \subset \mathcal{E}$ with the property that $\forall v \in \mathcal{E}$ there exists another net $\{a_n\} \subset \mathcal{A}$ satisfying*

$$v = \sum_n a_n e_n \quad \text{summing over a finite subnet}$$

The generating family $\{e_n\}$ is a **basis** *if its elements are $\mathcal{A}$-linearly independent.*

Definition 2.20 (Free module) *A module $\mathcal{E}$ which admits a basis*

Definition 2.21 (Module of finite type) *A module $\mathcal{E}$ which admits a basis of finite cardinality. In general this cardinality has* **not** *universal meaning, in that one is not assured there are no basis for $\mathcal{E}$ with different finite cardinality.*[13] *Modules of finite type are also called* **finite***.*

The prototype for a free finite module over the algebra $\mathcal{A}$ is $\mathcal{A}^m \cong \mathbb{C}^m \otimes \mathcal{A}$. the following holds

Proposition 2.3 *For* **any** *finite module $\mathcal{E}$ over the algebra $\mathcal{A}$, there is always $M \in \mathbb{N}$ and a morphism $\phi : \mathcal{A}^M \longrightarrow \mathcal{E}$ which is onto. Then ϕ maps a basis of $\mathcal{A}^M$ on a generating family of $\mathcal{E}$, the latter eventually lacking (when $\mathcal{E}$ is not free) of the linear independence of its elements.*

In the sequel we will deal only with finite modules, even when we do not specify it explicitly.

The fact a module $\mathcal{E}$ is not free is the translation of the non triviality of a vector bundle. The *canonical* example of this is the tangent bundle of the sphere S^2, which is a module over the algebra $\mathbf{C}^\infty(S^2)$, but does not admit a basis, since there does not exist two global independent vector fields .

Definition 2.22 (Projective module) *A left (right) module $\mathcal{E}$ over an algebra $\mathcal{A}$ which is a direct summand of a free module.*
Equivalently a module $\mathcal{E}$ is projective *if for every module $\mathcal{M}$, and every morphism $\phi : \mathcal{M} \longrightarrow \mathcal{E}$ which is onto, there exists its right inverse morphism $f : \mathcal{E} \longrightarrow \mathcal{M}$, i.e. $\phi \circ f = \mathbb{I}_{\mathcal{E}}$.*[14]
Moreover, it is equivalent to the fact that for any morphism $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ between

[13] This fact, true only for modules of finite type, depends on the algebra (more generally, on the ring) on which the module itself is defined. For a ring $\mathcal{R}$ with the *invariant basis property* the modules $\mathcal{R}^n$ and $\mathcal{R}^m$ are isomorphic only if $n = m$: so the cardinality of a basis for a module on such a ring defines an invariant of the module we call the *dimension* (or *rank*) of the module itself. In example this is the case for commutative rings, and for finite dimensional algebras, or whenever there is a ring map $\mathcal{R} \longrightarrow \mathbb{K}$ over a field, etc. For a C^*-algebra , the existence of a character, i.e. a $*$-morphism, and hence of a point in the geometry, entails the invariance of the dimension.

[14] This fact is also referred to as the morphism ϕ admitting a *split*, being f such a split.

modules, any morphism ψ admits a lift γ, i.e. the following diagram commutes

$$\begin{array}{ccc} & \gamma & \mathcal{F} \\ & \nearrow & \downarrow \phi \\ \mathcal{E} & \xrightarrow{\psi} & \mathcal{G} \end{array}$$

It can be shown of course that the three branches of the above definition are indeed equivalent statements.

For a finite projective module $\mathcal{E}$ over an algebra $\mathcal{A}$ one can show that, applying the definitions, and the (2.3), there exist an idempotent $\pi = \pi^2 : \mathcal{A}^M \longrightarrow \mathcal{A}^M$ such that $\mathcal{E} \cong \pi\mathcal{A}^M$. When the algebra $\mathcal{A}$ is a $*$-algebra (as is almost always our case), then it makes sense to define an Hermitian structure over the $\mathcal{A}$ modules, i.e. a sesquilinear form $\langle\cdot,\cdot\rangle : \mathcal{E}^2 \longrightarrow \mathcal{A}$ which is positive, i.e. $\langle\eta,\eta\rangle \geq 0$ and $\langle\eta,\eta\rangle = 0 \Leftrightarrow \eta = \underline{0}$. $\langle\cdot,\cdot\rangle$ is said to be *nondegenerate* if $\forall\zeta \in \mathcal{E}$ $\langle\zeta,\cdot\rangle : \mathcal{E} \longrightarrow \mathcal{E}'$ is an isomorphism between the module and its dual. If the finite projective module $\mathcal{E}$ admits an hermitian structure (i.e. is an *Hermitian module*), then the idempotent $\pi = \pi^2$ is a true projector (i.e. is self-adjoint as well).

The following theorem shows what is the relation between vector bundles and finite projective modules.

Theorem 2.2 (Serre-Swan) *Given a finite dimensional compact manifold* $\mathbf{M}$*, any module* $\mathcal{E}$ *over* $\mathbf{C}^\infty(\mathbf{M})$ *is isomorphic to the module of smooth sections of* some *bundle* $\mathbf{E} \longrightarrow \mathbf{M}$ *if and only if* $\mathcal{E}$ *is projective of finite type.*

2.5 Differential forms

Be $\mathcal{A}$ an (associative) algebra over $\mathbb{C}$. Then we put the following definition

Definition 2.23 (Universal differential forms) *It is the graded algebra*

$$\Omega\mathcal{A} \doteq \bigoplus_{p=0} \Omega^p\mathcal{A}$$

where we define[15]

- *The* 0 *degree is* $\Omega^0\mathcal{A} \doteq \mathcal{A}$

[15] Notice that each degree has a natural left $\mathcal{A}$ module structure.

- *The grading on the first degree $\delta : \mathcal{A} \longrightarrow \Omega^1\mathcal{A}$ which is a $\mathbb{C}$-linear map such that (Leibniz rule)*

$$\delta(ab) = \delta(a)b + a\delta(b)$$

- *The first degree $\Omega^1\mathcal{A}$ is the module generated by the image of the grading δ applied on the algebra $\mathcal{A}$*

- *Higher degrees are defined as*

$$\Omega^p\mathcal{A} \doteq \underbrace{\Omega^1\mathcal{A} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Omega^1\mathcal{A}}_{p}$$

the product being defined by simply writing all the factors in a row, and rearranging them using the Leibniz identity, so that e.g., we have $(a_1\delta a_2)(a_3\delta a_4) = a_1\delta(a_2a_3)\delta a_4 - a_1a_2\delta a_3\delta a_4$, $a_i \in \mathcal{A}$.

- *The grading δ is extended to the higher degrees by using the rule*

$$\delta(a_0\delta a_1 \cdots \delta a_p) \doteq \delta a_0\delta a_1 \cdots \delta a_p$$

Using in particular the last property, we find (this is a *consequence* of the definition)

$$\begin{array}{rcl} \delta(\omega_1\omega_2) & = & \delta(\omega_1)\omega_2 + (-1)^{deg(\omega_1)}\omega_1\delta(\omega_2) \\ \delta^2 & = & 0 \end{array}$$

Notice also that the *usual* rule for the commutation of differential forms simply does not make sense in this context.
If $\mathcal{A}$ has an involution $*$, then we can extend the differential algebra structure with

$$\begin{array}{rcl} (\delta a)^* & \doteq & -\delta(a^*) \\ (a_0\delta a_1 \cdots \delta a_q)^* & \doteq & (\delta a_q)^* \cdots (\delta a_1)^* a_0^* \end{array}$$

The usual cohomology is uninteresting here, because

$$\left\{ \begin{array}{rcccl} \frac{\mathbf{Ker}\ (\delta:\Omega^q\mathcal{A}\longrightarrow\Omega^{q+1}\mathcal{A})}{\mathbf{Im}\ (\delta:\Omega^{q-1}\mathcal{A}\longrightarrow\Omega^q\mathcal{A})} & = & 0 & \Longleftarrow & q \geq 1 \\ \mathbf{Ker}\ (\delta : \Omega^0\mathcal{A} \longrightarrow \Omega^1\mathcal{A}) & = & \mathbb{C} & & \end{array} \right. ,$$

i.e. it is trivial.

An interesting fact is now that the graded algebra we just defined is universal:

Proposition 2.4 *Suppose $\mathcal{A}$ is an associative algebra, and $(\bigoplus_n \Lambda^n, d)$ is a graded differential algebra; any morphism $\phi : \mathcal{A} \longrightarrow \Lambda^0$ can be extended in a unique way to a morphism $\psi : (\Omega\mathcal{A}, \delta) \longrightarrow (\bigoplus_n \Lambda^n, d)$ of graded differential algebras, in such a way that for every cell the following diagram commutes*

$$\begin{array}{cccc} \psi : & \Omega^q\mathcal{A} & \longrightarrow & \Lambda^q \\ & \delta \downarrow & & \downarrow d \\ \psi : & \Omega^{q+1}\mathcal{A} & \longrightarrow & \Lambda^{q+1} \end{array}$$

i.e. $d \circ \psi = \psi \circ \delta$.

The map is essentially defined by the following relation

$$\psi(a_0 \delta a_1 \cdots \delta a_q) = \phi(a_0) d\phi(a_1) \cdots d\phi(a_q)$$

Being $\mathcal{A}$ a unital algebra, we now instance the universal graded algebra $\Omega\mathcal{A}$, with the definitions:

$$\begin{array}{ccl} \Lambda^0 & \doteq & \mathcal{A} \\ \Lambda^1 & \doteq & \mathbf{Ker}\ \left(j : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \longrightarrow \mathcal{A}\ ,\ a \otimes_{\mathbb{C}} b \longmapsto ab \right) \\ d : \mathcal{A} & \longrightarrow & \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}\ ,\ a \longmapsto a \otimes_{\mathbb{C}} \mathbb{I} - \mathbb{I} \otimes_{\mathbb{C}} a \end{array}$$

We extend the above definition to the higher degrees by the immersion

$$\Lambda^q \doteq \underbrace{\Lambda^1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \Lambda^1}_{q} \subset \underbrace{\mathcal{A} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{A}}_{q+1}$$

such that[16]

$$a_0 da_1 \cdots da_q \simeq a_0 (a_1 \otimes_{\mathbb{C}} \mathbb{I} - \mathbb{I} \otimes_{\mathbb{C}} a_1) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (a_q \otimes_{\mathbb{C}} \mathbb{I} - \mathbb{I} \otimes_{\mathbb{C}} a_q)$$

and that, being $\bullet$ and $\cdot$ respectively the internal and external multiplications

$$\begin{array}{l} \omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_q \bullet \omega_{q+1} \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_{q+p} \doteq \omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_{q+p} \\ a \cdot \omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_q \doteq (a\omega_1) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_q \\ \omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} \omega_q \cdot a \doteq \omega_1 \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} (\omega_q a) \end{array}$$

The derivation d is extended by using the Leibniz rule and the usual identity

$$d(a_0 da_1 \cdots da_q) \doteq da_0 da_1 \cdots da_q$$

[16]Words like $a \cdot b \otimes_{\mathbb{C}} c \otimes_{\mathcal{A}} g$ are rewritten as $(ab) \otimes_{\mathbb{C}} (cg)$, because of the immersion.

just as it has been done for the universal differential forms. This can be accomplished due to the fact that

$$\omega \in \Lambda^1 \Leftrightarrow \omega = \sum_i a_i \otimes_{\mathbb{C}} b_i \quad \text{with} \quad \sum_i a_i b_i = 0$$

We can define a very simple graded universal algebra defined precisely along these lines, which is an algebra of functions on a generic manifold $\mathcal{M}$.

An algebra of functions.

We now want to present a simple example of the above kind, based on the algebra $\mathcal{A} = \mathbf{C}(\mathcal{M}, \mathbb{C})$ of functions on a space $\mathcal{M}$. This is done with the identification

$$\mathcal{A} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{A} \rightsquigarrow \mathbf{C}(\mathcal{M} \times \cdots \times \mathcal{M})$$

and the multiplications

$$(f \bullet g)(x_1, \dots, x_{q+p}) \doteq f(x_1, \dots, x_{q+1}) g(x_{q+1}, \dots, x_{q+p})$$
$$(h \cdot f)(x_1, \dots, x_{q+1}) \doteq h(x_1) f(x_1, \dots, x_{q+1})$$
$$(f \cdot h)(x_1, \dots, x_{q+1}) \doteq f(x_1, \dots, x_{q+1}) h(x_{q+1})$$

and the differential operator

$$df \doteq (\mathbb{I} \otimes_{\mathbb{C}} f - f \otimes_{\mathbb{C}} \mathbb{I})$$

which can be extended with

$$df(x_1, \dots, x_n) \doteq \sum_{i=1}^{q+1} (-1)^{i+1} f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{q+1})$$

on the spaces

$$\Lambda^q \doteq \{f \in \mathbf{C}(\mathcal{M} \times \cdots \times \mathcal{M}) \mid f(x_1, \dots, x_{i-1}, x, x, x_{i+1}, \dots, x_{q+1} = 0)_{i=1,\dots,q+1}\}$$

2.6 Spectral Triple

Definition 2.24 (Spectral Triple) *A triple $(\mathcal{A}, \mathcal{H}, \not{D})$ with $\mathcal{H}$ a Hilbert space, $\mathcal{A}$ a unital C^*-algebra of bounded operators*[17] *on $\mathcal{H}$,*[18] *and $\not{D}$ (the* Dirac operator*) a self adjoint operator on $\mathcal{H}$ satisfying*

[17] For technical reasons, we define here the spectral triple only on unital algebras. The modifications of the conditions for $\mathcal{A}$ nonunital may be found e.g. in [18], at section 3 and the following.

[18] Actually we should consider a generic representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on $\mathcal{H}$. We will usually omit the symbol π, except in some cases, just for the sake of simplicity. We just notice that the irreducible representations of $\mathcal{A}$ have a geometric meaning, see the discussion at section (2.3) of the generalisation of Gelfand-Naimark theorem to a noncommutative C^*-algebra .

1. $\forall z \notin \mathbb{R}\ ,\ (\not{D} - z\ \mathbb{I})^{-1} \in \mathcal{K}(\mathcal{H})$

2. $\forall a \in \mathcal{A}\ ,\ [\not{D}, a] \in \mathcal{B}(\mathcal{H})$

Definition 2.25 (Even Spectral triple) *A spectral triple* $(\mathcal{A}, \mathcal{H}, \not{D})$ *with a self-adjoint unitary operator* $\Gamma : \mathcal{H} \longrightarrow \mathcal{H}$ *satisfying* $\{\Gamma, \not{D}\} = 0$ *and* $\forall a \in \mathcal{A}\ ,\ [\Gamma, a] = 0$. *If the* grading Γ *does not exist, the triple is an* **odd triple**.

We now introduce the analog of *dimension* of a manifold into this abstract framework.

Definition 2.26 *The spectral triple* $(\mathcal{A}, \mathcal{H}, \not{D})$ *is said of* **dimension d** *if the operator* $||\not{D}||^{-d}$ *is an infinitesimal of first order (see def. (2.11)). The dimension is intended to be nonnegative.*

Definition 2.27 (Real Spectral triple) *An even spectral triple* $(\mathcal{A}, \mathcal{H}, \not{D}, \Gamma)$ *of dimension d, with an antilinear isometry* $J : \mathcal{H} \longrightarrow \mathcal{H}$ *respecting the following conditions*

1. $J^2 = \epsilon_1(d)\, \mathbb{I}$

2. $J\not{D} = \epsilon_2(d) \not{D} J$

3. *for even dimension* $J\Gamma = i^d\ \Gamma J$

4. $[a, Jb^* J^*] = 0 \qquad a, b \in \mathcal{A}$

5. $[[\not{D}, a], Jb^* J^*] = 0$

where the 8*-tuples* $\epsilon_{1,2}$ *are*[19]

$$\epsilon_1 = (1, 1, -1, -1, -1, -1, 1, 1) \qquad \epsilon_2 = (1, -1, 1, 1, 1, -1, 1, 1)$$

In particular the last condition of the above ones is called the *first order* axiom, i.e. it is the generalisation of the fact the Dirac operator $\not{D}$ is a first order differential operator. Usually one requires also that, defined the derivation

$$\delta(\bullet) \doteq [|\not{D}|, \bullet]$$

then $\forall a \in \mathcal{A}$ we have $a, [\not{D}, a] \in \bigcap_k \mathrm{Dom}(\delta^k)$. Since in the commutative case this entails $a \in \mathbf{C}^\infty$, this is called the *smoothness axiom.*

[19] The argument of $\epsilon_{1,2}$ is intended in $\mathbb{Z}_8$, of course.

2.7 Connes' differentials

Given the spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$, we define the following representation of the universal algebra $\Omega\mathcal{A}$, induced by the representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ of the algebra $\mathcal{A}$

$$\pi : \delta a \longmapsto [\not{D}, a]$$

where π is extended as a morphism (and due to the self-adjointness of Dirac operator, also a $*$-morphism) of the complex $\Omega\mathcal{A}$, thrown by π in $\mathcal{B}(\mathcal{H})$. The usual rules for the extension of the derivatives apply in this case as well, i.e.

$$\begin{array}{rcl} a_0 \delta a_1 \cdots \delta a_q & \longmapsto & \pi(a_0)[\not{D}, \pi(a_1)] \cdots [\not{D}, \pi(a_q)] \\ \pi(\delta(a_0 \delta a_1 \cdots \delta a_q)) & \longmapsto & [\not{D}, \pi(a_0)][\not{D}, \pi(a_1)] \cdots [\not{D}, \pi(a_q)] \end{array}$$

and the Leibniz rule.
But now some trouble occurs, because *it is not true* that the image of $\Omega\mathcal{A}$ by π is a correct algebra of forms. Indeed we notice that there exist forms for which $\pi(\omega) = 0$ and instead $\pi(\delta(\omega) \neq 0$, the so called **junk forms**. Since the things so far have been kept very abstract, it may be difficult to visualise these objects. Actually, in the commutative case, their arise is essentially due to the lack of noncommutativity of the substitute of the Grassman product, i.e. the formal product we obtained just writing the factors one by one, in a row. We want to show this concretely. Take a manifold $\mathcal{M}$ and the triple $(\mathcal{A} \doteq \mathbf{C}^\infty(\mathcal{M}), \mathcal{H} \doteq \mathbf{L}^2(\mathcal{M}, \mathcal{S}), \not{D} \doteq \gamma^\mu \partial_\mu)$, with $\mathcal{S}$ the space of spinors. The Dirac operator is just the usual one, well known from physics, with γ^μ the usual gamma matrices (actually sections of Clifford bundle over $\mathcal{M}$). It can be shown that this triple (once it is made even and real, according to our definitions before) represents the usual Riemann geometry of spin manifolds. We just want to show what junk forms are in this context. So we write (π is just the representation by mean of multiplication by a function)

$$\forall f \in \mathcal{A} \quad \pi(\delta f) = [\not{D}, f] = \gamma^\mu \partial_\mu f$$

A generic 1-form is

$$\omega(x) = \sum_k f_k(x) \gamma^\mu(x) \partial_\mu g_k(x)$$

we notice that the 1-form of $\Omega^1\mathcal{A}$

$$\omega_{\mathtt{junk}} \doteq f\delta f - (\delta f) f = 2f\delta f - \delta(f^2)$$

which is clearly non zero, is represented in this way

$$\pi(\omega_{\mathtt{junk}}) = f\gamma^\mu\partial_\mu f - (\gamma^\mu\partial_\mu f)f = 0$$

and its derivative instead is

$$\pi(\delta\omega_{\mathtt{junk}}) = 2[\not{D}, f][\not{D}, f] = 2\gamma^\mu\gamma^\nu\partial_\mu f\partial_\nu f = -4g^{\mu\nu}\partial_\mu f\partial_\nu f \;\; \mathbb{I}$$

which is clearly nonzero for non-constant $f(x)$. Notice that in the classical Grassmann product this "symmetric" part simply does not appear. The idea now is to eliminate such terms, in order to make noncommutative differential geometry analog to ordinary one.

Definition 2.28 (Connes' Differential Forms) *The graded algebra defined by*

$$\Omega_{\not{D}}\mathcal{A} \doteq \frac{\Omega\mathcal{A}}{\mathcal{J}_0 + \delta\mathcal{J}_0}$$

where we meant by $\mathcal{J}_0$ the following object[20]

$$\mathcal{J}_0 \doteq \bigoplus_q \{\omega \in \Omega^q\mathcal{A} | \pi(\omega) = 0\}$$

The representation of $\frac{\Omega\mathcal{A}}{\mathcal{J}_0+\delta\mathcal{J}_0}$ is just $\frac{\pi(\Omega\mathcal{A})}{\pi(\delta\mathcal{J}_0)}$, so that taking the quotient is the same as to eliminate the forms $\delta\omega$ for which $\pi(\omega) = 0$ (because $\omega \in \mathcal{J}_0$) and $\pi(\delta\omega) \neq 0$.
It can be shown rigorously that the Connes' algebra $\Omega_{\not{D}}\mathcal{A}$ is isomorphic in the commutative case to the Grassmann algebra of differential forms, and so it is the noncommutative generalisation of the latter.

2.8 Connections and Gauge *fields*

Let us take a right finite projective $\mathcal{A}$-module $\mathcal{E}$. For the Serre-Swan theorem, when $\mathcal{A}$ is commutative, $\mathcal{E}$ is the module of sections of some bundle. Even when the algebra is noncommutative, we want to give a meaning to the concept of connections on such "bundles".

[20] It is easy to show that $\mathcal{J}_0 + \delta\mathcal{J}_0$ is a two sided ideal, with differential grading, so that the quotient keeps the property of $\Omega\mathcal{A}$.

Definition 2.29 (Universal Connection) *It is a* $\mathbb{C}$*-linear map*

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^q \mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{q+1} \mathcal{A}$$

which in addition follows the Leibniz rule

$$\forall \omega \in \Omega\mathcal{A}, v \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^q \mathcal{A} \qquad \nabla(v \cdot \omega) = \nabla v \cdot \omega + (-1)^q v \cdot \delta\omega$$

Usually in field theory also the curvature needs to be defined. It is

$$\mathbf{R} \doteq \nabla^2|_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^2 \mathcal{A}$$

It is easy to show that ∇^2 is also $\mathcal{A}$-linear

$$\nabla^2(v \cdot \omega) = (\nabla^2 v) \cdot \omega$$

and satisfies the *Bianchi identity*

$$[\nabla, \mathbf{R}] = 0$$

We want to explicitly write an alternative view over the connection ∇. We could view it as the map

$$[\nabla, \bullet] : End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega^q \mathcal{A} \longrightarrow End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega^{q+1} \mathcal{A}$$

This is customary when physicists say something about covariant derivatives in the *Noncommutative field theory.* Now we state the important theorem

Theorem 2.3 *Any module* $\mathcal{E}$ *is projective if and only if it admits a connection.*

For any finite projective $\mathcal{A}$-module $\mathcal{E}$ it is defined a natural connection, the so called *Grassmann connection*; given the surjection $\phi : \mathcal{A}^M \longrightarrow \mathcal{E}$ as in the proposition (2.3), and its right inverse $\psi : \mathcal{E} \longrightarrow \mathcal{A}^M = \mathbb{C}^M \otimes_{\mathbb{C}} \mathcal{A}$, and the projection $p : \mathcal{A}^M \longrightarrow \mathcal{E}$, we may define[21]

$$\nabla_{\mathtt{Gr}} \doteq p \circ (\,\mathbb{I} \otimes \delta) \circ \psi \;\; : \;\; \mathcal{E} \otimes_{\mathcal{A}} \Omega^q \mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{q+1} \mathcal{A}$$

with δ the universal differential grading, and ψ as well as p have been extended in the obvious way in order to be defined on the tensor products with the algebra $\Omega\mathcal{A}$. One could even write, for short $\nabla_{\mathtt{Gr}} = p\delta$. It is easy to see that the difference between two

[21] Both ψ and p exist due to the definition (2.22)

connections, just like in ordinary geometry, is an $\mathcal{A}$-linear operator, i.e. it belongs to $End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$, so that we could write a generic universal connection as

$$\nabla = \nabla_{\mathtt{Gr}} + A \qquad A \in End_{\mathcal{A}}\mathcal{E} \otimes_{\mathcal{A}} \Omega\mathcal{A}$$

where we give A the natural name of *gauge potential.*

When the module $\mathcal{E}$ is given an Hermitian structure, we may demand the connection to be **compatible** with this structure. This is the requirement

$$\forall \eta, \zeta \in \mathcal{E} \qquad \delta\langle \eta, \zeta\rangle = \langle \eta, \nabla\zeta\rangle - \langle \nabla\eta, \zeta\rangle$$

and the sesquilinear form has been extended in the obvious way to the tensor product $\mathcal{E} \otimes_{\mathcal{A}} \Omega^1\mathcal{A}$. It is easy to see that the Grassmann connection is compatible, and that for a general connection given by $\nabla_{\mathtt{Gr}} + A$ the compatibility requires that the gauge potential be hermitian: $A = A^*$.

What we have done so far was aimed at the connections coming from the universal calculus. But the same formal things can be re-done verbatim for the Connes' calculus, due to the universality properties previously stated. So to deal with the differential calculus it is enough to consider connections as maps

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^q_{D\!\!\!/}\mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^{q+1}_{D\!\!\!/}\mathcal{A}$$

following the rule

$$\forall \omega \in \Omega_{D\!\!\!/}\mathcal{A}, v \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^q_{D\!\!\!/}\mathcal{A} \qquad \nabla(v \cdot \omega) = \nabla v \cdot \omega + (-1)^q v \cdot d\omega$$

with $d(\bullet) \doteq [D\!\!\!/, \bullet]$. A generic (compatible) connection is now of course

$$\nabla = \nabla_{\mathtt{Gr}} + A = pd + A \qquad \text{with} \quad A = A^*$$

and so on.

2.8.1 Gauge transformations, Diffeomorphisms

Given a (left) finite projective $\mathcal{A}$-module $\mathcal{E}$, the $\mathcal{A}$ linear transformations of $\mathcal{E}$ to itself form the algebra of *Endomorphisms* of the module $\mathcal{E}$. The latter is called $End_{\mathcal{A}}\,\mathcal{E}$

$$End_{\mathcal{A}}\,\mathcal{E} \doteq \{\phi : \mathcal{E} \longrightarrow \mathcal{E} \mid \forall a \in \mathcal{A} v \in \mathcal{E}\,,\ \phi(a \cdot v) = a \cdot \phi(v)\}$$

If the module is Hermitian we can define $End_{\mathcal{A}}\,\mathcal{E}$ as an involutive algebra, with an involution $*$ given by the usual rule

$$\langle v_1, Bv_2\rangle \doteq \langle B^* v_1, v_2\rangle$$

Given the canonical isomorphism $\mathcal{E} \simeq p\mathcal{A}^M$, with hermitian idempotent p, this algebra is clearly isomorphic to the "projected" algebra $p(\mathcal{A} \otimes_{\mathbb{C}} {}_M\mathbb{C}_M)p = p_M\mathcal{A}_M p$, so that we could identify the endomorphisms of $\mathcal{E}$ as all the matrices $b \in {}_M\mathcal{A}_M$ which commute with the projector $p \circ b = b \circ p$.

The algebra $End_{\mathcal{A}}\,\mathcal{E}$ has a subgroup formed by all the unitary endomorphisms (so that they are automorphisms)

$$\mathcal{U}(\mathcal{E}) \doteq \{u \in End_{\mathcal{A}}\,\mathcal{E} \mid u^*u = \mathbb{I} = uu^*\}$$

For $\mathcal{U}(\mathcal{E})$ in particular is true that given a finite projective $\mathcal{A}$-module $\mathcal{E}$

$$\mathcal{E} \simeq p\,\mathcal{A}^M p, \qquad \mathcal{U}(\mathcal{E}) \simeq p\,\mathcal{U}(\mathcal{A}^M)p$$

The action of the unitary group $\mathcal{U}(\mathcal{E})$ on the universal compatible connection ∇ is given by the natural law

$$u : \nabla \longmapsto u\nabla u^*$$

It follows that the curvature transform in the same way as well

$$u : \nabla^2 \longmapsto u\nabla^2 u^*$$

The gauge potential instead transforms[22]

$$u : A \longmapsto uAu^* + up\delta u^*$$

Of course this is true also when instead of the universal connection one considers the Connes' connection, just in the same way it has been done above. For the potential in particular we rewrite the above transformation rule as

$$u : A \longmapsto uAu^* + updu^*$$

Now we take the unital C^*-algebra $\mathcal{A}$ and consider its group of automorphisms, $Aut(\mathcal{A})$. This group has a normal (i.e. invariant[23]) subgroup, made up by automorphisms of the form

$$\forall a \in \mathcal{A} \quad \phi_u : a \longmapsto uau^* \qquad u \in \{u \in \mathcal{A} \mid uu^* = \mathbb{I} = u^*u\}$$

This normal subgroup is the group of *Inner automorphisms* $Inn(\mathcal{A}) \lhd Aut(\mathcal{A})$. To interpret the role of this automorphisms, we now get a commutative unital C^*-algebra

[22] We use short notations, in which $\mathcal{E}$ has been identified with $p\mathcal{A}^M$

[23] In the sense it is left invariant by any automorphism of $\mathcal{A}$

$\mathbf{C}^\infty(\mathcal{M})$ for some compact manifold $\mathcal{M}$. It can be proved by considering the appropriate pullbacks, that the automorphisms in this case are just the diffeomorphisms, i.e.:

$$Aut(\mathbf{C}^\infty(\mathcal{M})) \simeq Diff(\mathcal{M})$$

Of course, being this a commutative C^*-algebra , all the automorphisms are outer ones $Out(\mathcal{A}) \doteq Aut(\mathcal{A})/Inn(\mathcal{A}) = Aut(\mathcal{A})$, since $Inn(\mathcal{A})$ is trivial; for a noncommutative C^*-algebra the correct analog of the diffeomorphisms are the outer automorphisms, indeed the normal subgroup $Inn(\mathcal{A})$ leaves invariant each irreducible representation of $\mathcal{A}$ on $\mathcal{H}$, i.e. any "point" in the (noncommutative) space.
Anyway, given a real spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ of dimension d with real structure J, where the C^*-algebra $\mathcal{A}$ is represented by $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$, we can see that any $u \in \mathcal{U}(\mathcal{A})$ generates an isomorphism with the new spectral triple $(\mathcal{A}, \mathcal{H}, \not{D}+u[\not{D}, u^*]+\epsilon_2(d)Ju[\not{D}, u^*]J^*)$ where the C^*-algebra $\mathcal{A}$ is represented by another representation, namely the composition of the old one with the inner automorphism generated by u, $\pi' = \pi \circ \phi_u$. This gives an interpretation of the inner automorphisms as "gauge transformations" of the noncommutative geometry, and in turn of the gauge degrees of freedom as inner fluctuations of the noncommutative geometry.

2.9 Integration, or (Dixmier) Trace

Let $T \in \mathcal{K}(\mathcal{H})$ be a compact operator on some Hilbert space. As in definition (2.11), we can classify T by the decay rate of the eigenvalues $\{\gamma_m(T)\}$ of its *norm operator* $\sqrt{T^*T}$. If T_1 and T_2 are two infinitesimals of order respectively μ_1 and μ_2, then the operator $T_1\,T_2$ is of order not greater than $\mu_1+\mu_2$. Moreover the space of infinitesimal on $\mathcal{H}$ form a two-sided ideal of $\mathcal{B}(\mathcal{H})$ Now consider the (generally divergent) sequence of partial sums

$$\langle T \rangle_M \doteq \sum_{n=0}^{M} \gamma_m(T)$$

For first order infinitesimals, the above sequence is logarithmically divergent. We want our "noncommutative integral" to have non vanishing value only for infinitesimal of first order. The first step is to define it on positive infinitesimals of first order. The one can extend it by linearity, because of the fact the ideal of first order infinitesimal is generated by its positive part. Now let T be such a positive infinitesimal, we can define an interpolation to non integer values of $\langle T\rangle_M$, and then the Cesaro mean

$$\mathbf{tr}_{\Lambda}(T) \doteq \frac{1}{\log\Lambda}\int_e^{\Lambda}\frac{dt}{t}\frac{\langle T\rangle_t}{\log t}$$

which is bounded due to the fact

$$\langle T\rangle_t \leq C\log t$$

and moreover is *asymptotically linear* in the sense that

$$|\ \mathbf{tr}_{\Lambda}(T_1+T_2)-\mathbf{tr}_{\Lambda}(T_1)-\mathbf{tr}_{\Lambda}(T_2)\ | \leq B\frac{\log\log\Lambda}{\log\Lambda}$$

So any limit point of $\mathbf{tr}_{\Lambda}(T)$ defines a linear positive trace, vanishing for infinitesimals of order greater than one. In most cases of physical interest (like Yang-Mills and Gravity), $\mathbf{tr}_{\Lambda}(T)$ converges, so that the integral does not depend on the limit point one choses (see [8] chapter VIII and [28] section 6.2 and 6.3).

Chapter 3

Landau levels

In this chapter we will analyse the problem of electrons confined to move in a plane, interacting with an orthogonal magnetic field (see [6]). We will start with the usual one body problem, and then we will show in detail the projection to the lowest level states, together with its generalisation to the lowest $N+1$ states, in particular explaining the consequences in terms of noncommutativity. Then we will show one more deformation of the algebra defining the Landau levels, which introduces a noncommutative geometry as well, and presents some interesting physical features for a system in a noncommutative space, for the sake of physical intuition.

3.1 The one body problem

First we need the hamiltonian for an electron in a uniform constant magnetic field. The hamiltonian is written in the standard fashion

$$\mathfrak{H} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 \tag{3.1}$$

We will choose the so-called symmetric gauge, for it keeps manifest the azimuthal symmetry of the problem. So, in cartesian coordinates

$$\mathbf{A} = \frac{B}{2}(-x_2, x_1)$$

We may as well suppress the $\hat{\mathbf{x}}_3$ coordinate along which the magnetic field is directed. The momentum operator

$$\mathbf{p} = -i\nabla$$

will be considered acting on the first two coordinates of wave functions. Let us introduce complex coordinates now. We pose

$$\begin{cases} z &= x_1 + i x_2 \\ \bar{z} &= x_1 - i x_2 \\ \partial &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) \\ \bar{\partial} &= \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \end{cases}$$

It is also customary to use *magnetic units*, defined by:

$$\hbar = 1 \quad c = 1 \quad \ell = \sqrt{\frac{2\hbar c}{eB}} = 1$$

The last quantity is called the *magnetic length*. It is a length scale of the problem introduced by the presence of the magnetic field.
The hamiltonian may be written in a harmonic oscillator form, introducing the ladder operators[1]

$$a \doteq \frac{z}{2} + \bar{\partial} \qquad a^\dagger \doteq \frac{\bar{z}}{2} - \partial \tag{3.2}$$

They satisfy the usual commutation relation

$$\left[a\,,\,a^\dagger\right] = 1 \tag{3.3}$$

So the hamiltonian takes the form

$$\mathfrak{H} = 2\;a^\dagger\,a\;+\;1 \tag{3.4}$$

There is another conserved quantity, the angular momentum, which is conserved due to the rotational invariance, it is . To write it, we introduce two more ladder operators, commuting with the a's

$$b \doteq \frac{\bar{z}}{2} + \partial \qquad b^\dagger \doteq \frac{z}{2} - \bar{\partial} \tag{3.5}$$

They satisfy the equation

$$\left[b\,,\,b^\dagger\right] = 1 \tag{3.6}$$

These operators can be shown to be the generators of the magnetic translations[17, 15]. The algebra of the latter ones is

$$[\hat{\chi}_i, \hat{\chi}_j] = iq\;\epsilon_{ijk}\;B_k \qquad \text{where} \;\; B_k = \mathbf{B}\cdot\hat{\mathbf{x}}_\mathbf{k} \;\; \text{is the magnetic field}$$

[1] The a and $a^\dagger$ operators are manifestly the covariant derivatives.

which in our two dimensional case, with $\mathbf{B} = B\,\hat{\mathbf{x}}_\mathbf{3}$, becomes[2]

$$[\hat{\chi}_1, \hat{\chi}_2] = i\, q\, B \qquad\qquad \text{and } \chi_3 \approx 0$$

or in complex coordinates

$$[\hat{\chi}, \hat{\bar{\chi}}] = \frac{B}{2} \qquad \begin{cases} \hat{\chi} & \doteq & \frac{1}{2}(\hat{\chi}_1 + i\hat{\chi}_2) \\ \hat{\bar{\chi}} & \doteq & \frac{1}{2}(\hat{\chi}_1 - i\hat{\chi}_2) \end{cases}$$

In magnetic units this commutator becomes that of b operators (3.6) Now we can write the angular momentum

$$\mathfrak{J} = \; b^\dagger\, b \; - \; a^\dagger\, a \tag{3.7}$$

We see that $[\mathfrak{H}, \mathfrak{J}] = 0$ so that a base for the Hilbert space is given in term of simultaneous eigenstates of both the operators, in this form

$$\psi_{mn} \doteq \frac{b^{\dagger m}}{\sqrt{m!}} \frac{a^{\dagger n}}{\sqrt{n!}}\; \psi_0 \tag{3.8}$$

with

$$\begin{cases} \mathfrak{H}\psi_{mn} & = & (2n+1)\; \psi_{mn} \\ \mathfrak{J}\psi_{mn} & = & (m-n)\; \psi_{mn} \end{cases} .$$

The states ψ_{mn} are normalized by

$$\langle \psi_{mn} | \psi_{kl} \rangle = \int d^2 z\, \psi^*_{mn}(z, \bar{z}) \psi_{kl}(z, \bar{z}) e^{-|z|^2}$$

The basic wavefunction $\psi_0(z, \bar{z}) = \langle z, \bar{z} \,|\, \psi_0 \rangle$ is solution of

$$a|\psi_0\rangle = 0 \qquad , \qquad b|\psi_0\rangle$$

and therefore is gaussian:

$$\langle z, \bar{z} \,|\, \psi_0 \rangle = \psi_0(z, \bar{z}) = \frac{1}{\sqrt{\pi}}\; e^{-\frac{|z|}{2}^2} \qquad\qquad \|\psi_0\|^2 = 1.$$

We see that each energy level (Landau levels) is infinitely degenerate. Let us give a look to the lowest Landau level, the level with $n = 0$. The wave functions of these states are

$$\psi_{m0}(z, \bar{z}) = \frac{1}{\sqrt{\pi}} \frac{z^m}{\sqrt{m!}}\; e^{-\frac{|z|}{2}^2}$$

[2]The $\chi_3 \approx 0$ constraint is actually a secondary constraint coming out by requiring the hamiltonian to be first class.

These are the wave functions of particles localized in a "fuzzy" annulus, because the probability distribution is angle-independent and peaked at $|z|^2 = m$. So the lowest level is made up by concentric layers. In the higher Landau levels, the wave functions present, besides the power factor, a generalised Laguerre polynomial factor.

We may count the states in each Landau level, in a disc[3] of radius R, their number being $n_e = \frac{R^2}{\ell^2} = \frac{\Phi}{\Phi_0}$ being $\Phi = \pi R^2 B$ the magnetic flux through the disc and $\Phi_0 = \pi \ell^2 B$ the *quantum of magnetic flux*. So we may say that in each Landau level there is one state for each *flux quantum* through the disc.

3.2 W_∞ algebra

By using the fact that the generators of magnetic translation b, $b^\dagger$ commute with the hamiltonian $\mathfrak{H}$, we can construct several obviously conserved quantities [6]

$$\mathcal{L}_{nm} \doteq (b^\dagger)^n b^m \tag{3.9}$$

We may ask now which Nöther symmetry they generate. Their algebra is

$$[\mathcal{L}_{nm}, \mathcal{L}_{kl}] = \sum_{i=1}^{m \curlywedge k} \frac{m!k!}{(m-i)!(k-i)!i!} \mathcal{L}_{n+k-i,m+l-i} - \left({}^{m \leftrightarrow l}_{n \leftrightarrow k} \right) \tag{3.10}$$

which, up to higher quantum corrections (we restore for a moment $\hbar$), reads

$$[\mathcal{L}_{nm}, \mathcal{L}_{kl}] = \hbar(mk - nl)\mathcal{L}_{n+k-1,m+l-1} + O(\hbar^2) \tag{3.11}$$

This is known to be the algebra of (classical) area preserving diffeomorphisms, or w_∞. The algebra defined by (3.10), like all the quantum generalisations of (3.11), is called W_∞ algebra.

3.2.1 Second quantization

We can give now a more intuitive description of the generators of W_∞ algebra (see [6] and refs. therein) by using second quantization. Namely, given the wavefunctions (3.8), we define the field operators

$$\hat{\phi}(z, \bar{z}) \doteq \sum_{ln} c_{ln} \psi_{ln}(z, \bar{z})$$

[3]This configuration is known as the *Corbino disc geometry*

where we have used the Fock (fermionic) operators

$$[c_{ln}, c^\dagger_{km}]_+ = \delta_{lk}\delta_{nm}$$

as usual in field theory, acting on a Hilbert space defined from a vacuum $|0\rangle$ as the (closure of the) linear span of the set

$$\{\prod_i c^\dagger_{k_i n_i}|0\rangle\}$$

The second quantized version of the $\mathcal{L}_{st}$ operators is

$$\mathcal{L}_{st} \doteq \int d^2z\, \hat{\phi}^\dagger(z,\bar{z})(b^\dagger)^n b^m \hat{\phi}(z,\bar{z}) = \sum_{n=0}^{\infty}\sum_{l=s}^{\infty} c^\dagger_{l,n} c_{t+l-s,n} \frac{\sqrt{l!(t+l-s)!}}{(l-s)!} =$$
$$= \sum_{n=0}^{\infty}\sum_{l=t}^{\infty} c^\dagger_{l,n} c_{s+l-t,n} \frac{\sqrt{l!(s+l-t)!}}{(l-t)!}$$

It is manifest that Landau levels with different principal quantum numbers (the number of $a^\dagger$.s in the state) are not connected by the $\mathcal{L}_{st}$ operators. Each term of the sum

$$c^\dagger_{l,n} c_{s+l-t,n} \frac{\sqrt{l!(s+l-t)!}}{(l-t)!}$$

simply shuffles the particles within the same (n-th) level, varying their angular momentum. When an electron is shifted on an orbital with larger radius, then its angular momentum is increased, while it decreases if the radius of the final orbital is smaller.

We are only interested now in the lowest Landau level ($n = 0$), and in the action of $\mathcal{L}_{st}$ on the ground state. The latter is the state with the minimum angular moment, which is simply, for N particles

$$|\Omega\rangle \doteq c^\dagger_{N,0} \cdots c^\dagger_{0,0}|0\rangle$$

Applying a generator of W_∞ to $|\Omega\rangle$, we notice immediately that it vanishes identically if $s < t$, while it reduces to a number in the case $s = t$

$$\begin{cases} \mathcal{L}_{st}|\Omega\rangle = 0 & \Leftarrow\ t > s \\ \mathcal{L}_{ss}|\Omega\rangle = \frac{(N+1)!}{(s+1)(N-s)!}|\Omega\rangle & \end{cases}$$

So the only nontrivial case is when $s > t$, in which case its effect on the ground state is that of increasing the angular momentum of the ground state $|\Omega\rangle$ by shifting electrons from inside the *Fermi sphere* to more external orbitals. So the incompressibility of the ground state is simply due to the fact it is the state with minimum angular momentum, and can be written by the *highest weight-like* conditions ([6])

$$\mathcal{L}_{st}|\Omega\rangle = 0 \Leftarrow s < t \tag{3.12}$$

We stress here that the commutation relations close within the set of $\mathcal{L}_{st}$ with $s < t$. So the whole Lie subalgebra generated by $\{\mathcal{L}_{st}\}_{s<t}$ annihilates the ground state.

3.3 The truncation to a finite number of Landau levels

3.3.1 The theory in the lowest level

For $B \to \infty$, we may wish to consider only the states belonging to the lowest Landau level. They have been written above, as a gaussian times an entire function of z. Now we characterise them by a projector that maps any wavefunction to its $n = 0$ component. Similarly any operator is sandwiched between two copies of the projector

$$\mathbb{I}_0 \doteq \sum_{m=0}^{\infty} \psi_{m0} \circ \psi^{\dagger}_{m0} \tag{3.13}$$

This operator projects on the levels with $n \leq N$. To pick out the lowest at all, we put $N = 0$.

We can see that the commutation relations *are not* left unchanged by this (nonunitary) transformation. In particular, we may compute that:

$$\begin{array}{lcl} [z, \bar{z}] & \rightsquigarrow & [z, \bar{z}]_N = -\sum_{m=0}^{\infty} \psi_{m0} \circ \psi^{\dagger}_{m0} \\ [\partial, \bar{\partial}] & \rightsquigarrow & [\partial, \bar{\partial}]_N = -\frac{1}{4} \sum_{m=0}^{\infty} \psi_{m0} \circ \psi^{\dagger}_{m0} \end{array} \tag{3.14}$$

We see that the algebra of functions of the coordinates of the problem, abelian at the beginning, is made noncommutative by this projection, as well as the algebra generated by the derivatives. To be more specific, we have obtained the algebra of the noncommutative plane, generated by the projected operators $z_0 \doteq \mathbb{I}_0 z\, \mathbb{I}_0$ and $\bar{z}_0 \doteq \mathbb{I}_0 \bar{z}\, \mathbb{I}_0$ which satisfies

$$[z_0, \bar{z}_0] = -\,\mathbb{I}_0$$

For the derivative operators it is

$$[\partial_0, \bar{\partial}_0] = -\frac{1}{4}\,\mathbb{I}_0$$

3.3.2 Projection to the first $N+1$ Landau levels

This projector is generalised to higher levels (see also [32, 31]) with the definition

$$\mathbb{I}_N \doteq \sum_{n=0}^{N}\sum_{m=0}^{\infty} \psi_{mn} \circ \psi_{mn}^{\dagger} \tag{3.15}$$

In this way we find the following results

$$\begin{cases} [z,\bar{z}]_N &= -(N+1)\sum_{m=0}^{\infty} \psi_{mN} \circ \psi_{mN} = -(N+1)(\mathbb{I}_N - \mathbb{I}_{N-1}) \\ \left[\partial,\bar{\partial}\right]_N &= -\frac{N+1}{4}\sum_{m=0}^{\infty} \psi_{mN} \circ \psi_{mN} = -\frac{N+1}{4}(\mathbb{I}_N - \mathbb{I}_{N-1}) \\ [\partial,z]_N &= \left[\bar{\partial},\bar{z}\right]_N = \mathbb{I}_N - \frac{N+1}{2}\sum_{m=0}^{\infty} \psi_{mN} \circ \psi_{mN} = \frac{N-1}{2}\,\mathbb{I}_N + \frac{N+1}{2}\,\mathbb{I}_{N-1} \\ [\partial,\bar{z}]_N &= \left[\bar{\partial},z\right]_N = 0 \end{cases} \tag{3.16}$$

In this equations we do not find any longer the noncommutative plane algebra, because the commutator $[z_N, \bar{z}_N]$ is not proportional to the identity anymore. We notice that as the number N is sent to infinity, the sequence of a generic projected operator $A_N \doteq \mathbb{I}_N A\,\mathbb{I}_N$ *does not* converge operatorially to anything, as can be seen by the fact the norm of the operator $\sum_{m=0}^{\infty} \psi_{mN} \circ \psi_{mN}$ equals one for each N. Anyway, it converges weakly, i.e., the convergence is limited to any matrix element between normalizable states. This is also a consequence of the fact one cannot define a derivative on a finite rank matrix algebra, e.g. take X an hermitian $N\times N$ matrix generating the algebra of (formal) power series

$$\mathcal{A} = \{\sum_n a_n X^n\}$$

and take the derivative ∂_X be such that

$$[\partial_X, X] = \mathbb{I}$$

with $\mathbb{I}$ the $N\times N$ identity matrix. Then taking the trace of the above equation we have a $0 = 1$ inconsistency, because the trace of a finite rank commutator vanishes, while this is obviously not the case for the identity matrix. The only way out from this, exists when the matrices are "infinite" dimensional so that the trace is divergent (protecting the commutator, roughly speaking, by the usual linear manipulations).

3.4 Deformed Landau levels

This section is inspired by a work of Nair and Polychronakos [38] about quantum mechanics on noncommutative plane, we introduce effects of a noncommutative geometry in the well known physical problem of the quantum mechanics of Landau levels. We reconsider the algebra of the ladder operators $a, a^\dagger$ and $b, b^\dagger$, and generalise it as follows

$$\begin{cases} \left[a, a^\dagger\right] &= 1 \\ \left[b, b^\dagger\right] &= \beta \ \in \mathbb{R}_0^+ \\ \left[\, a, b \,\right] &= 0 = \left[a, b^\dagger\right] \end{cases} . \tag{3.17}$$

We want to keep the interpretation of this algebra as that of the quantum mechanics on a plane thread by the magnetic field; therefore we take the $a, a^\dagger$ operators as the kinematic momenta with which the Hamilton operator is made, and the $b, b^\dagger$ as the magnetic translations on the plane. So we have

$$\mathfrak{H} = 2a^\dagger a + 1 \qquad [b, \mathfrak{H}] = 0 = [b^\dagger, \mathfrak{H}]$$

We still have an Hilbert space built starting from a vacuum $|\psi_0\rangle$, by the application of both a and b. We use the same notation we employed before in the "ordinary" case, (3.8).

We can fix the form of the coordinate operators in terms of the a's and b's by considering what the commutation relations of the latter with $z, \bar{z}$ must be. We have the requirements

$$[z, a] = 0 \ , \ [z, a^\dagger] = 1$$

just as in the ordinary case, and

$$[b^\dagger, z] = 0 \ , \ [b, z] = 1$$

because of the transformation rules of the coordinates under magnetic translations. These relations fix the coordinates $z, \bar{z}$ to be

$$\begin{cases} z &\doteq b^\dagger / \beta + a \\ \bar{z} &\doteq b / \beta + a^\dagger \end{cases} \tag{3.18}$$

Since we want the rotational symmetry in our problem we must fix the form of the angular momentum, $\mathfrak{J}$, such that it both commutes $\mathfrak{H}$, and transforms the coordinates in the natural (vector) fashion, i.e.

$$\left[\mathfrak{J}, \mathfrak{H}\right] = 0 \qquad \left[\mathfrak{J}, z\right] = z \qquad \left[\mathfrak{J}, \bar{z}\right] = -\bar{z}$$

With this properties $\mathfrak{J}$ can be found to be

$$\mathfrak{J} = \frac{b^\dagger b}{\beta} - a^\dagger a$$

Of course the normalized eigenvectors of $\mathfrak{H}$ and $\mathfrak{J}$ are modified in the following way

$$\psi_{mn} \doteq \frac{b^{\dagger m}}{\sqrt{m!\beta^m}} \frac{a^{\dagger n}}{\sqrt{n!}}\, \psi_0 \tag{3.19}$$

From the (3.18) the noncommutativity relation of the coordinates can be computed to be

$$[z, \bar{z}] = 1 - \frac{1}{\beta}$$

Of course when $\beta = 1$ the original commutative theory is recovered. When $\beta \neq 1$, these coordinates do not have a straightforward meaning, because they are not c-numbers: let us discuss this point in more detail. In the study the quantum mechanics of a point charge in ordinary Landau levels. Usually what one does, is to pick up a pair of functions from $\mathcal{A}$ (since we are on a plane), and identify any value of the pair of coordinates, with a point on the plane. In the quantum theory, there exists the position operator, and to each point of the plane corresponds to a vector in an orthonormal complete set $\{|z, \bar{z}\rangle\}$ of eigenstates of position operator. As we have said in the section 2.2, in the more abstract algebraic framework, a point on a space is basically an equivalence class of irreducible representations of the algebra $\mathcal{A}$ of ($\mathbf{C}^{r\geq 0}$) functions on that space. From the same point of view of the above lines, each one of these equivalence classes is labelled by the eigenvalues of the coordinate operator, which are just c-numbers. The operators (3.18), do not form a complete system of operators, because they cannot be simultaneously diagonalized, and do not lead to pairs of coordinates. Hence, one obtains a less detailed information from the coordinates only.

3.4.1 The Weyl transform

When noncommutativity of coordinates has been introduced, we cannot describe physical quantities using pairs of coordinates. An idea is to consider Wigner functions. Basically we want to study the matrix element

$$\left(\psi_{l,0}\,,\, ⸬\delta(p - \bar{z})\delta(q - z)⸬\, \psi_{m,0}\right)$$

between two one-particle states of the lowest Landau level; here $⸬ \cdots ⸬$ means we are taking the symmetric (Weyl) ordering, that avoids ambiguities in the definition of the above equation. Another reason is the following. Let us introduce now the Weyl transform which maps functions to operators. Take the algebra of functions on the plane, and take the algebra (noncommutative plane) $\mathcal{A}_\theta$ generated by the operators x^i satisfying

$$\left[x^i, x^j\right] = \theta\epsilon^{ij}, \quad i,j = 1,2$$

We can associate to each function $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$ on the plane the operator of $\mathcal{A}_\theta$

$$U[f] \doteq \frac{1}{(2\pi)^2}\int d^2k \int d^2\xi \, e^{ik\cdot(x-\xi)} f(\xi) \tag{3.20}$$

This is a "noncommutative generalisation" of the Dirac δ relation

$$f(x) = \int dy \, \delta(x-y) f(y)$$

so that we can write, using complex coordinates:

$$U[f] = \int d^2\xi \; f(\xi) \; {}^\circ_\circ \delta(\xi_z - z)\delta(\xi_{\bar{z}} - \bar{z}) {}^\circ_\circ \tag{3.21}$$

This formula gives a precise meaning to the idea of "substituting" an operator for a coordinate in an ordinary function; indeed it allows us to write each operator of $\mathcal{A}_\theta$ in an unambiguous form. Moreover the equation (3.20) does automatically the job of ordering operator monomials in the most symmetric way. Now one can ask if it is possible to rewrite the product of two such operators in the same way, i.e. as a "operatorial" kernel smeared with a "classical" function. In particular we would like the product of the smearing functions be at least associative. By plugging in the definition of the Weyl operators, and using the Campbell-Baker-Hausdorff lemma, we can find that it is indeed possible, and that the product is given in terms of the following convolution

$$U[f]U[g] = \frac{1}{(2\pi)^2}\int d^2k \, e^{ik\cdot x} \int d^2\xi \, e^{-ik\cdot\xi} f \star g(\xi) = U[f \star g] \; , \tag{3.22}$$

which defines the Moyal product $\star$

$$f \star g(\xi) \doteq f(\xi) \; e^{\frac{\theta}{2}\overleftarrow{\partial_i} \, \epsilon^{ij} \, \overrightarrow{\partial_j}} \; g(\xi)$$

and the derivatives are meant to act on ξ variables according to the direction of the overset arrows. Of course this product is not commutative.

Every operatorial ordering of (3.21) defines a different quantization of the algebra of regular functions on the plane, but all of these quantizations are equivalent. Thus, we are free to chose the symmetric ordering, being the most natural one. In this case we have the algebra generated by the operators $z, \bar{z}$ satisfying

$$[z, \bar{z}] = 1 - \frac{1}{\beta}$$

The expression of the matrix element is:

$$\begin{aligned}\left(\psi_{l,0}\,,\,\circ\!\!\!\circ\,\delta(q-z)\delta(p-\bar z)\,\circ\!\!\!\circ\,\psi_{m,0}\right) &\doteq \int\frac{dxdy}{(2\pi)^2}\left(\psi_{l,0}\,,\,e^{i(qx+py)-i(zx+\bar z y)}\psi_{m,0}\right) =\\ &= \int\frac{dxdy}{(2\pi)^2}\,e^{i(qx+py)}\left(e^{i\bar y a}\psi_{l,0}\,,\,e^{-\frac{i}{\beta}yb}e^{-\frac{i}{\beta}xb^\dagger}e^{-ixa}\psi_{m,0}\right)e^{-\frac{xy}{2}}e^{\frac{xy}{2\beta}} =\\ &= \int\frac{dxdy}{(2\pi)^2}\,e^{i(qx+py)}\left(e^{\frac{i}{\beta}\bar y b^\dagger}\psi_{l,0}\,,\,e^{-\frac{i}{\beta}xb^\dagger}\psi_{m,0}\right)e^{-\frac{xy}{2}}e^{\frac{xy}{2\beta}}\,. \qquad (3.23)\end{aligned}$$

Here we used the fact that in the lowest Landau level the a operator vanishes, $a\psi_{l,0}=0$. The computation leads for the matrix elements

$$\left(e^{\frac{i}{\beta}\bar y b^\dagger}\psi_{l,0}\,,\,e^{-\frac{i}{\beta}xb^\dagger}\psi_{m,0}\right) = \sqrt{\frac{l!m!}{\beta^{l+m}}}\sum_{s=0}^{m\curlywedge l}\frac{\beta^s(-ix)^{l-s}(-iy)^{m-s}}{(l-s)!(m-s)!s!}\;e^{-\frac{xy}{\beta}} \qquad (3.24)$$

where $m \curlywedge l \doteq \min\{m,l\}$. Notice that this is just a polynomial in x and y times the overall exponential. Now we can put it back into (3.23) and take the Fourier transform obtaining

$$\begin{aligned}&\left(\psi_{l,0}\,,\,\circ\!\!\!\circ\,\delta(q-z)\delta(p-\bar z)\,\circ\!\!\!\circ\,\psi_{m,0}\right) =\\ &= \frac{1}{\pi}\left|\frac{2\beta}{1+\beta}\right|\sqrt{\frac{l!m!}{\beta^{l+m}}}\sum_{s=0}^{m\curlywedge l}\frac{\beta^s}{(l-s)!(m-s)!s!}\left(-\frac{\partial}{\partial q}\right)^{l-s}\left(-\frac{\partial}{\partial p}\right)^{m-s}e^{-\frac{2\beta}{1+\beta}pq} \qquad (3.25)\end{aligned}$$

We can go on computing an alternative form that does not contain derivatives

$$\begin{aligned}&\left(\psi_{l,0}\,,\,\circ\!\!\!\circ\,\delta(q-z)\delta(p-\bar z)\,\circ\!\!\!\circ\,\psi_{m,0}\right) =\\ &= (-1)^{l+m}\frac{|2\beta|}{\pi|1+\beta|}\sqrt{\frac{l!m!}{\beta^{l+m}}}\sum_{s=0}^{m\curlywedge l}\sum_{t=0}^{(m\curlywedge l)-s}\frac{\beta^s\left(-\frac{2\beta}{1+\beta}\right)^{m+l-2s-t}p^{l-s-t}\,q^{m-s-t}}{(l-s-t)!\,(m-s-t)!\,s!\,t!}\;e^{-\frac{2\beta}{1+\beta}pq}\end{aligned} \qquad (3.26)$$

The above formula allow us to write any expectation value of the form $\left(\psi_{l,0}\,,\,U[f]\,\psi_{m,0}\right)$ as an integral on a "quasiclassical phase space" (q,p)

$$\begin{aligned}&\left(\psi_{l,0}\,,\,U[f]\,\psi_{m,0}\right) =\\ &\quad= \frac{\sqrt{l!m!}}{\pi}\frac{|2\beta|}{|1+\beta|}\sum_{s=0}^{m\curlywedge l}\sum_{t=0}^{(m\curlywedge l)-s}\frac{(-1)^t\beta^{s-\frac{l+m}{2}}\left(\frac{2\beta}{1+\beta}\right)^{m+l-2s-t}}{(l-s-t)!\,(m-s-t)!\,s!\,t!}\times\\ &\qquad\times\int dq\,dp\;f(q,p)\;e^{-\frac{2\beta}{1+\beta}pq}p^{l-s-t}\,q^{m-s-t} \qquad (3.27)\end{aligned}$$

Now, let us write down the expression of the wavefunctions of the first Landau level for $\beta = 1$

$$\psi_{l,0}(z,\bar{z}) = \frac{z^l}{\sqrt{\pi\, l!}}\; e^{-\frac{|z|}{2}^2}$$

After rescaling of the last integral, we can recognise it as the matrix element between the wavefunctions of appropriate states in the lowest Landau level for $\beta = 1$

$$\int dq\, dp\, f(q,p)\, e^{-\frac{2\beta}{1+\beta}pq} p^{l-s-t}\, q^{m-s-t} = \pi\sqrt{(l-s-t)!\,(m-s-t)!}\left(\frac{2\beta}{1+\beta}\right)^{s+t-\frac{m+l}{2}} \times$$
$$\times \left|\frac{1+\beta}{2\beta}\right| \int d\zeta\, d\bar{\zeta}\; \psi^{\beta=1}_{l-s-t,0}(\zeta,\bar{\zeta})^*\; f\left(\sqrt{\frac{1+\beta}{2\beta}}\zeta, \sqrt{\frac{1+\beta}{2\beta}}\bar{\zeta}\right)\; \psi^{\beta=1}_{m-s-t,0}(\zeta,\bar{\zeta}) \tag{3.28}$$

We see that (3.27) can be written as a linear combination of the analogous matrix element for $\beta = 1$, involving just the states with lower angular momentum ($\psi_{l',0}$ with lower l'). This implies that the deformation of the algebra considered here, does not violate the incompressibility defined in terms of W_∞ algebra (see section (3.2)): the matrix elements of any observables are indeed written in terms of $\beta = 1$ matrix elements between states of equal or lower angular momentum. In this interpretation, putting

$$\tilde{f}(\zeta,\bar{\zeta}) \doteq f\left(\sqrt{\frac{1+\beta}{2\beta}}\zeta, \sqrt{\frac{1+\beta}{2\beta}}\bar{\zeta}\right)$$

we can write

$$\left(\psi_{l,0}\, ,\, U[f]\;\psi_{m,0}\right) =$$
$$= \sqrt{l!m!}\sum_{s=0}^{l\curlywedge m}\;\sum_{t=0}^{(l\curlywedge m)-s} \frac{(-1)^t\;\left(\frac{1+\beta}{2}\right)^{s-\frac{l+m}{2}}}{s!\,t!\sqrt{(m-s-t)!(l-s-t)!}}\;\left(\psi^{\beta=1}_{l-s-t,0}\, ,\, \tilde{f}\;\psi^{\beta=1}_{m-s-t,0}\right) \tag{3.29}$$

3.4.2 Second quantization and density

Now we come back for a moment to the $\beta = 1$ situation, i.e. to the commutative case, for the theory projected to the first Landau level. In this context, one has the wavefunctions

$$\psi^{\beta=1}_{l,0}(z,\bar{z}) = \frac{z^l}{\sqrt{\pi\, l!}}\; e^{-\frac{|z|}{2}^2}$$

As discussed earlier, we introduce the second quantization, using a set of ladder (fermionic) operators

$$c_l\,,\; c_l^\dagger \qquad \{c_l, c_m^\dagger\} = \delta_{lm}$$

and the Fock space generated starting from the vacuum state $|0\rangle$ as the closure of the span of

$$\left\{ \prod_l c^\dagger_{k_l} \, |0\rangle; \; \forall l, \; k_l \in \mathbb{N}, \; k_{l-1} < k_l \right\}$$

with $\langle 0|0\rangle = 1$. The field $\phi(z,\bar{z})$ in second quantization is

$$\phi(z,\bar{z}) = \sum_l c_l \, \psi_{l,0}(z,\bar{z})$$

We need the ground state of the incompressible fluid of $N+1$ electrons

$$|\Omega\rangle \doteq c^\dagger_N \cdots c^\dagger_0 |0\rangle$$

Now we want to evaluate the expectation value of the density operator ρ of the field ϕ on this fundamental state $|\Omega\rangle$. The density is

$$\rho(z,\bar{z}) \doteq \phi^\dagger \phi(z,\bar{z}) = \sum_{kl} c^\dagger_l c_k \, \psi^*_{l,0}(z,\bar{z}) \psi_{k,0}(z,\bar{z})$$

For its expectation value one finds

$$\begin{aligned} \langle \Omega | \rho(z,\bar{z}) | \Omega \rangle &\doteq \sum_{kl} \psi^*_{l,0}(z,\bar{z}) \psi_{k,0}(z,\bar{z}) \, \langle 0 | c_0 \cdots c_N \, c^\dagger_l c_k \, c^\dagger_N \cdots c^\dagger_0 |0\rangle = \\ &= \sum_{l=0}^{N} \psi^*_{l,0}(z,\bar{z}) \psi_{l,0}(z,\bar{z}) \end{aligned}$$

This can be written as

$$\langle \Omega | \rho(z,\bar{z}) | \Omega \rangle = \sum_{l=0}^{N} \int d^2\zeta \;\, \psi_{l,0}(\zeta,\bar{\zeta}) \, \delta(\zeta - z) \delta(\bar{\zeta} - \bar{z}) \, \psi_{l,0}(\zeta,\bar{\zeta}) = \sum_{l=0}^{N} \left(\psi_{l,0} \, , \, \delta_z \delta_{\bar{z}} \psi_{l,0} \right)$$

For $\beta \neq 1$, we repeat the previous steps, obtaining the following relation

$${}_\beta\langle \Omega | U[\rho(\eta,\bar{\eta})] | \Omega \rangle_\beta = \sum_{k=0}^{N} \left(\psi_{k,0} \, , \, \circ\!\!\!\circ\, \delta_\eta \delta_{\bar{\eta}} \,\circ\!\!\!\circ\, \psi_{k,0} \right)$$

where η is a complex number which represents the point where we computed the density in the $\beta = 1$ framework of above. We can apply now our formula (3.29) to get the result after some manipulation

$${}_\beta\langle \Omega | U[\rho(\eta,\bar{\eta})] | \Omega \rangle_\beta = \frac{1}{\pi} \left| \frac{2\beta}{1+\beta} \right| \sum_{k=0}^{N} \sum_{s=0}^{k} \binom{k}{s} \left(\frac{2}{1+\beta} \right)^{k-s} \frac{\mathcal{U}\left(s-k, 1, \left| \frac{2\beta}{1+\beta} \right| \eta\bar{\eta} \right)}{(k-s)!} \, e^{-\left| \frac{2\beta}{1+\beta} \right| \eta\bar{\eta}} \tag{3.30}$$

where $\mathcal{U}(a, c, z)$ is the Tricomi function (hypergeometric confluent of the second kind).

We can now put the complex coordinates of any point in the place of η and $\bar{\eta}$, so that we can see that the expectation value of the density on the lowest Landau level is rotational invariant. We can plot it for various values of β and at fixed N (see figure 3.1).

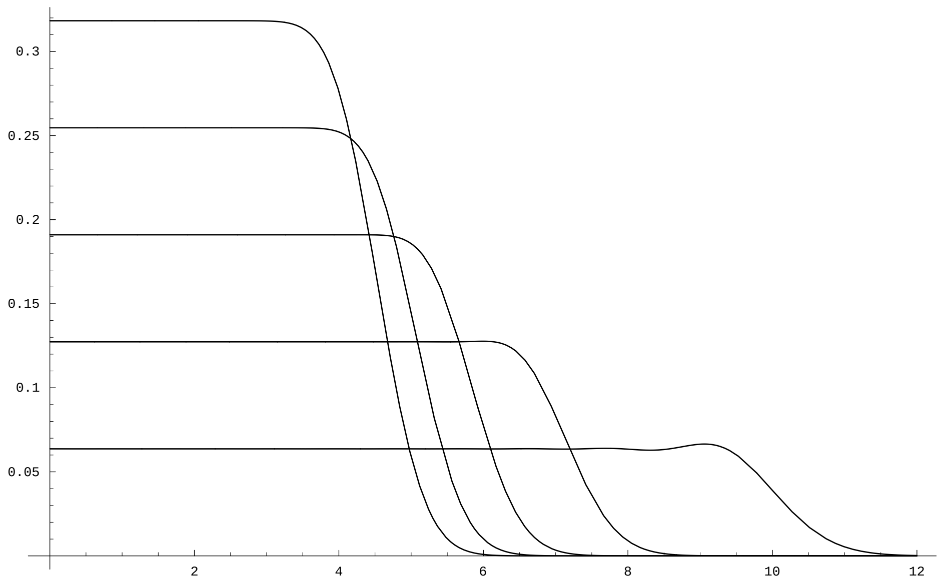

Figure 3.1: Density plot for various values of β

When one varies the number of particles, we expect that the droplet expands without changing its plateaux density, because the filling fraction of the deformed Landau level is $\frac{\beta}{1-2\beta}$. We can see this to happen when $\beta = \frac{1}{2}$ in figure 3.2.

The correlation function $\langle \rho(x)\rho(y) \rangle$

We turn back for a moment to $\beta = 1$, in order to show the form of the density-density correlation function on the incompressible ground state $\langle \Omega | \rho(z_1)\rho(z_2) | \Omega \rangle$. We will work out a form which holds also for $\beta \neq 1$, and now try to compute the correlation function $\langle \rho(x)\rho(y) \rangle_\Omega$. A straightforward computation leads for generic β,

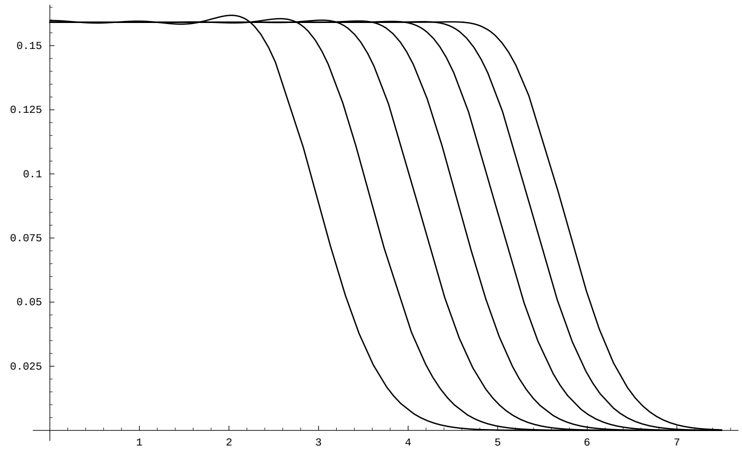

Figure 3.2: Density plot for various numbers of particles

in the same way as before, to the result

$$
\begin{aligned}
&\langle\Omega|U[\rho(z_1,\bar z_1)]U[\rho(z_2,\bar z_2)]|\Omega\rangle = \\
&\quad = \sum_{klmn} \psi^*_{l,0}(z_1,\bar z_1)\psi_{k,0}(z_1,\bar z_1)\psi^*_{m,0}(z_2,\bar z_2)\psi_{n,0}(z_2,\bar z_2)\,\langle 0|c_0\cdots c_N\, c^\dagger_l c_k\, c^\dagger_m c_n\, c^\dagger_N\cdots c^\dagger_0|0\rangle = \\
&\quad = \delta^2(z_1-z_2)\langle U[\rho(z_1,z_1)]\rangle + \\
&\qquad - \sum_{l\neq m}^{N}\big(\psi_{l,0}\,,\,{}^{\circ}_{\circ}\delta(z_1-z)\delta(\bar z_1-\bar z){}^{\circ}_{\circ}\psi_{m,0}\big)\big(\psi_{m,0}\,,\,{}^{\circ}_{\circ}\delta(z_2-z)\delta(\bar z_2-\bar z){}^{\circ}_{\circ}\psi_{l,0}\big) + \\
&\qquad + \sum_{l\neq m}^{N}\big(\psi_{l,0}\,,\,{}^{\circ}_{\circ}\delta(z_1-z)\delta(\bar z_1-\bar z){}^{\circ}_{\circ}\psi_{l,0}\big)\big(\psi_{m,0}\,,\,{}^{\circ}_{\circ}\delta(z_2-z)\delta(\bar z_2-\bar z){}^{\circ}_{\circ}\psi_{m,0}\big)
\end{aligned}
$$

Operating on this expression, we can see the last two terms are

$$
\begin{aligned}
&\sum_{l\neq m}^{N}\big(\psi_{l,0}\,,\,{}^{\circ}_{\circ}\delta(z_1-z)\delta(\bar z_1-\bar z){}^{\circ}_{\circ}\psi_{m,0}\big)\big(\psi_{m,0}\,,\,{}^{\circ}_{\circ}\delta(z_2-z)\delta(\bar z_2-\bar z){}^{\circ}_{\circ}\psi_{l,0}\big) = \\
&\quad = \sum_{l\neq m}^{N} l!m!\,\frac{1}{\pi^2}\left|\frac{2\beta}{1+\beta}\right|^2 e^{-\frac{2\beta}{1+\beta}(z_2\bar z_2+z_1\bar z_1)}\times \\
&\qquad \times\left[\sum_{s=0}^{l\wedge m}\sum_{t=0}^{l\wedge m-s}\frac{(-1)^t\left(\frac{2}{1+\beta}\right)^{\frac{l+m}{2}-s}}{(l-s-t)!(m-s-t)!s!t!}\left(z_1\sqrt{\frac{2\beta}{1+\beta}}\right)^{m-s-t}\left(\bar z_1\sqrt{\frac{2\beta}{1+\beta}}\right)^{l-s-t}\right]\times \\
&\qquad \times\left[\sum_{s=0}^{l\wedge m}\sum_{t=0}^{l\wedge m-s}\frac{(-1)^t\left(\frac{2}{1+\beta}\right)^{\frac{l+m}{2}-s}}{(l-s-t)!(m-s-t)!s!t!}\left(z_2\sqrt{\frac{2\beta}{1+\beta}}\right)^{l-s-t}\left(\bar z_2\sqrt{\frac{2\beta}{1+\beta}}\right)^{m-s-t}\right]
\end{aligned}
$$

and

$$\sum_{l\neq m}^{N}\left(\psi_{l,0}\,,\,\circ\!\!\circ\,\delta(z_1-z)\delta(\bar z_1-\bar z)\circ\!\!\circ\,\psi_{l,0}\right)\left(\psi_{m,0}\,,\,\circ\!\!\circ\,\delta(z_2-z)\delta(\bar z_2-\bar z)\circ\!\!\circ\,\psi_{m,0}\right)=$$

$$=\sum_{l\neq m}^{N} l!m!\,\frac{1}{\pi^2}\left|\frac{2\beta}{1+\beta}\right|^2 e^{-\frac{2\beta}{1+\beta}(z_2\bar z_2+z_1\bar z_1)}\times$$

$$\times\left[\sum_{s=0}^{l}\sum_{t=0}^{l-s}\frac{(-1)^t\left(\frac{2}{1+\beta}\right)^{l-s}}{((l-s-t)!)^2 s!t!}\left(\frac{2\beta}{1+\beta}\right)^{l-s-t}(z_1\bar z_1)^{l-s-t}\right]\times$$

$$\times\left[\sum_{s=0}^{m}\sum_{t=0}^{m-s}\frac{(-1)^t\left(\frac{2}{1+\beta}\right)^{m-s}}{((m-s-t)!)^2 s!t!}\left(\frac{2\beta}{1+\beta}\right)^{m-s-t}(z_2\bar z_2)^{m-s-t}\right]$$

One can see that the two terms above are both real, and moreover they are both invariants under simultaneous rotations of z_1 and z_2 on the complex plane

$$\begin{cases} z_i & \longmapsto & e^{i\phi} z_i \\ \bar z_i & \longmapsto & e^{i\phi}\bar z_i \end{cases}$$

We can considerably simplify the formula for the correlation function by computing it for $z_1 = 0$ and with z_2 on the real line $\eta = z_2 = \bar z_2$, away from the origin $\eta = 0$. We obtain

$$\langle\Omega|U[\rho(0)]U[\rho(\eta,\bar\eta)]|\Omega\rangle =$$

$$=\frac{1}{\pi^2}\left|\frac{2\beta}{1+\beta}\right|^2\sum_{m=0}^{N}\left\{\frac{1+\beta}{2}\left(1-\left(\frac{\beta-1}{\beta+1}\right)^{N+1}\right)-\left(\frac{\beta-1}{\beta+1}\right)^{m}\right\}\times$$

$$\times\sum_{s=0}^{m}\binom{m}{s}\left(\frac{2}{1+\beta}\right)^{m-s}\frac{\mathcal{U}(s-m,1,\frac{2\beta}{1+\beta}|\eta|^2)}{(m-s)!}\,e^{-\frac{2\beta}{1+\beta}|\eta|^2}\qquad(3.31)$$

The shape of the function as we vary the number of particles N, is left basically invariant within a characteristic length, the latter being basically the only object which varies with N. This is exactly what happens in commutative case. As it is apparent from figures 3.4 and 3.5, in the noncommutative case ($\beta\neq 1$) the two points correlation function of the density has an uncommon feature near the origin, because it becomes negative. This is an effect of noncommutative deformation of the algebra of Landau levels. To understand it in physical terms, we can do the following: we switch on a small perturbation, in the form of a two body potential

$$\hat V(x,y)\doteq\hat\psi(x)\hat\psi^*(x)V(x-y)\hat\psi(y)\hat\psi^*(y)$$

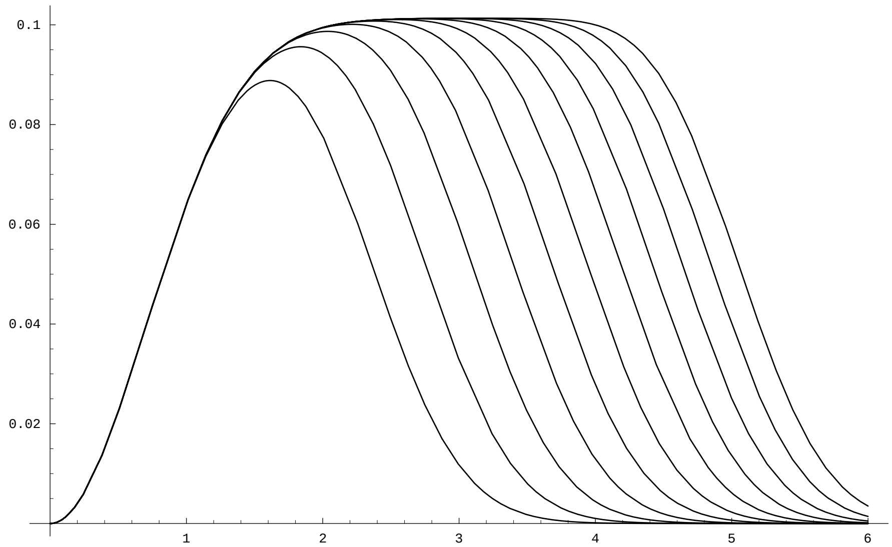

Figure 3.3: Plot of the correlation function of the density with itself for various numbers of particles for $\beta = 1$ (commutative case).

and we compute the first order perturbation on the unperturbed ground state. The result is (for simplicity we do the computation at $x = 0$, $y = \bar{y} = r$)

$$\mathcal{V}(r) \doteq \mathcal{V}(0, r) = \langle \Omega | U[\rho(0)] V(0, r) U[\rho(r)] | \Omega \rangle$$

In the case of the harmonic potential $V(0, r) = \frac{1}{2} r^2$, we obtain for the effective potential $\mathcal{V}(r)$ a shape which has a minimum at $r \neq 0$, as shown by figures (3.6) and (3.7). It means that the attraction between the particles due to $\hat{V}$ is balanced by an effective "repulsion" that is related to the loss of localization on the noncommutative plane (see also the introductory chapter of this thesis and [28]).

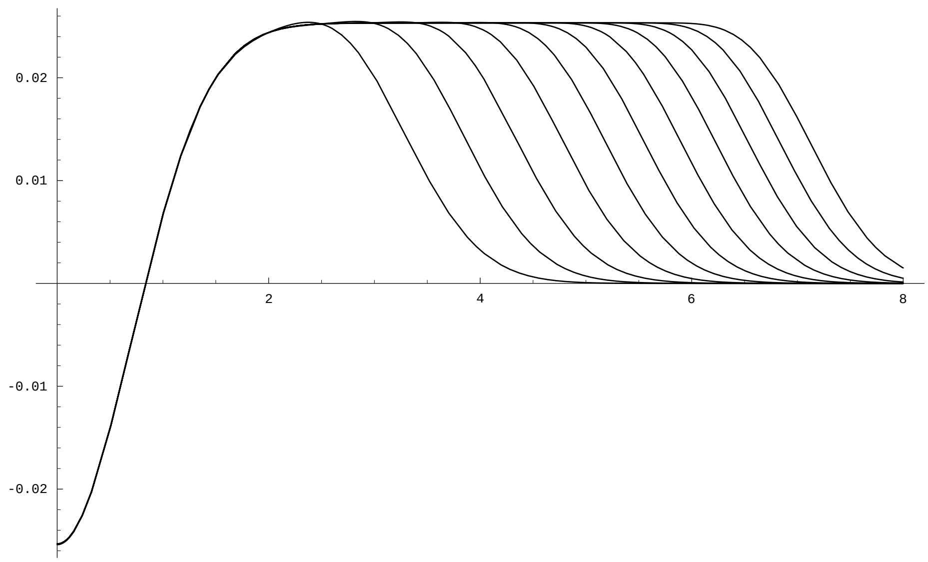

Figure 3.4: Plot of the correlation function of the density with itself for various numbers of particles $\beta = \frac{1}{2}$ (noncommutative case with $\theta = -1$).

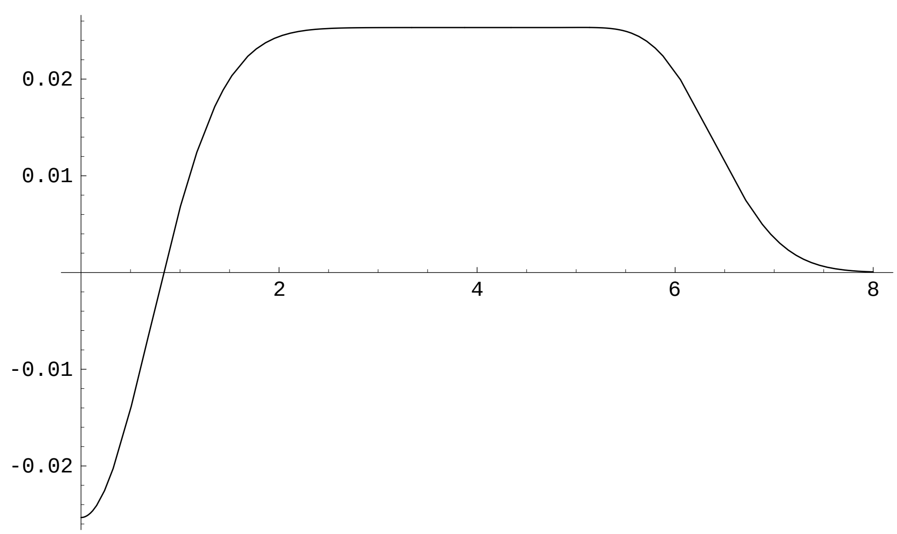

Figure 3.5: Plot of the correlation function of the density with itself for $N = 20$ particles, $\beta = \frac{1}{2}$.

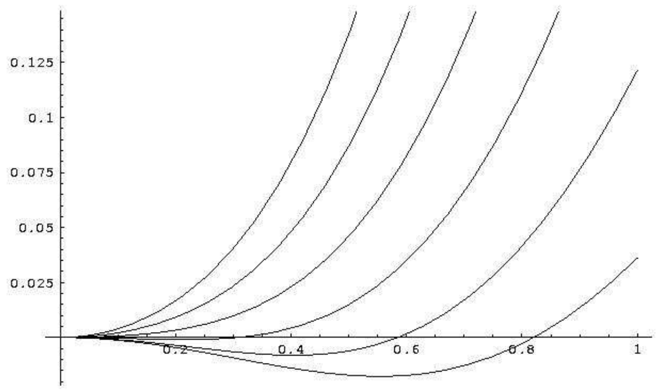

Figure 3.6: Effective potential for various $\beta \in [0.5, 1.5]$

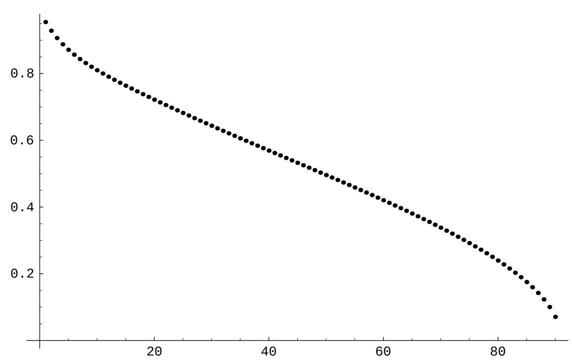

Figure 3.7: Locations of minima of the effective potential as a function of $\beta \in [0.2, 0.99]$.

Chapter 4

From Lagrange incompressible fluid to Noncommutative Chern-Simons theory

4.1 Incompressible fluid

At high values of the magnetic field, and at low temperature, the two-dimensional electrons of the quantum Hall effect form an incompressible fluid, the density of which is uniform and corresponds to the observed plateaus [44, 34]. The fluid is incompressible because the density waves have a gap, which is infinite in the limit of infinite magnetic field [47]. We shall be interested in this limit only.

The incompressibility of the fluid implies that the theory is invariant under transformations leaving invariant the volume element, i.e. under the Area-preserving Diffeomorphisms. Having a granular picture of the fluid in mind, one may view this transformations just as a relabelling of the particles of the fluid.

4.2 Lagrange coordinates

The basic object in the Lagrange description of fluids [25] is the set of *"comoving coordinates"* $\mathbf{X}(t, \mathbf{x})$, i.e. a set of vector valued functions of the time, each of them "following" the time-evolution of a particle. Each function of this set is labelled by an initial condition for the position of the particle it is following, i.e. $\mathbf{X}(t = 0, \mathbf{x}) = \mathbf{x}$ (see fig. 4.1).

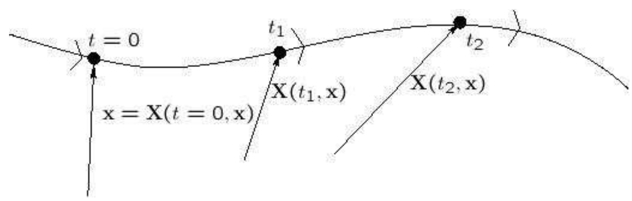

Figure 4.1: Comoving coordinates following a particle

Using the comoving coordinates we may express quantities depending on fixed points in the space[1], such as the fluid density and the current, in the following way

$$\rho(t,\mathbf{r}) = \int d\mathbf{x}\, \rho_0(\mathbf{x})\, \delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r}) = \frac{\rho_0}{\det\left(\frac{\partial \mathbf{X}(t,\mathbf{x})}{\partial \mathbf{x}}\right)_{\mathbf{x}=\chi(t,\mathbf{r})}} \tag{4.1}$$

$$j(t,\mathbf{r}) = \rho_0 \int d\mathbf{x}\, \dot{\mathbf{X}}(t,\mathbf{x})\, \delta(\mathbf{X}(t,\mathbf{x}) - \mathbf{r}) \tag{4.2}$$

Here ρ_0 is a reference uniform density in the *"space of labels"*. We now write the Lagrangian for the incompressible fluid in comoving coordinates

$$\mathcal{L}_0 = \int d\mathbf{x}\, \rho_0 \left\{ \frac{1}{2}\dot{\mathbf{X}}^2(t,\mathbf{x}) - V\left[\det\left(\frac{\partial \mathbf{X}(t,\mathbf{x})}{\partial \mathbf{x}}\right)\right]\right\} \tag{4.3}$$

We can see that the theory defined by this lagrangian is invariant under transformations such that

$$\begin{cases} \mathbf{x} \longmapsto \mathbf{x} + \mathbf{f}(\mathbf{x}) \\ \mathbf{X}(t,\mathbf{x}) \longmapsto \mathbf{X}(t,\mathbf{x}) + (\mathbf{f}\cdot\nabla)\mathbf{X}(t,\mathbf{x}) \end{cases} \quad \text{with} \quad f^i(\mathbf{x}) \propto \epsilon^{ij}\partial_j \phi(\mathbf{x}) \tag{4.4}$$

We see that $\mathbf{X}(t,\mathbf{x})$ are scalar fields under these transformations. In two dimensions these ones are the most general area-preserving transformations.

4.3 Interaction with a magnetic field

We are now going to add to this lagrangian the interaction with an external magnetic field

$$\mathcal{L}' = \frac{eB}{2} \int d\mathbf{x}\, \rho_0\, \epsilon^{ab}\, \dot{X}_a X_b \tag{4.5}$$

[1] This is the standard **Euler description** of fluid mechanics, in which the observer just "sits" at a point in the space, measuring the quantities of the fluid as this passes by.

It's easy to see that $\mathcal{L}'$ is invariant under (4.4) as well. Using Nöther theorem we may see that the conserved quantity deriving from this symmetry is

$$\det \frac{\partial \mathbf{X}}{\partial \mathbf{x}}(t,\mathbf{x}) = \frac{1}{2}\epsilon^{ab}\epsilon^{ij}\frac{\partial}{\partial x_i}X_a\frac{\partial}{\partial x_j}X_b$$

This is proportional to the inverse of the density of the fluid, in comoving coordinates. So we can say that the density of an element of the fluid is a constant when we follow its motion. At the equilibrium we put as a constraint

$$\det \frac{\partial \mathbf{X}}{\partial \mathbf{x}}(t,\mathbf{x}) = 1 \tag{4.6}$$

In the limit in which the magnetic field is infinite only the magnetic term and the constraint survive. So we are left with the lagrangian

$$\mathcal{L} = \frac{eB}{2}\rho_0 \int d\mathbf{x} \left[\left(\dot{X}_a - \{X_a, A_0\}\right) X_b\epsilon^{ab} + \frac{2}{\rho_0}A_0\right] \tag{4.7}$$

where we have introduced the Poisson brackets notation

$$\{A,B\} \doteq \frac{1}{\rho_0}\epsilon^{ij}\frac{\partial}{\partial x_i}A\frac{\partial}{\partial x_j}B \tag{4.8}$$

and the Lagrange multiplier $A_0(\mathbf{x})$. The term in curly brackets plays the role of a covariant time derivative: it is part of the Gauss-law constraint (4.6), that will be important to ensure the invariance of the theory.

4.3.1 Introduction of noncommutativity

Some considerations are in order. The number of states of two-dimensional electrons in each Landau level is the ratio between the total magnetic flux threading the surface and the *elementary quantum of flux* $\Phi_0 \doteq \frac{2\pi\hbar c}{e}$. This means that, in each Landau level, there is a state for each one of these *"fluxons"*, so that the fluid shows somehow a "granularity". In the sections 2.2 and 2.3, we saw that the introduction of a noncommutative algebra in the place of the algebra of functions on a manifold, causes the loss of the notion of points of the space. Actually one is left with the classes of irreducible representations of the algebra itself, which contain less information. We may view this also from a simple point of view, going back to the basic interpretation of quantum theory: one considers the uncertainty relation generated by the commutator of the coordinates of the noncommutative plane

$$[x,y] = i\theta \qquad (\Delta x)^2(\Delta y)^2 \gtrsim \frac{1}{2}\theta^2$$

This means that in a noncommutative theory the points are somehow "blurred", or fuzzy. Thus introducing the noncommutativity is a clean way to introduce a sort of delocalization of points of the space. One way to achieve this, is to consider the Lagrange coordinates as time dependent matrices. In this approach, the integral is substituted by the trace, and the role of the Poisson brackets (4.8) is taken by the commutator

$$\{,\} \rightsquigarrow i[,]$$

After rescaling of the Lagrangian, we obtain with these substitutions

$$\mathcal{L}(X_a, A_0) = \frac{B}{2}\mathbf{tr}\left[\left(\dot{X}_a + i[X_a, A_0]\right) X_b \epsilon^{ab} + 2\theta A_0\right] \tag{4.9}$$

where we have introduced the constant $\theta = 1/\rho_0$. The equation of motion for A_0 is just a constraint, because it appears into the Lagrangian without time derivatives, like a Lagrange multiplier, and it is the so said *Gauss law constraint*

$$X_a X_b - X_b X_a = i\theta\,\epsilon_{ab}. \tag{4.10}$$

which has to be read as a matrix equation. Because of the reality of the coordinates $\mathbf{X}(t, \mathbf{x})$, the matrices which substitute them has to be taken hermitian. The model (4.9) can be called *Chern-Simons Matrix Quantum mechanics.* The reason of this name can be understood in the following way [4]. Take the theory (4.9) as a theory of fluctuations of the matrices X_a on a fixed background x_a. So we could write, introducing the fluctuation matrices A_a

$$X_a = x_a + \theta\,\epsilon^{ab} A_b \tag{4.11}$$

where the background x_a satisfy the commutation relation $[x_1, x_2] = i\theta$. We could also view this form of X_a as if we had written the displacement from the initial reference positions of each fluid element by the displacement vector $\theta\epsilon^{ab}A_b$. So we just need to substitute (4.11) into (4.9), to obtain

$$\mathcal{L}(A) = \frac{B\theta}{2}\mathbf{tr}\left\{\epsilon^{\mu\nu\sigma}\left(A_\mu \partial_\nu A_\sigma + \frac{2}{3}A_\mu A_\nu A_\sigma\right)\right\} \tag{4.12}$$

where the derivative ∂_μ has been defined as

$$\partial_0 \cdot \doteq \frac{\partial}{\partial t}\,\cdot$$
$$\partial_i \cdot \doteq \left[-i\frac{x_i}{\theta}, \cdot\right] \qquad i = 1, 2$$

As it is apparent, equation (4.12) is the noncommutative generalization of the Chern-Simons Lagrangian. We could rewrite the action in terms of the appropriate Moyal product in this way

$$\mathcal{S}'(A) = \frac{B\theta}{2} \int d^3x \; \epsilon^{\mu\nu\sigma} \left(\tilde{A}_\mu \star \partial_\nu \tilde{A}_\sigma + \tilde{A}_\mu \star \tilde{A}_\nu \star \tilde{A}_\sigma \right)$$

where now the $\tilde{A}_\mu$ are space-time functions, in the spirit of section 3.4.1. We can now write down the equations of motion for $\tilde{A}_\mu$, up to the first order in θ to find [47]

$$\epsilon^{ab} \left(\partial_a A_b - \frac{1}{2} \{A_a, A_b\} \right) = 0$$

This is the equation of the fluid whose dynamics is given by the action (4.7) in which we substituted (4.11) meant as a commutative expression, i.e. in terms of commutative $\mathbf{x}$ and $\mathbf{A}(\mathbf{x}, t)$. Of course this fact says nothing about the inverse path from the commutative to the noncommutative theory. This must be done, as we did, choosing the most natural (in a sense minimal) matrix action. Having said all this about the action (4.12), we are free to come back to the form (4.9) instead, which will be considered in the following. This is the so called "Chern-Simons Matrix Quantum Mechanics".

4.4 Matrix Chern-Simons theory

We will use, in the sequel, the Matrix Chern-Simons theory defined by the action

$$\mathcal{S}(X_a, A_0) = \frac{B}{2} \int dt \; \mathbf{tr} \left[\left(\dot{X}_a + i[X_a, A_0] \right) X_b \epsilon^{ab} + 2\theta A_0 \right] \qquad (4.13)$$

As mentioned before, the equation of motion of the Lagrange multiplier A_0 is just the Gauss law constraint

$$X^a X^b - X^b X^a - i\theta \, \epsilon^{ab} \approx 0 \qquad (4.14)$$

or in components

$$G_{ik} \doteq X^1_{il} X^2_{lk} - X^2_{il} X^1_{lk} - i\theta \, \delta_{ik} \approx 0 \qquad (4.15)$$

The canonical coordinate-momentum pairs obtained from the above first-order lagrangian are

$$(X^{(1)}_{ij}, \frac{B}{2} X^{(2)}_{ji}) \approx (X^{(1)}_{ij}, -i\frac{\delta}{\delta X^{(1)}_{ij}}) \quad \text{and} \quad (X^{(2)}_{ij}, \frac{B}{2} X^{(1)}_{ji}) \approx (X^{(2)}_{ij}, -i\frac{\delta}{\delta X^{(2)}_{ij}})$$

We have to perform a choice of polarization by adding to the kinetic term of the lagrangian a total derivative term. In this way we have the canonical pairs

$$(X^{(1)}_{ij}, BX^{(2)}_{ji}) \approx (X^{(1)}_{ij}, -i\frac{\delta}{\delta X^{(1)}_{ij}})$$

The canonical commutation relations are consequently

$$[X^{(1)}_{ik}, X^{(2)}_{lm}] = \frac{1}{B}\delta_{im}\delta kl$$

The constraint (4.14) cannot be solved by finite rank matrices. To see this it is enough taking the trace of both sides of the equation, and noticing that the trace of a commutator between finite rank matrices is always zero. So we must search the solutions in the space of operators or, to say this roughly, of infinite rank matrices. Of course this can mean problems, when one needs to do actual computations, because we need at least some condition about the behaviour of the matrix elements restricted to orthogonal complements of increasing codimension in the Hilbert space on which operators are defined. Indeed we will see in the next section a constructive way of truncating the coordinate operators while keeping a consistent Gauss' law constraint.

Chapter 5

Finite N Noncommutative Chern-Simons

In view of the problems mentioned in section 4.4, we need a truncation of the model to finite dimensional N. We now will follow the work [42], but using functional integral techniques. The action for infinite N is (remember that X_a are hermitian matrices)

$$\mathcal{S}(X_a, A_0) = \frac{B}{2} \int dt \ \mathbf{tr} \left[\left(\dot{X}_a + i[X_a, A_0] \right) X_b \epsilon^{ab} - 2\theta A_0 \right]$$

and the equation of the motion of the Lagrange multiplier A_0, i.e. the Gauss law constraint, is

$$G_{ik} \doteq X^1_{il} X^2_{lk} - X^2_{il} X^1_{lk} - i\theta \, \delta_{ik} \approx 0$$

or

$$X^a X^b - X^b X^a - i\theta \, \epsilon^{ab} \approx 0$$

We now modify this equation in the following way

$$X^a X^b - X^b X^a - i\theta \, \epsilon^{ab} - K \approx 0 \tag{5.1}$$

where all the matrices here are N-dimensional, and we have introduced K such that the finite N inconsistency disappears

$$\mathbf{tr} \, K = -i\theta \, N$$

An important thing to notice, before we pass to the functional integral for this model, is that the above constraint is invariant under the $U(N)$ gauge group if together with the X_a, we vary the K matrix itself, in the same matrix fashion

$$K \longmapsto U \, K \, U^\dagger$$

To modify the action in order to obtain the consistent Gauss law constraint, we could add to the action a term of the form

$$-i \int dt \; \mathbf{tr} \, \{K \, A_0\}$$

At this point, we want to write the partition functional for the theory, considering for now K as an external field. We now will use the action for the X_a matrices

$$S_{CS}[X,Y] = B \int dt \; \mathbf{tr} \left(\dot{X} Y\right) \qquad \text{with} \; \begin{cases} X \doteq X_1 \\ Y \doteq X_2 \end{cases}$$

which correspond to the choice of polarization corresponding to the canonical pairs

$$(X_{ij}, BY_{ji}) \approx (X_{ij}, -i\frac{\delta}{\delta X_{ij}})$$

and the commutation relations

$$[X_{ik}, Y_{lm}] = \frac{1}{B}\delta_{im}\delta_{kl}$$

The action S_{CS} differs from the kinetic term of (4.13) for a total time derivative. We must constrain the functional integration only to matrices satisfying the Gauss law constraint (5.1). This is done by integrating out the Lagrange multiplier A_0, thus obtaining a Dirac delta function into the (reduced) partition functional

$$\widetilde{\mathcal{Z}}[K] = \int \mathfrak{D}X\mathfrak{D}Y e^{iS_{CS}[X,Y]}\delta\big[[X,Y] - i\theta \cdot -K\big] \tag{5.2}$$

Now we notice that the action, still preserving global gauge invariance (i.e. invariance under time independent $U(N)$ transformations), is not invariant under an $U(t) \in U(N)$ depending on time. Infact, if we perform a (time dependent) $U(N)$ gauge transform $U(t)$ on the X, Y we obtain

$$\begin{aligned} \mathcal{S}_{CS}[X,Y] \longmapsto & B \int dt \; \mathbf{tr} \left[\frac{d}{dt}\left(UXU^\dagger\right) UYU^\dagger\right] = \\ =& B \int dt \; \mathbf{tr} \, \dot{X}Y + B \int dt \; \mathbf{tr} \, U^\dagger \dot{U}[X,Y] = \\ =& \mathcal{S}_{CS}[X,Y] + B \int dt \; \mathbf{tr} \, U^\dagger \dot{U} K + iB\theta \int dt \; \mathbf{tr} \, U^\dagger \dot{U} \end{aligned} \tag{5.3}$$

Where we used the constraint and the usual properties of the trace. The last integral is

$$\mathcal{I}_U = iB\theta \int dt \; \mathbf{tr} \, U^\dagger \dot{U} = iB\theta \int dt \, \frac{d}{dt} \ln \det U = -B\theta \; \arg \det U \Big|_{t_1}^{t_2}$$

where we could eventually put the integration limits to infinity. If we take a gauge transformation trivial at the limit times, then $\mathcal{I}_U$ is just an integer multiple of 2π, being it just a natural representation of $\pi_1(U(N)) = \mathbb{Z}$. Since in the path integral this term enters as the argument of an exponential, the product $B\theta$ must be an integer number, in order to have a consistent definition of an $U(N)$-invariant measure of the path integral: we have just found the condition for the quantization of the number $B\theta$, analogous of the level of the ordinary Chern-Simons field theory. From the original definition of θ in terms of the Lagrange fluid, we find

$$\mathbb{Z} \ni B\theta = \frac{B}{\rho_0} = \frac{1}{\nu}$$

where ν is the filling fraction of the Hall fluid, i.e. the density rescaled by the square of the magnetic length $\ell^2 = B$. So in physical terms, we find the quantization of the filling fraction.

Now let us look at the second integral in the last row of (5.3)

$$B \int dt \,\mathbf{tr}\, \dot{U} K U^\dagger$$

We write K as a matrix which columns are arbitrary linear combinations of, say, M vectors of $\mathbb{C}^N$. We only need to make it an anti-hermitian matrix; we write K in the form

$$K \doteq iA \cdot J \cdot A^\dagger \qquad A \in {}_N\mathbb{C}_M,\ J \in {}_M\mathbb{C}_M \tag{5.4}$$

For simplicity we take $J = \mathbb{I}$. Under a gauge transformation on X, Y the rectangular matrix A changes by a left translation

$$\begin{cases} X & \stackrel{U}{\longmapsto} UXU^\dagger \\ A & \stackrel{U}{\longmapsto} UA \end{cases} \tag{5.5}$$

In this way the constraint is left invariant. Consider now the action [42, 35]

$$\mathcal{S}'[A] \doteq iB \int dt \,\mathbf{tr}\, A^\dagger \dot{A}$$

This form of the action of $(A, A^\dagger)$ gives as canonical pairs

$$(A_{ia}, BA^*_{ia}) \approx (A_{ia}, -i\frac{\delta}{\delta A_{ia}})$$

so the relative commutation relations are[1]

$$[A_{ia}, A^*_{kb}] = \frac{1}{B}\delta_{ik}\delta_{ab}$$

[1]Notice that in case of a more general F we would rather have $[A_{ia}, \sum_c F_{bc}A^*_{kc}] = \frac{1}{B}\delta_{ik}\delta ab$ being $B\sum_c F_{bc}A^*_{kc}$ the canonical momentum of A_{kb}.

Its change after a gauge transformation (5.5) is given by

$$\int dt\ \mathbf{tr}\, A^\dagger \dot{A} \longmapsto \int dt\ \mathbf{tr}\, A^\dagger \dot{A} + \int dt\ \mathbf{tr}\, \dot{U} A A^\dagger U^\dagger$$

We can see that the last term is exactly the change of $\mathcal{S}_{CS}$ we saw in (5.3). Now, anyway, we will consider only the "minimal" version of the substitution (5.4), i.e. when $A \in \mathbb{C}^N$ [2]

$$K \doteq i\,\Psi\Psi^\dagger \qquad K_{ij} = i\,\psi_i \psi_j^*$$

and the action we should add to $\mathcal{S}_{CS}$ to render it invariant is

$$\mathcal{S}_B[\Psi] = -iB \int dt\ \mathbf{tr}\ \Psi^\dagger \dot{\Psi}$$

So that the total action is given by

$$\mathcal{S}_T = \mathcal{S}_{CS}[X,Y] + \mathcal{S}_B[\Psi] \tag{5.6}$$

Of course in the partition functional we must consider also the integration over these new degrees of freedom, the Ψ's. We choose for them a flat measure, so that

$$\mathcal{Z} = \int \mathfrak{D}\Psi\, \mathfrak{D}\Psi^\dagger\, \mathfrak{D}X\, \mathfrak{D}Y\ e^{i\mathcal{S}_{CS}[X,Y] + i\mathcal{S}_B[\Psi]}\, \delta\big[[X,Y] - i\theta \cdot -i\,\Psi\Psi^\dagger\big] \tag{5.7}$$

This is the Chern-Simons MAtrix Model introduced by Polychronakos in [42]. This functional integral is invariant under $U(N)$ gauge transformations, intended as acting in the following way

$$\begin{cases} X & \stackrel{U}{\longmapsto} UXU^\dagger \\ Y & \stackrel{U}{\longmapsto} UYU^\dagger \\ \Psi & \stackrel{U}{\longmapsto} U\Psi \end{cases} \qquad U = U(t) \in U(N) \tag{5.8}$$

5.1 Faddeev-Popov quantization

We want to treat the partition functional (5.7) with the standard techniques of Gauge Field Theory [12]. To do this, we first need a convenient gauge choice. The first coming to mind is obviously the gauge in which matrices are diagonal. Anyway this is not completely possible. We cannot diagonalize both X and Y (at fixed t) with a single $U(N)$ transformation, because of the constraint. Hence we will choose

[2] See [35] for a more complete discussion of the general substitution for K, and its interpretation in terms of multi-layered fluid physics.

to diagonalize just one matrix, and the other one will have some degrees of freedom fixed by the constraint, while the integration will be free on the others.
As a starting point, we will use the following identity

$$1 = \int \mu(U)\delta[UXU^\dagger - \Lambda]\Delta_{FP}[\Lambda] \tag{5.9}$$

where $\mu(U)$ is an invariant measure on the group of unitary time depending transformations, Λ is a diagonal matrix which is the gauge fixed form of X. By standard Field Theory arguments, the *Faddeev-Popov* determinant depends only on gauge invariant quantities.
We insert now the identity (5.9) in the integral (5.7), and we obtain

$$\mathcal{Z} = \int \mu(U) \int \mathfrak{D}\Psi\,\mathfrak{D}\Lambda\,\mathfrak{D}Y\; e^{i\mathcal{S}_{CS}[\Lambda,UYU^\dagger]+i\mathcal{S}_B[U\Psi]}\,\delta\big[[\Lambda,UYU^\dagger]-i\theta\cdot - i\,U\Psi\Psi^\dagger U^\dagger\big]\Delta_{FP}[\Lambda]$$

where we used the invariance of the total action (provided the quantization condition on $B\theta$ is met) and of the Dirac delta. Now we can use the fact the measures on X, Y and Ψ can be defined to be unitary invariant, and write the functional in the gauge fixed form

$$\mathcal{Z} = \left(\int \mu(U)\right) \int \mathfrak{D}\Psi\,\mathfrak{D}\Lambda\,\mathfrak{D}Y\; e^{i\mathcal{S}_{CS}[\Lambda,Y]+i\mathcal{S}_B[\Psi]}\,\delta\big[[\Lambda,Y] - i\theta\cdot - i\,\Psi\Psi^\dagger\big]\Delta_{FP}[\Lambda] \tag{5.10}$$

Of course the volume of the gauge group is not relevant for the physics. We need now to compute the determinant $\Delta_{FP}[\Lambda]$. It can be done rewriting the identity (5.9) in the following way

$$1 = \int \mu(U)\delta[UXU^\dagger - \Lambda]\Delta_{FP}[\Lambda] = \int \mu(U)\delta[UX - \Lambda U]\Delta_{FP}[\Lambda]$$

using the invariance of right shift of the argument of the delta.[3]
Now the above integral can be recognized, in the standard way, to be

$$\Delta_{FP}[\Lambda] = \left[\int \mu(U)\delta[UX - \Lambda U]\right]^{-1} = \mathcal{D}et'\frac{\delta(UX - \Lambda U)}{\delta U}$$

The functional derivative of the gauge fixing functional is

$$\left(\frac{\delta(UX - \Lambda U)(t)_{ab}}{\delta U(t')^{ik}}\right)_{(ik)(ab)} = \delta_{ai}(X_{kb} - \lambda_a\delta_{kb})\delta(t - t')$$

[3]Any Jacobian arising by the shift can be reabsorbed in the measure $\mu(U)$, in that it depends only by the transformation U, and not by Λ.

The determinant can be easily seen (e.g. introducing ghost fields) to be

$$\mathcal{D}et'\left(\delta_{ai}(X_{kb}-\lambda_a\delta_{kb})\delta(t-t')\right) = \exp\left\{\int dt\ \ln\prod_a\prod_{k\neq a}\det{}'(X-\lambda_a\cdot)\right\} = $$
$$= \exp\left\{\int dt\ \ln\prod_{i\leq k}(\lambda_i(t)-\lambda_k(t))^2\right\}$$

where $\det'$ means we are excluding null modes, i.e. the determinant $\det'$ is not performed on the eigenspaces relative to eigenvalue λ_a. This is the obvious generalization of the Vandermonde determinant $\prod_{i\leq k}(\lambda_i-\lambda_k)^2$ for time depending matrices. We can put this result into the partition function, obtaining

$$\mathcal{Z} = \Omega_{U(N)}\int \mathfrak{D}\Psi\,\mathfrak{D}\Lambda\,\mathfrak{D}Y\ e^{i\mathcal{S}_{CS}[\Lambda,Y]+i\mathcal{S}_B[\Psi]}\,\delta\big[[\Lambda,Y]-i\theta\cdot-i\,\Psi\Psi^\dagger\big]\times$$
$$\times\exp\left\{\int dt\ \ln\prod_{i\leq k}(\lambda_i(t)-\lambda_k(t))^2\right\}$$

We can still elaborate on this expression, rewriting the argument of the Dirac delta as follows

$$\big([\Lambda,Y]-i\theta\cdot-i\,\Psi\Psi^\dagger\big)_{ik} = (\lambda_i-\lambda_k)Y_{ik}-i\theta\delta_{ik}-i\,\psi_i\psi_k^* =$$
$$=\begin{cases}(\lambda_i-\lambda_k)(Y_{ik}-i\,\frac{\psi_i\psi_k^*}{\lambda_i-\lambda_k}) & \text{for}\quad i\neq k\\ -i\theta\delta_{ik}-i\psi_i\psi_k^* & \text{for}\quad i=k\end{cases}$$

In this way we may well see that the the delta decomposes into a "diagonal" part depending just on the absolute values of the components of Ψ and θ, and a more complicated "off-diagonal" part

$$\prod_i\delta[\theta+|\psi_i|^2]\ \prod_{i\neq k}\delta\big[(\lambda_i-\lambda_k)(Y_{ik}-i\frac{\psi_i\psi_k^*}{\lambda_i-\lambda_k})\big]$$

The first factor simply constrains the integral over Ψ, to be an N-fold integration over complex unimodular numbers. The second part is now a constraint for the off-diagonal entries of the matrix Y. But it is not linear in Y_{ik}. We can rewrite it in a way it be of the form $\delta(y-a)$

$$\prod_{i\neq k}(\mathcal{D}et\delta(t-t')(\lambda_i-\lambda_k))^{-1}\delta\big[Y_{ik}-i\frac{\psi_i\psi_k^*}{\lambda_i-\lambda_k}\big] =$$
$$= \exp\{-\int dt\ \ln\prod_{i\neq k}(\lambda_i(t)-\lambda_k(t))\}\delta\big[Y_{ik}-i\frac{\psi_i\psi_k^*}{\lambda_i-\lambda_k}\big]\}$$

This factor cancel the Faddeev-Popov determinant, to give the gauge fixed partition function[4]

$$\mathcal{Z} = \Omega_{U(N)} \int \mathfrak{D}\Psi \prod_i \delta[\theta + |\psi_i|^2]\, e^{i\mathcal{S}_B[\psi_i]} \int \prod_i \mathfrak{D}\lambda_i\, \mathfrak{D}y_i\, e^{i\mathcal{S}_{CS}[\lambda_i, y_i]} \prod_{i \neq k} \delta\left[Y_{ik} - i\frac{\psi_i \psi_k^*}{\lambda_i - \lambda_k}\right] \tag{5.11}$$

where the Chern-Simons action is

$$\mathcal{S}_{CS}[\Lambda, Y] = \mathcal{S}[\lambda_i, y_i] = B \sum_i \int dt\ \dot{\lambda}_i\, y_i$$

The correlation function of any gauge invariant functional of the X, Y, in particular the addition to the action of any invariant potential, can be obtained by insertion into the (5.11). Notice that the Hamiltonian of this Chern-Simons Matrix Quantum Mechanics, just as in the ordinary case, is vanishing; moreover we see that (5.11) is just a (constrained) phase space path integral, in the conjugate coordinates (λ_i, y_i). We will discuss the meaning of these conjugate pairs afterwards.
Considering the fact this model is the matrix model generalization of an incompressible fluid, and that we just cut it off to have a finite number of degrees of freedom, physically we expect that in absence of a confining potential, the density of the particles (or quasi-particles), which are in a finite number, must fall off to zero, because they are spread on an noncompact space. The simplest confining potential is the quadratic one (see [42])

$$\mathcal{V}[X, Y] = \frac{\omega}{2} \int dt \mathbf{tr}\,(X^2 + Y^2) \tag{5.12}$$

This is manifestly $U(N)$ invariant. In our Λ gauge, it can be written

$$\mathcal{V}[\Lambda, Y] = \frac{\omega}{2} \sum_i \int dt\ (y_i^2 + \lambda_i^2) + \frac{\omega}{2} \sum_{i \neq k} \int dt\ \frac{\theta^2}{(\lambda_i - \lambda_k)^2} \tag{5.13}$$

where we have imposed also the gauge condition $|\psi_i|^2 = -\theta$. Inserting this into (5.11), we can see that the partition functional for the problem with the confining potential becomes

$$\mathcal{Z}[\mathcal{V}] = \Omega_{U(N)}\, \mathcal{Z}_\Psi \int \prod_i \mathfrak{D}\lambda_i\, \mathfrak{D}y_i\ e^{i\mathcal{S}_{CS}[\lambda_i, y_i] - i\mathcal{V}[\lambda_i, y_i]} \tag{5.14}$$

By direct inspection, we see that the dynamics of the eigenvalues is given in terms of the *Calogero model* Hamiltonian $\mathcal{V}[\lambda_i, y_i]$ of the (unidimensional) problem described by the conjugate pairs of coordinates $(q_i, p_i) \doteq (\lambda_i, y_i)$.

[4] For a different computation of the Calogero model in the framework of 1-dimensional matrix models, the reader can see the first part of the paper [20].

5.2 Scalar product

By using the same technology we used to compute the functional integral of the model, we can compute also the scalar product between states of the matrix model. We are not interested now in the actual explicit expression of the wavefunctions satisfying the constraint (5.1), because in the next section we will have them computed in a more interesting gauge choice, namely by passing to complex coordinates: that will be the point at which the comparison with Laughlin theory of quantum Hall effect will be made.
The scalar product between two states $|\Phi_1\rangle$ and $|\Phi_2\rangle$ can be rewritten in the form

$$\langle\Phi_1|\Phi_2\rangle = \int \prod_{ij} dX_{ij}\ \langle\Phi_1|X\rangle\langle X|\Phi_2\rangle$$

where it has been inserted in the scalar product the resolution of identity in terms of coherent states[5]

$$\mathbb{I} = \int \prod_{ij} dX_{ij}\ |X\rangle\langle X| \qquad |X\rangle \doteq e^{\mathbf{tr}\, X\widehat{X}_1}|0\rangle \tag{5.15}$$

Applying the same machinery we used before, the scalar product reduces to the integral over eigenvalues

$$\langle\Phi_1|\Phi_2\rangle = \int \prod_i d\lambda_i \prod_{i<j}(\lambda_i-\lambda_j)^2\ \langle\Phi_1|\{\lambda_l\}\rangle\langle\{\lambda_l\}|\Phi_2\rangle \tag{5.16}$$

The Vandermonde factor in the integrand can be cast in the definition of the wavefunctions in this way

$$\Phi[X] \rightsquigarrow \prod_{i<j}(\lambda_i-\lambda_j)\Phi[X] \tag{5.17}$$

so that the wavefunctions changes it symmetry under exchange of eigenvalues (i.e. particles). With this identification the scalar product reduced to the eigenvalues becomes an integral with a flat $\mathbb{R}^N$ measure.

The (5.16) can be used to compute the Green function by functional integration in the usual way. The basic expression is of the form[6]

$$G[X',X,t] = \langle X',t|X,0\rangle = \langle X'|e^{i\mathcal{H}t}|X\rangle$$

[5] According to the typographical traditions of Quantum Mechanics the hat in the symbol $\widehat{X}_1$ is showing its operatorial nature as opposed to the c-number nature of X.

[6] We allow here also for the inclusion of a potential as in (5.12) and (5.13) in the hamiltonian of the system.

Now we insert the identities

$$\mathbb{I} = \int \prod_{ij} dQ_{ij} \; |Q\rangle\langle Q| \qquad |Q\rangle \doteq e^{\mathbf{tr}\, Q\hat{X}_1}|0\rangle$$

$$\mathbb{I} = \int \prod_{ij} d\Pi_{ij} \; |\Pi\rangle\langle \Pi| \qquad |\Pi\rangle \doteq e^{\mathbf{tr}\, \Pi\hat{X}_2}|0\rangle$$

$$\text{with } \langle Q|\Pi\rangle = \frac{1}{2\pi} e^{i\mathbf{tr}\, \Pi Q}$$

at intermediate times $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = t$ and the constraint (5.1), which must be cast essentially in the definition of the evolution operator in the obvious way[7]

$$e^{i\mathcal{H}t} \doteq \mathbb{P}_{Phys} \; e^{iHt} \; \mathbb{P}_{Phys}$$

What we obtain is of course nothing else than the partition functional we had in (5.11) or (5.14) (see also [20]).

[7]This is the evolution operator making only the gauge invariant (physical) states evolve.

Chapter 6

Complex Coordinates for the Chern-Simons Matrix Model

In the previous chapter we have found the path integral of truncated Chern-Simons matrix model. In particular the fact that a 2-dim model ends up in the quantization of a 1-dim one may be puzzling. Anyhow, in a noncommutative geometry, one usually cannot give sense to the concept of localization of points in terms of n-tuples of coordinates, in the same way as in the phase space of a system after quantization. As in the ordinary Quantum Mechanics, discussed in chapter 3, one may adopt Wigner representation (i.e. Weyl transform and Moyal product) in order to use coordinates in the description of physical problems on a noncommutative geometry.

In the previous chapter we performed the quantization of the model in hermitian coordinates. The reduction to the eigenvalues led to a model described in terms of real, one dimensional coordinates of the electrons. To have a more direct physical interpretation of the result, we prefer working with complex eigenvalues. Hence we need to introduce the analogous of the quantization in complex coordinates.

What we now describe is the "Holomorphic quantization" of the model [5]. We can define complex coordinates for our matrix model

$$X \doteq X_1 + iX_2 \qquad X^\dagger \doteq X_1 - iX_2$$

With these coordinates the action becomes

$$S_{CS}[X, X^\dagger] + \mathcal{S}_B[\Psi] = \frac{iB}{2}\int dt\ \mathbf{tr}\left(\dot{X}X^\dagger\right) - \frac{iB}{2}\int dt\ \mathbf{tr}\ \Psi^\dagger\dot{\Psi} \tag{6.1}$$

while the constraint is

$$G \doteq [X, X^\dagger] - 2\theta - \Psi \circ \Psi^\dagger \approx 0 \tag{6.2}$$

and as before the consistency condition on Ψ is

$$2N\theta + \sum_i \psi_i^* \psi_i = 0$$

As a result the partition functional is

$$\mathcal{Z} = \int \mathfrak{D}\Psi \, \mathfrak{D}\Psi^\dagger \, \mathfrak{D}X \, \mathfrak{D}X^\dagger \; e^{i\mathcal{S}_{CS}[X,X^\dagger]+i\mathcal{S}_B[\Psi]} \, \delta\big[[X,X^\dagger] - 2\theta \cdot - \Psi\Psi^\dagger\big] \tag{6.3}$$

This path integral is still $U(N)$ invariant. But now, for a general $\theta \neq 0$, any matrix satisfying the Gauss' law constraint cannot be diagonalized by a $U(N)$ gauge transformation. This is because of the fact that necessary and sufficient condition for a complex matrix X to be diagonalizable by $U(N)$ transformation is $[X, X^\dagger] = 0$, i.e. X is a *Normal Matrix*.

When $\theta = 0$, instead, we have classically from the Gauss' law constraint

$$[X, X^\dagger] = \Psi \circ \Psi^\dagger \quad \text{and} \quad \sum_i |\psi_i|^2 = 0$$

So $\Psi \equiv 0$ and $[X, X^\dagger]$ i.e. X is normal.

Thus the classical expectation is that for $\theta = 0$ our path integral becomes an integral over normal matrices. Though, as we will see in the sequel, the natural measure we will be lead to is not that induced from the flat measure over complex matrices by the natural inclusion. This is of course due to the presence of the Dirac δ function enforcing the constraint, which in turn naturally arises as a result of integration over the Lagrange multiplier A_0.

6.1 Diagonalization

In the space of complex matrices the subset of matrices with distinct eigenvalues is the highest dimensional invariant subset. This means, the sets of matrices with two or more degenerate eigenvalues is negligible in the sense of Lebesgue measure.
Since a matrix $X \in \ _N\mathbb{C}_N$ with N distinct eigenvalues is always diagonalizable by an invertible transformation of basis vectors, we can write any complex matrix X, besides a null-measure set of matrices, as

$$X = V\Lambda V^{-1} \quad V \in GL(N, \mathbb{C}) \quad \Lambda = diag(\lambda_1, \ldots, \lambda_N) \tag{6.4}$$

6.1.1 Canonical coordinates

The kinetic first order action integral

$$\frac{iB}{2}\int dt\ \mathbf{tr}\left(\dot{X}X^{\dagger}\right)-\frac{iB}{2}\int dt\ \mathbf{tr}\ \Psi^{\dagger}\dot{\Psi}$$

implies the canonical hamiltonian coordinates to be

$$(X_{ij},\frac{iB}{2}X^{*}_{ij})\approx(X_{ij},-i\frac{\delta}{\delta X_{ij}})$$
$$(\psi_{l},-\frac{iB}{2}\psi^{*}_{l})\approx(\psi_{l},-i\frac{\delta}{\delta\psi_{l}})$$

These last equations in turn imply the commutation relations

$$[X_{ik},X^{*}_{lm}]=\frac{2}{B}\delta_{il}\delta_{km}$$
$$[\psi_{k},\psi^{*}_{l}]=-\frac{2}{B}\delta_{kl}$$

With these relations one may rewrite the Gauss' law constraint (5.1) in Schrödinger representation in the following normal ordered form

$$G_{ik}=X_{is}\frac{\delta}{\delta X_{ks}}-X_{sk}\frac{\delta}{\delta X_{si}}-2B\theta\delta_{ik}-\psi_{i}\frac{\delta}{\delta\psi_{k}} \tag{6.5}$$

Performing the diagonalization (6.4) we obtain the obvious result on X_{ij} and ψ_l:

$$X_{ij}=V_{il}\lambda_{l}V^{-1}_{lj}$$
$$\psi_{k}=V_{kl}\phi_{l}$$

These factorisations induce a decomposition on the cotangent space at a point in (X,Ψ) manifold. These decomposition may be found, and in turn this implies the decomposition of derivative (momentum) operators on tangent space. The decomposition on cotangent space is

$$dX_{ij}=V_{il}(d\lambda_{l}\delta_{lm}-[\Lambda,dv]_{lm})V^{-1}_{mj}$$
$$d\psi_{k}=V_{kl}(d\phi_{l}+dv_{lm}\phi_{m})$$

where we defined

$$dv_{ij}\doteq V^{-1}_{il}dV_{lj}$$

Requiring that the canonical pairing is invariant what we obtain for the momenta is[1]

$$\begin{aligned}\frac{\delta}{\delta X_{ij}} =&V^{-1}_{li}\left[\delta_{lm}\frac{\delta}{\delta\lambda_l}+\frac{1-\delta_{lm}}{\lambda_l-\lambda_m}\left(\phi_l\frac{\delta}{\delta\phi_m}-\frac{\delta}{\delta v_{lm}}\right)\right]V_{jm}\\ \frac{\delta}{\delta\psi_k} =&V^{-1}_{lk}\frac{\delta}{\delta\phi_l}\end{aligned}$$

Since the canonical pairing is left invariant, also the commutators of the new variables are still canonical.

One could be puzzled by the fact that with this diagonalization we have twisted the usual (and handy) relation between the matrix X and its hermitian conjugate $X^\dagger$. This can be seen as though the introduction of noncommutativity (i.e. switching on $\theta \neq 0$) made up for the appearance of a background into the first-quantized theory. On the other hand, in the next section the path integral formulation will make clear that the hermiticity of the hamiltonian of the present system is spoilt by this diagonalization. This is not bad, in regard of the unitarity of the model and positivity of the norm (e.g. see [3]), as far as the hamiltonian preserves $\mathcal{PT}$ symmetry. The non-hermiticity of the hamiltonian causes by itself the fact the conjugate of the matrix X after the diagonalization is not the conjugate of the diagonalized matrix anymore (see e.g. [14]). We will not consider this in the more general framework now, because it goes beyond the scope of the present work.

However, we have recovered the rule of conjugation in the form of a covariant derivative term, similar to the effect of some background introduced by noncommutativity. We will see how the expressions for the new canonical operators will work properly: in particular the operators that are the matrix extensions of the generators of W_∞ (see section 3.2) will have the correct behaviour under complex conjugation, when the covariant derivative term is taken into account.

An important thing to notice is that the Gauss' law constraint (6.2) is reduced, after normal ordering, to the following form

$$G_{ij} = V_{il}\left[\delta_{lm}\left(-\phi_l\frac{\delta}{\delta\phi_l}-2B\theta\right)-(1-\delta_{lm})\frac{\delta}{\delta v_{lm}}\right]V^{-1}_{mj} \tag{6.6}$$

Applying the constraint to physical states we get

$$G_{ij}|Phys\rangle = 0 \Longleftrightarrow \left\{\begin{array}{rcll}\left(\phi_l\frac{\delta}{\delta\phi_l}+2B\theta\right)|Phys\rangle &=& 0 & ,\quad \forall\, l\\ \frac{\delta}{\delta v_{lm}}|Phys\rangle &=& 0 & \Longleftarrow\quad l\neq m\end{array}\right.$$

[1] These results are nothing more than the *contragredient rule* plus the fact the decomposition by an invertible matrix cause a nontrivial form of parallel transport.

The non-diagonal components fixes the covariance of physical states under $GL(N,\mathbb{C})$ transformations, precisely they imply that the wavefunctions of physical states must depend on V only by terms of the form $\det V^n$; the diagonal ones are the remaining nontrivial part of the constraint, and we will give more on this later.

In case of a more general auxiliary term A_{ia}, as in (5.4), we can perform the previous decomposition *mutatis mutandis*, to obtain as a result for the diagonal part of the constraint

$$\forall\, l\,, \qquad \left(\sum_s A_{ls}\frac{\delta}{\delta A_{ls}} + 2B\theta\right)|Phys\rangle = 0$$

6.1.2 Path Integral (Faddeev-Popov adapted)

For $\theta \neq 0$, the $GL(N,\mathbb{C})$ diagonalization we just performed is not a gauge transformation in the path integral, indeed any gauge invariant term of the action transforms non-trivially; e.g. any term of the kind of

$$\mathbf{tr}\, XX^\dagger = \mathbf{tr}\, \Lambda(V^\dagger V)^{-1}\Lambda^\dagger(V^\dagger V)$$

The point here is to make a $GL(N,\mathbb{C})$ transformation inside the path integral re-adsorbing in some way the non-invariant term. The strategy to do this is the following:

$$\mathcal{J} = \int \mathfrak{D}X\mathfrak{D}X^\dagger \mathcal{F}[X,X^\dagger] = \int \mathfrak{D}X\mathfrak{D}X^\dagger \int_{\text{diag}} \mathfrak{D}\Lambda\mathfrak{D}\Lambda^\dagger\, \Delta_{FP}\times$$
$$\times \int_{GL(N,\mathbb{C})} \mu(V)\, \delta_{\mathbb{C}}[V^{-1}XV - \Lambda]\delta_{\mathbb{C}}[V^\dagger X^\dagger V^{\dagger -1} - \Lambda^\dagger]\mathcal{F}[X,X^\dagger]$$

Here, in the Fadeev-Popov identity the $\delta_{\mathbb{C}}$ functions are intended not as the usual δ, but as a "complex argument" $\delta_{\mathbb{C}}$, defined in a way that

$$\delta_{\mathbb{C}}[M]\delta_{\mathbb{C}}[M^\dagger] \equiv \delta[M]$$

Working out the integral above we obtain [12]

$$\mathcal{J} = \int_{GL(N,\mathbb{C})} \mu(V) \int_{\text{diag}} \mathfrak{D}\Lambda\mathfrak{D}\Lambda^\dagger \Delta_{FP}[\Lambda,\Lambda^\dagger]\mathcal{F}[\Lambda,(V^\dagger V)^{-1}\Lambda^\dagger(V^\dagger V)]\times$$
$$\times \int \mathfrak{D}X\mathfrak{D}X^\dagger \delta_{\mathbb{C}}[X-\Lambda]\delta_{\mathbb{C}}[V^\dagger V X^\dagger (V^\dagger V)^{-1} - \Lambda^\dagger] =$$
$$= \int_{GL(N,\mathbb{C})} \mu(V) \int_{\text{diag}} \mathfrak{D}\Lambda\mathfrak{D}\Lambda^\dagger \int \mathfrak{D}\tilde{\Lambda}\, \Delta_{FP}\, \delta_{\mathbb{C}}[\tilde{\Lambda} - (V^\dagger V)^{-1}\Lambda^\dagger(V^\dagger V)]\mathcal{F}[\Lambda,\tilde{\Lambda}]$$

The quantity Δ_{FP} may be computed in the usual way being it a jacobian determinant, just keeping in mind that now the coordinates are complex, so that we essentially obtain the square of the determinant we already had for hermitian matrices, namely

$$\Delta_{FP} = \prod_t \prod_{i<j} |\lambda_i - \lambda_j|^4$$

Now we only need to integrate out the $GL(N, \mathbb{C})$ transformation. When we do this integration, the integration over $\tilde{\Lambda}$ is reduced to the integration over matrices which have the same eigenvalues of $\Lambda^\dagger$ (because it is an integral over a conjugacy class of $\Lambda^\dagger$). The determinant we obtain performing the integration

$$\int \mu(V) \int \mathfrak{D}\tilde{\Lambda}\, \delta_{\mathbb{C}}[\tilde{\Lambda} - W\Lambda^\dagger W^{-1}] \qquad \text{with} \qquad W \doteq V^\dagger V$$

is

$$\prod_t \prod_{i<j} (\lambda_i^* - \lambda_j^*)^{-2}$$

This cancels part of the Faddeev-Popov determinant, actually the one depending on $\{\lambda_i^*\}$ only; let us write now the complete path integral, putting altogether

$$\int \mathfrak{D}\Phi\mathfrak{D}\tilde{\Phi} \int \mathfrak{D}\Lambda\mathfrak{D}\tilde{\Lambda} \prod_t \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{iS_{CS}[\Lambda,\tilde{\Lambda}] + iS_{\mathcal{B}}[\Phi,\tilde{\Phi}]} \delta[[\Lambda, \tilde{\Lambda}] - 2\theta - \Phi \circ \tilde{\Phi}] \quad (6.7)$$

where the change of variables from Ψ to Φ has been made in order to re-adsorb the change of the action after the $GL(N, \mathbb{C})$ transformation. Another consequence of this change of variables, as is already clear from the general framework of this theory, is to keep the constraint invariant.[2]

A clarification on the δ function in the complete partition function is in order here. The original path integral had in it

$$\delta[[X, X^\dagger] - 2\theta - \Psi \circ \Psi^\dagger] = \int \mathfrak{D}M e^{i\mathbf{tr}\, M([X,X^\dagger] - 2\theta - \Psi\circ\Psi^\dagger)}$$

Since the argument of the δ is an hermitian matrix, the matrix M in the integral is hermitian as well. When one conjugates the argument by an invertible matrix, the proof of the invariance of the δ goes on by conjugating M by the inverse of the matrix. Since the top form

$$[dM] \doteq \bigwedge_{i,j} dM_{ij}$$

[2]In terms of the equivalent Chern-Simons noncommutative field theory, both these effects are expressions of the gauge invariance of the A_α CS action.

is invariant under conjugation, and the eigenvalues of M are invariant as well, the integral defining the delta function still makes sense; so does the δ function. One may consider it as a trivial example of holomorphic path integral [29]. We can write down the δ function on the diagonal gauge according to the components of the argument

$$\delta[[\Lambda,\tilde{\Lambda}]-2\theta-\Phi\circ\tilde{\Phi}]=\prod_{i\neq j}\delta[(\lambda_i-\lambda_j)\tilde{\Lambda}_{ij}-\phi_i\tilde{\phi}_j]\prod_i\delta[2\theta+\phi_i\tilde{\phi}_i]=$$
$$=\prod_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\delta[\tilde{\Lambda}_{ij}-\frac{\phi_i\tilde{\phi}_j}{\lambda_i-\lambda_j}]\prod_i\delta[2\theta+\phi_i\tilde{\phi}_i]$$

So the jacobian determinant coming from the δ cancels the part of Δ_{FP} which depends on $\{\lambda_i\}$, that's to say the Faddeev-Popov determinant, just as in the hermitian case, gets completely cancelled out at last. Hence the path integral (6.7) at the very end of the diagonalization process is

$$\int\mathfrak{D}\Phi\mathfrak{D}\tilde{\Phi}\int\mathfrak{D}\Lambda\mathfrak{D}\tilde{\Lambda}e^{iS_{CS}[\Lambda,\tilde{\Lambda}]+iS_{\mathcal{B}}[\Phi,\tilde{\Phi}]}\prod_{i\neq j}\delta[\tilde{\Lambda}_{ij}-\frac{\phi_i\tilde{\phi}_j}{\lambda_i-\lambda_j}]\prod_i\delta[2\theta+\phi_i\tilde{\phi}_i] \qquad (6.8)$$

where

$$S_{CS}[\Lambda,\tilde{\Lambda}]=i\frac{B}{2}\int dt\sum_i\dot{\lambda}_i\tilde{\lambda}_i$$

and

$$S_{\mathcal{B}}[\Phi,\tilde{\Phi}]=-i\frac{B}{2}\int dt\sum_i\dot{\phi}\tilde{\phi}$$

After the elimination of the ϕ and $\tilde{\phi}$ auxiliary fields, whose dynamics is actually completely constrained, we see that the above is the path integral of the theory of the electrons with the coordinates $\{(\lambda_i,\tilde{\lambda}_i\}$, projected at the lowest Landau level [5].

6.2 The Physical Hilbert space

We already mentioned the fact that the constraint

$$\begin{cases}\left(\phi_l\frac{\delta}{\delta\phi_l}+2B\theta\right)|Phys\rangle = 0 \quad , \quad \forall\, l\\ \frac{\delta}{\delta v_{lm}}|Phys\rangle = 0 \Longleftarrow l\neq m\end{cases}$$

implies strong restrictions on the form of the wavefunctions of physical states. First of all, making use of the relation

$$\frac{\partial}{\partial v_{ij}}\det(V-z\,\mathbb{I})=\det(V-z\,\mathbb{I})\left(\frac{V}{V-z\,\mathbb{I}}\right)_{ji} \qquad (6.9)$$

we easily show that the only V-depending covariant factor (besides the constant) we may find in a wavefunction is $\det V$, indeed, putting $z = 0$ in (6.9)

$$\frac{\partial}{\partial v_{ij}} \det V = \det V \left(\frac{V}{V} \right)_{ji} = \det V \delta_{ij} = 0 \Longleftarrow i \neq j$$

A result similar to (6.9)

$$\frac{\partial}{\partial v_{ij}} \det(V^n - z\,\mathbb{I}) = n \det(V^n - z\,\mathbb{I}) \left(\frac{V^n}{V^n - z\,\mathbb{I}} \right)_{ji}$$

can be used to show that in the general case the same vanishing property holds for $\det V^n$ terms only.

The diagonal components imply instead that the wavefunction is homogeneous of degree $-2B\theta$ of any of the ϕ_i, so a generic wavefunction is of the form[3]

$$\Xi[X, V, \Phi] = \det V^k \left(\prod_i \phi_i \right)^k \chi[X] \qquad k \doteq -2B\theta$$

We have to impose further restrictions on the reduced wavefunction $\chi[X]$ in order to determine it completely. To find out how, we must impose the physical condition of incompressibility, which is doable in terms of representations of W_∞ algebra.

6.3 Incompressibility

6.3.1 Matrix W_∞ algebra

We now want to revive the discussion of section 3.2 regarding how to impose the physical condition of incompressibility on the Hilbert space of states of the system. What is in order now is to define the matrix substitute of the algebra of area-preserving diffeomorphisms. We recall the definition of the generators $\mathcal{L}_{st}$ of W_∞ in terms of the generators of magnetic translations

$$\mathcal{L}_{st} \doteq (b^\dagger)^s b^t$$

Since Chern-Simons matrix model lives in the limit $B \longrightarrow \infty$, we need only to deal with the restriction to the lowest Landau level of the $\mathcal{L}_{st}$ operators; in complex coordinates their many body first quantized expression is

$$\mathcal{L}_{st}|_{n=0} = \sum_\alpha z_\alpha^s \left(\frac{d}{dz_\alpha}\right)^t$$

[3]Recall that due to the level quantization condition, $B\theta$ must be an integer number.

The proposed matrix generalisation of the above operator is the following[4]

$$\mathcal{L}_{st}^{(\mathtt{mat})} \doteq :\mathbf{tr}\, X^s (X^\dagger)^t:$$

which in the case of normal matrices reduces to

$$\mathcal{L}_{st}^{(\mathtt{mat})} = \sum_k z_k^s \bar{z}_k^t \approx \sum_i z_k^s \left(\frac{\partial}{\partial z_k}\right)^t$$

after canonical quantization. So we see that our definition of $\mathcal{L}_{st}^{(\mathtt{mat})}$ reduces to the many body definition of the ordinary $\mathcal{L}_{st}$ in the lowest Landau level.

We can generalise the incompressibility conditions (3.12) to our new operators $\mathcal{L}_{st}^{(\mathtt{mat})}$: they must be imposed on the ground state of the theory as a necessary condition for describing the quantum Hall ground state

$$\mathcal{L}_{st}^{(\mathtt{mat})} \Xi_{GS} = 0 \qquad s < t$$

In the case of the matrix model generators, the algebra which turns out computing the commutators is different by (3.10), and does not close by itself, indeed the commutator

$$[\mathcal{L}_{st}^{(\mathtt{mat})}, \mathcal{L}_{mn}^{(\mathtt{mat})}]$$

gets several corrections, which we interpret as finite size corrections,[5] which are products of terms having the following form

$$:\Psi^\dagger X^{s'} (X^\dagger)^{t'} \Psi: \tag{6.10}$$

The exact form of the W_∞ algebra in this matrix version is still unknown, due to the ever increasing complexity of the direct computation of commutators between higher order generators.

One can easily see that when performing the commutator, if we started with two operators $\mathcal{L}_{st}^{(\mathtt{mat})}$ and $\mathcal{L}_{mn}^{(\mathtt{mat})}$ to the both of which the highest weight condition (3.12) applies (i.e. $s < t$ and $m < n$), then, in the result, each factor of any addend would present X and $X^\dagger$ in the form

$$\cdots X^{s'} (X^\dagger)^{t'} \cdots \qquad \text{with } s' < t'$$

[4]The expression is normal ordered ($::$) in view of canonical quantization.

[5]The fact (6.10) are finite size correction of W_∞ algebra can be seen working in a gauge in which the vector Ψ has the form $(0, \cdots, 0, NB\theta)$. In this gauge only the entries of the last row and column of $X^s(X^\dagger)^t$ matrix enter into the $\mathcal{P}_{st}^{(\mathtt{mat})}$ operators. In a proper $N \to \infty$ weak limit only the "bulk" of the matrix will contribute, and moreover the Ψ vector need to disappear from the expressions, so do the $\mathcal{P}_{st}^{(\mathtt{mat})}$ operators.

Since we want the Lie subalgebra generated by the $\mathcal{L}^{(\mathtt{mat})}_{st}$ with $s < t$ to vanish on the ground state for consistency, we must join to the set of generators $\mathcal{L}^{(\mathtt{mat})}_{st}$ the set $\mathcal{P}^{(\mathtt{mat})}_{s't'} \doteq :\Psi^\dagger X^{s'} (X^\dagger)^{t'} \Psi:$, with the conditions

$$\begin{aligned} \mathcal{L}^{(\mathtt{mat})}_{st} \Xi_{GS} &= 0 \qquad s < t \\ \mathcal{P}^{(\mathtt{mat})}_{st} \Xi_{GS} &= 0 \qquad s < t \end{aligned} \tag{6.11}$$

Regarding the normal ordering, we can see that if both the (6.11) are met then they hold also for any other normal ordering. This is due to the fact that in each reordering term

$$(X^\dagger)^t X^s = (X^\dagger)^{t-1} X X^\dagger X^{s-1} - (X^\dagger)^{t-1} (\theta + \Psi\Psi^\dagger) X^{s-1}$$

the powers of X and $X^\dagger$ are decreased by the same amount, except for the leading term that, at the end of the computation, becomes $X^s (X^\dagger)^t$. So when imposing the conditions of incompressibility (6.11) in a different ordering, one may work recursively, at each step just having to care for the leading order condition, because lower order ones are already satisfied in virtue of the previous steps.

6.3.2 Wavefunctions for the CS Matrix Model

In order to find a general covariant expression (i.e. one which depends on $X, X^\dagger$ and $\Psi, \Psi^\dagger$) one can try to solve the constraint (6.5), or alternatively one can use the group theoretical properties of the states, exploiting the invariance under $SU(N)$ algebra generated by the hamiltonian constraint G, as has been done in [42, 24]; one recovers in such a way the wavefunctions [24]

$$\Xi[X, \Psi] = \mathbf{tr}\, X^{c_1} \cdots \mathbf{tr}\, X^{c_k} \left[\epsilon^{i_1 \cdots i_N} (X^0 \Psi)_{i_1} \cdots (X^{N-1} \Psi)_{i_N} \right]^k \tag{6.12}$$

Here the $\mathbf{tr}\, X^{c_\bullet}$ factors in the wavefunction creates the excitations, while the ϵ part of the function is the wavefunction of the ground state

$$\Xi_{GS}[X, \Psi] = \epsilon^{i_1 \cdots i_N} (X^0 \Psi)_{i_1} \cdots (X^{N-1} \Psi)_{i_N} \tag{6.13}$$

These expressions are obtained simply by contracting all the indexes of any monomial of the form

$$\prod_a X_{i_a j_a} \psi_{l_a}$$

with the invariant tensors of $SU(N)$, namely δ and ϵ, to form a gauge invariant expression. Moreover the Gauss' law constraint (5.1) requires the total number of ψ

to be k, fixing ultimately the form of the wavefunction.

Now we show our incompressibility conditions (6.11) are met. It is more easy to work with antinormal ordered operators (we already showed after (6.11) this can be safely done), writing ($s < t$)

$$\mathbf{tr}\, X^s (X^\dagger)^t\ \epsilon^{i_1 \cdots i_N} (X^0 \Psi)_{i_1} \cdots (X^{N-1}\Psi)_{i_N}$$

where we use coordinate representation, i.e. $X^\dagger_{ij} \approx \frac{\delta}{\delta X_{ji}}$. When all of the derivatives have acted on the factors of the determinant, the indexes in the matrix products get rearranged. But since there are more derivatives with respect to X_{ij} than multiplications by X_{ij}, the total degree in X of the determinant is decreased. So in each addend of the resulting polynomial at least two of the columns of the matrix $[(X^{i-1}\Psi)j]_{i,j=1,\dots,N}$ are made equal, making the determinant vanish.

A similar argument is true for the finite size corrections:

$$\Psi^\dagger X^s (X^\dagger)^t \Psi\ \epsilon^{i_1 \cdots i_N} (X^0 \Psi)_{i_1} \cdots (X^{N-1}\Psi)_{i_N}$$

where in coordinate representation $\psi^*_i \approx \frac{\delta}{\delta \psi_i}$. The only difference is that here after the rearrangement of indexes some factor loses some extra power of X because there appear terms of the form $\mathbf{tr}\, X^h$ for some h, due to the presence of Ψ and derivatives with respect to Ψ: of course the total degree in X of each addend of the resulting polynomial is decreased and so they vanish since they are determinants with two or more equal columns.

Notice that if $s > t$, the previous argument does not apply. So one can easily see by direct computation that $\mathcal{L}^{(\mathtt{mat})}_{st}$ and $\mathcal{P}^{(\mathtt{mat})}_{st}$, with $s > t$, when applied on ground state wavefunction make factors of $\mathbf{tr}\, X^h$ for some h appear, so generating linear combinations of states like (6.12) from the ground state (6.13), i.e. creating excitations on the ground state. This is exactly what happens in the framework of chapter 3 when acting on the ground state with a generator $\mathcal{L}_{st}$ with $s > t$. Therefore the Hilbert spaces of physical states is a representation of the matrix W_∞ algebra.

Working out the ground state wavefunction (6.13), we find its behaviour under a similarity transformation $X \longmapsto VXV^{-1}$

$$\Xi_{GS}[X, \Psi] \longmapsto \Xi_{GS}[VXV^{-1}, V\Psi] = \det V^k\ \Xi_{GS}[X, \Psi]$$

One thing clear here from the transformation rule is that the symmetry of the wavefunction under exchange of two coordinates is given essentially by the parity of the exponent k.

Using the above rule, by diagonalizing X, we find Ξ_{GS} to be equal to

$$\Xi_{GS}[X,\Psi] = \det V^k \; \Xi_{GS}[\Lambda,\Phi] = \det V^k \left(\prod_i \psi_i\right)^k \prod_{i<j}(\lambda_i - \lambda_j)^k$$

Therefore, after the reduction to the eigenvalues, in physical complex coordinates, the wavefunction (6.13) manifestly reduces to a Laughlin wavefunction.

6.3.3 Scalar product

In complex coordinate the scalar product between wavefunctions is expressed in a different form than that of section 5.2. What makes the difference is actually the form the amplitudes of the coherent states assume when expressed in terms of the basic operators X and $X^\dagger$, namely while in section 5.2 we had

$$|Q\rangle \doteq e^{\mathbf{tr}\, Q\widehat{X}_1}|0\rangle \qquad \langle Q|Q'\rangle = \delta[Q-Q']$$
$$|\Pi\rangle \doteq e^{\mathbf{tr}\, \Pi\widehat{X}_2}|0\rangle \qquad \langle \Pi|\Pi'\rangle = \delta[\Pi-\Pi']$$
$$\text{and the wavefunction is } \langle Q|\Pi\rangle = \frac{1}{2\pi}e^{i\mathbf{tr}\, \Pi Q}$$

now that we have switched to complex quantization we get

$$|Q^\dagger\rangle \doteq e^{\mathbf{tr}\, Q^\dagger\widehat{X}}|0\rangle$$
$$\langle Q| \doteq \langle 0|e^{\mathbf{tr}\, Q\widehat{X}^\dagger}$$
$$\text{the wavefunction equals } \langle Q_1|Q_2^\dagger\rangle = e^{\mathbf{tr}\, Q_1Q_2^\dagger}$$

Moreover we will need to incorporate in the scalar product the Ψ, $\Psi^\dagger$ as well. We do it by defining

$$|\Psi^\dagger\rangle \doteq e^{\mathbf{tr}\, \Psi^\dagger\widehat{\Psi}}|0\rangle$$
$$\langle \Psi| \doteq \langle 0|e^{\mathbf{tr}\, \Psi\widehat{\Psi}^\dagger}$$
$$\text{with} \qquad \langle \Psi_1|\Psi_2^\dagger\rangle = e^{\mathbf{tr}\, \Psi_1\Psi_2^\dagger}$$

When we diagonalize the matrix X, and correspondingly transform the matrix $X^\dagger$ as

$$X \;\;= V\Lambda V^{-1}$$
$$X^\dagger = V\tilde{\Lambda} V^{-1}$$

then the coherent states gets defined by the chain of relations

$$\langle X| \doteq \langle 0|e^{\mathbf{tr}\, X\widehat{X}^\dagger} = \langle 0|e^{\mathbf{tr}\,\Lambda(V^{-1}\widehat{X}^\dagger V)} \doteq \langle\langle\Lambda|$$
$$|X^\dagger\rangle \doteq e^{\mathbf{tr}\, X^\dagger\widehat{X}}|0\rangle = e^{\mathbf{tr}\,\tilde{\Lambda}(V^{-1}\widehat{X}V)}|0\rangle \doteq |\tilde{\Lambda}\rangle\rangle$$

where we have used the transformation rule on the canonical momentum operators $X^\dagger$ caused by the diagonalization of coordinate X as was shown in section 6.1.1. In the same section we saw that this transformation leaves the canonical coordinates invariant, being it the quantum version of a canonical transformation. So all the properties of coherent states are still in charge with the gauge fixed coordinates and operators, we just need to keep track of the $GL(N, \mathbb{C})$ conjugations.

Differently from the hermitian coordinate situation, here we have to insert the projector on the physical states in the scalar product itself. This is due to the fact that, in covariant notation (i.e. before fixing the gauge) the dependence of a wavefunction on the matrix X, on imaginary conjugation $|\rangle^* = \langle|$ is switched to the dependence on $X^\dagger$, so that the resolution of identity by coherent states involves both the coordinates and the momenta at fixed time. So the equation we must work on is now

$$\langle 1|2\rangle = \int\prod_i d\psi_i d\psi_i^* \int\prod_{ij} dX_{ij}dX_{ij}^\dagger\; e^{-\mathbf{tr}\, XX^\dagger - \Psi^\dagger\Psi}\langle 1|X^\dagger,\Psi^\dagger\rangle\langle X,\Psi|2\rangle\delta[[X,X^\dagger]-2\theta-\Psi\circ\Psi^\dagger]$$

We have for the scalar product

$$\begin{aligned}\langle 1|2\rangle = &\int\prod_i d\phi_i d\tilde{\phi}_i e^{-Nk}\prod_i\delta[\phi_i\tilde{\phi}_i + 2\theta]\times\\ &\times\int d\Lambda d\tilde{\Lambda}\; e^{-\mathbf{tr}\,\Lambda\tilde{\Lambda}}\,\langle 1|\tilde{\Lambda},\tilde{\Phi}\rangle\rangle\,\langle\langle\Lambda,\Phi|2\rangle\prod_{i\neq j}\delta[\tilde{\Lambda}_{ij} - \frac{\phi_i\tilde{\phi}_j}{\lambda_i-\lambda_j}]\end{aligned} \tag{6.14}$$

From here we can see that the property

$$\langle\Xi_1|\mathbf{tr}\, X^s(X^\dagger)^t\Xi_2\rangle = \langle\mathbf{tr}\, X^t(X^\dagger)^s\Xi_1|\Xi_2\rangle$$

beholds also in the diagonal gauge, when we account for the nontrivial covariant derivative.

We can write in example the scalar product (6.14) in the case of $N = 2$, obtaining

$$\langle GS_k|GS_k\rangle = \mathcal{N}\int d\lambda_1 d\lambda_2 d\tilde{\lambda}_1 d\tilde{\lambda}_2\; e^{-\Sigma_{i=1}^2\lambda_i\tilde{\lambda}_i}\left(\tilde{\lambda}_1 - \tilde{\lambda}_2 + \frac{2k}{\lambda_1-\lambda_2}\right)^k(\lambda_1-\lambda_2)^k$$

where $\mathcal{N}$ is the constant obtained by the integration over Φ and $\widetilde{\Phi}$. The form of the above integral, due to the already mentioned twisted conjugation (i.e. the presence of a nontrivial parallel transport term), differs from the standard form for the overlaps of Laughlin wavefunctions in the Quantum Hall effect:

$$\langle GS_k|GS_k\rangle = \mathcal{N}\int d\lambda_1 d\lambda_2 d\bar{\lambda}_1 d\bar{\lambda}_2 \; e^{-\sum_{i=1}^{2}\lambda_i\bar{\lambda}_i}\left(\bar{\lambda}_1-\bar{\lambda}_2\right)^k(\lambda_1-\lambda_2)^k \; .$$

Anyhow the quantities which possess a physical meaning are the values of the (normalized) scalar products. These are numbers, and only in their terms one can compare the Matrix Model with the physics of Laughlin wavefunctions. In addition we notice that for $\theta \longrightarrow 0$ we have

$$\tilde{\lambda}_i \longrightarrow \bar{\lambda} \; ,$$

but the scalar product does not manifestly reduce itself to the ordinary normal matrix integral ([33, 19, 54])

$$\int d^2\lambda_1 d^2\lambda_2 \;\left(\bar{\lambda}_1-\bar{\lambda}_2\right)(\lambda_1-\lambda_2) \; .$$

This is due to the presence of the δ function enforcing the constraint. Usually when defining the normal matrix integral, it is said that the conditions

$$[X, X^\dagger]_{ij} = 0$$

are not independent. Indeed if one uses a δ function to enforce the above condition, one gets very soon into troubles. These troubles are apparent already in the $N = 2$ model, let us use the following decomposition for X ([54, 52])

$$X = U(\Lambda + R)U^\dagger \qquad \text{with } U \in U(2),\; \Lambda = diag(\lambda_1,\lambda_2),\; R = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$$

to write the above constraint

$$[X, X^\dagger] = U\begin{pmatrix} |r|^2 & r(\bar{\lambda}_2-\bar{\lambda}_1) \\ \bar{r}(\lambda_2-\lambda_1) & -|r|^2 \end{pmatrix}U^\dagger$$

Manifestly the four conditions are not independent. If we chose the of diagonal ones, to put into a Dirac δ function, we would obtain a jacobian determinant which is the same we obtained in our computation $|\lambda_1-\lambda_2|^{-2}$, which cancels the Vandermonde determinant of the measure induced by the immersion of the set of normal matrices into the bigger set of arbitrary complex ones [33, 19]. If instead one uses one of the diagonal entries of the constraint, one gets no jacobian, and so there is still a measure in the integrand, which is just the usual Vandermonde. When computing the integral over normal matrices, one can overcome this ambiguity just inducing the measure from the flat measure of complex matrices, by the inclusion

$$\Lambda = diag(\lambda_i) \hookrightarrow X = U\Lambda U^\dagger$$

But this just corresponds to a peculiar matrix quantum mechanics. Indeed in our model we are forced to put a Dirac δ function into the integral in order to enforce the constraint, because it is a remnant of the Chern-Simons gauge theory in the temporal gauge, in which the component A_0 of the CS gauge field acts as a Lagrange multiplier (see (4.7), (4.9) and (4.12), and discussion about them). Integration over A_0 gives the Dirac delta. With the introduction of the Ψ the entries of the constraint become independent. Indeed we see for $N = 2$ the constraint is

$$\begin{pmatrix} |r|^2 - |\psi_1|^2 & r(\bar{\lambda}_2 - \bar{\lambda}_1) - \psi_1\psi_2^* \\ \bar{r}(\lambda_2 - \lambda_1) - \psi_2\psi_1^* & -|r|^2 - |\psi_2|^2 \end{pmatrix} = 0$$

This breaks up into four δ-like factors

$$\prod_{ij} \delta([X, X^\dagger]_{ij} - \psi_i\psi_j^*) = \delta(|r|^2 - |\psi_1|^2)\delta(|r|^2 + |\psi_2|^2)\delta_{\mathbb{C}}(r(\bar{\lambda}_2 - \bar{\lambda}_1) - \psi_1\psi_2^*)\delta_{\mathbb{C}}(\bar{r}(\lambda_2 - \lambda_1) - \psi_2\psi_1^*)$$

Now all the entries of the constraint matrix are independent from each other, and we see the last two δ factors drop a jacobian. Therefore, as we stated above, the integrand in the limit in which θ vanishes does not reduce itself to the integrand of usual normal matrix model, but we must consider only the values of the integrals normalized to the norm of the ground state as the correct physical quantities to be compared with those coming from Laughlin theory of the Quantum Hall effect. As we already showed in the section 6.3.1, the Hilbert space of physical states realizes a representation of matrix W_∞ algebra. Thus we state that the scalar products can be computed algebraically by using the commutation relations defining the matrix W_∞ algebra itself.

As argued by other authors [27, 26], the expression for the wavefunction of the ground state in hermitian coordinates can be written in terms of

$$X_{ij}^\dagger = X_{ij}^{(1)} - \frac{2}{B}\frac{\partial}{\partial X_{ji}^{(1)}}$$

in the same form of (6.13) because of the antisymmetry of the expression, being it a determinant, so that in terms of the eigenvalues of $X^{(1)}$, in the hermitian gauge we have the wave function

$$\Phi[X^{(1)}] \propto \prod_{i<j}(x_i - x_j)^k$$

which becomes, after the shift (5.17)

$$\Phi[X^{(1)}] \rightsquigarrow \prod_{i<j}(x_i - x_j)^{k+1}$$

So we find in this hermitian gauge the shift several authors already found [42, 43, 4] in other ways[6].

6.4 Conclusions

We have studied here the matrix model derived from the many body action of electrons in the first Landau level, both in hermitian and in complex coordinates. We worked out the path integral and the scalar product of the theory in both cases: in the latter one, in particular, we performed a holomorphic quantization in order to specify the diagonal gauge choice; it is useful to stress once more the importance of introducing the constraint in the definition of physical scalar product. The constraint in this gauge has been solved explicitly, showing the general form of the gauge fixed wavefunctions.

In this diagonal gauge the derivative (momentum) operators get a term of parallel transport, as we saw in section 6.1.1; this term spoils the explicit form of hermitian conjugation, as we saw, the off-diagonal entries of $X^\dagger$ on the diagonal gauge are fixed by the constraint in terms of the physical degrees of freedom (i.e. the eigenvalues of X, $\{\lambda_i\}$ and the auxiliary ϕ and $\tilde{\phi}$), while the diagonal ones keep the dynamical meaning of canonical momenta of the reduced system.
The off-diagonal entries are generated geometrically as a nontrivial parallel transport in the manifold of the variables $(X\Psi)$ when the diagonalization is performed: they appear in gauge fixed variables as some sort of background the which arises when noncommutativity is switched on. Let us look at the gauge fixed action

$$S_{CS}|_{gf} \propto \int dt \sum_i \dot{\lambda}_i \tilde{\lambda}_i \qquad (X, X^\dagger) \rightsquigarrow (\Lambda, \widetilde{\Lambda} = diag(\tilde{\lambda}_i) + \mathcal{A}(\Lambda, \Phi\widetilde{\Phi}))$$

As we saw also before, the noncommutativity of coordinates gives rise to a "background" $\mathcal{A}(\Lambda, \Phi, \widetilde{\Phi})$ which is just the expression of the fact we cannot find a basis of simultaneous eigenvectors of both X and $X^\dagger$. This background terms, do not enter into the action directly, since their canonically conjugated variables, i.e. the

[6] Notice, however, that the wavefunction here is expressed in terms of the x_i, the real eigenvalues of $X^{(1)}$ only, which is hermitian, not in terms of the eigenvalues λ_i of the complex matrix X.

off-diagonal entries of Λ, vanish due to the gauge condition (auxiliary condition in the sense of ref. [12]). Hence we are left with the action of electrons in the lowest Landau level, with coordinates $\{(\lambda_i, \tilde{\lambda}_i)\}$ [5].
One more key to understand the appearance of this "covariant derivative term" is the analogy with the so called *statistical interaction* of Fradkin and Lopez [16].

Moreover we analysed a matrix version of the W_∞ algebra generators, rephrasing the incompressibility conditions in terms of them, keeping track of the finite size corrections. The already known ground state wavefunction [24], which are recognised to be Laughlin states, turned out to be incompressible in the sense of these new operators. The incompressibility conditions can be imposed in the model at fixed (diagonal) gauge. The direct computation proves itself very hard to perform; one can anyway explicitly see it at $N = 2$ for example, and low exponents s, t in $\mathcal{L}^{(\mathtt{mat})}_{st}$, finding for the simplest cases [5]

$$\mathcal{L}^{(\mathtt{mat})}_{n1} = \sum_i \lambda_i^n \frac{\partial}{\partial \lambda_i}$$

$$\mathcal{L}^{(\mathtt{mat})}_{n2} = \sum_i \lambda_i^n \left[\frac{\partial^2}{\partial \lambda_i^2} - \sum_{n \neq i} \frac{k^2 + k}{(\lambda_n - \lambda_i)^2} \right]$$

applying these operators to the ground state

$$\Xi_{GS} \propto \prod_{i<j} (\lambda_i - \lambda_j)^\ell$$

we see the incompressibility conditions require that $\ell = k + 1$. The apparent discrepancy of the above with the Gauss law constraint may be solved by considering that the outcome of the diagonalization we performed is a nonlinear expression in the physical variables: there can be subtleties about the proper normal ordering of the operator at fixed gauge. Indeed, if one considers, into the operator $\mathcal{L}^{(\mathtt{mat})}_{n2}$ above when reduced to its gauge fixed form, the action of $\frac{\partial}{\partial v_{ij}}$ on the V_{kl} and V^{-1}_{st} employed for the diagonalization, one can check that it annihilates the ground state with $\ell = k$. On this argument about normal ordering troubles, see also [5].
We stress once more that there is not yet a complete control of the algebra of the $\mathcal{L}^{(\mathtt{mat})}$ and $\mathcal{P}^{(\mathtt{mat})}$, and that the latter ones in particular can arise as descendants of the operators $\mathcal{L}^{(\mathtt{mat})}$ into the incompressibility conditions.
We focused here on the properties of the ground state, but we saw also that the excited states can be obtained by applying nontrivial $\mathcal{L}^{(\mathtt{mat})}_{st}$ and $\mathcal{P}^{(\mathtt{mat})}_{st}$ operators. So the overlaps can be computed in principle in a purely algebraic way. The manifest

form of the integrand of the scalar product could be formally changed, so one cannot unambiguously identify the physical feature of the model without actually performing the integrals: only the amplitudes indeed are physically sensitive objects; they can be computed completely using the fact the physical Hilbert space is a representation of W_∞ algebra, thus obtaining a coordinate invariant description of the physics of the present model. In this way one can compare the results coming from the matrix model and the standard Quantum Hall computations, completely clarifying the physical content of Chern-Simons matrix model.

Index

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