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Matrix Effective Theories of the Quantum Hall Effect

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Introduction

The quantized Hall effect was discovered by Klaus Von Klitzing in 1980 in the experimental setting of fig.1, involving a two-dimensional electron gas placed into a strong magnetic field $\mathbf{B}$. For certain values of the field, the current in the x axis vanishes and the component R_{xy} of the resistivity (Hall resistivity) is quantized [1],

$$\begin{aligned} R_H &\equiv R_{xy} = \sigma_{xy}^{-1} = \nu^{-1}\frac{h}{e^2}, \\ R_{xx} &= \sigma_{xx} = 0, \end{aligned} \tag{1}$$

where ν is the so called filling fraction, that can be integer (fig.2) or fractional (fig.3) (Integer and Fractional Quantum Hall effect, respectively).

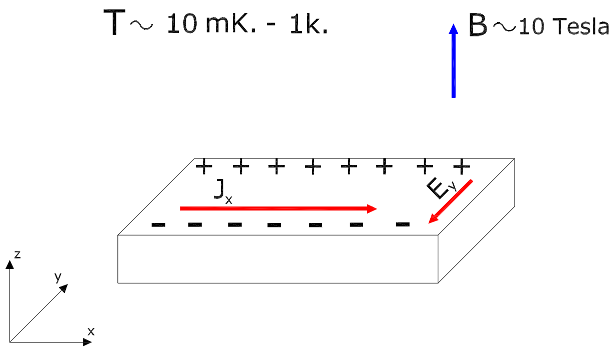

Figure 1: Scheme of the experiment.

The regimes in which the values of the resistivity are given by (1) are called plateux of the Quantum Hall effect. At the plateux, the ground states are gapful and very stable, with uniform density $\overline{\rho} = \frac{e\mathbf{B}}{hc}\nu$, i.e. the system of electrons behaves like a fluid with characteristic quantum effects [1][2][3]. The elementary excitations of the ground states correspond to local fluctuations in the density, and are called quasi-holes and quasi-particles of the QHE [1]. The low-energy excitations are gapful and thus the quantum fluid is incompressible.

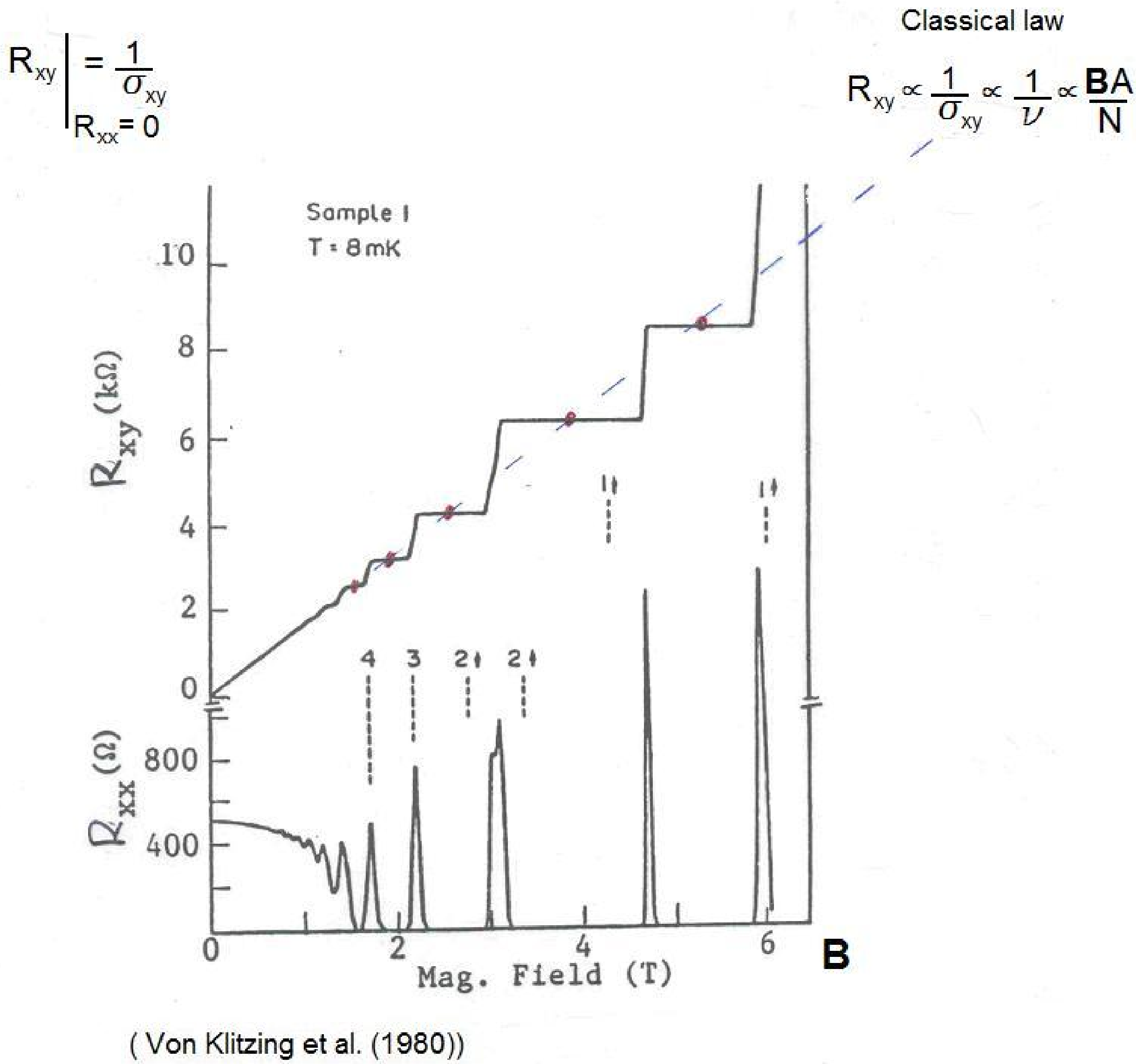

Figure 2: Diagonal and Hall resistivity in a sample of IQHE.

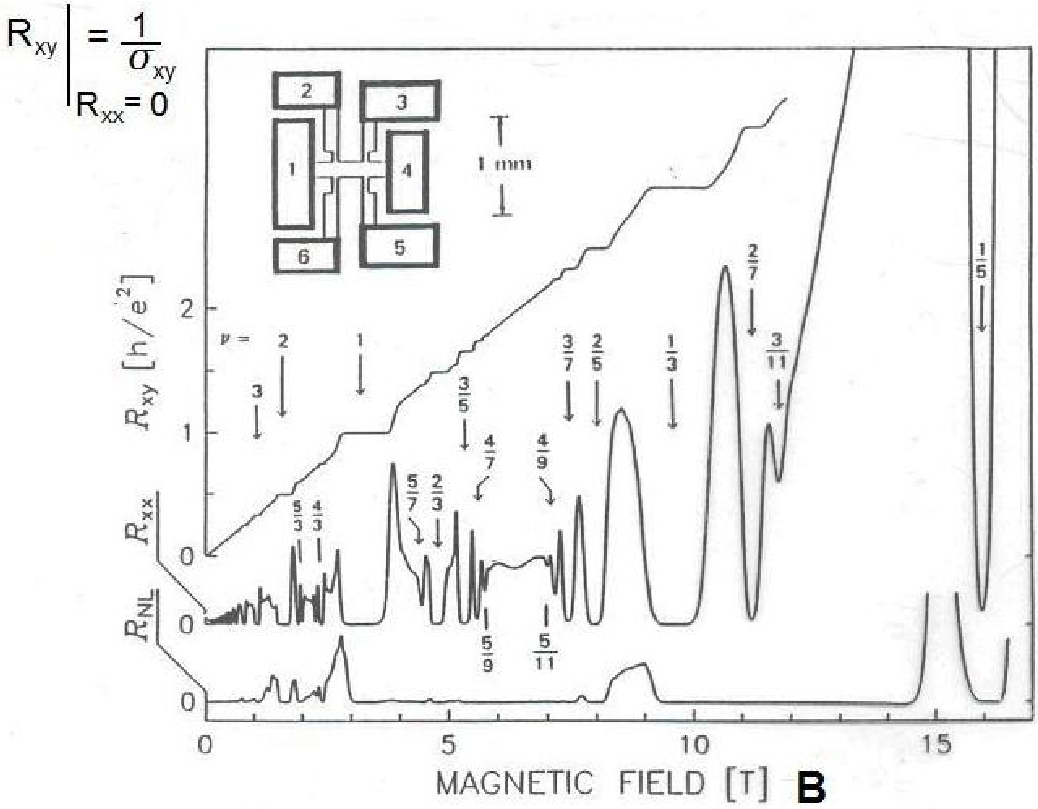

Figure 3: Diagonal and Hall resistivity in a sample of FQHE.

The IQHE can be described in terms of free electrons in the Landau levels, while the FQHE requires interacting electrons. It was discovered by Tsui et al. [4] in 1982. In 1983 Laughlin proposed a phenomenological theory of the FQHE, for the fillings, $\nu = \frac{1}{2k+1}$, with k a positive integer [2]. The Laughlin theory was confirmed by experiments in 1997 [3].

There are other filling fractions not contained in the Laughlin theory, that are observed experimentally [4], given by the more general expression $\nu = \frac{n}{2nk+1}$ [1][5], where n and k are positive integers. J.Jain explained these filling fractions [5] by proposing that electrons condensate in new particles, called composite fermions. Based on this idea, he conjectured that the ground states for fractional quantum Hall states are equivalent to integer quantum Hall states of composite fermions [5]. The Jain idea, of weakly-interacting composite fermions, is in good agreement with the experiments [6]. E.Fradkin and A.Lopez [7], and others [1][9], formalized the Jain idea in two dimensional quantum field theory by letting the electrons to interact with a "statistical" Chern-Simons gauge field. They studied the theory within the mean field approximation and reproduced the Jain ground states; however, the extension of their approach beyond mean field theory presents some limitations [1][10].

In this thesis we study other types of effective theories for the FQHE, that are based on matrix models or, more precisely, matrix quantum mechanics [12]. The main part of our work is devoted to the original proposal of the Maxwell Chern-Simons matrix theory and to the analysis of its possible ground states [13].

The next chapter contains a brief introduction to the QHE: we recall the Landau levels and define the Integer and Fractional QHE. We review the Laughlin theory [2], the Jain interpretation of the FQHE [5] and the mean field theory proposed by E.Fradkin and A.Lopez [7].

The second chapter deals with the Chern-Simons matrix model, and reviews the work by Susskind and Polychronakos [14], [15]. The use of noncommutative and matrix theories was initiated by Susskind [14], who observed that two-dimensional semiclassical incompressible fluids in strong magnetic fields can be described by the noncommutative Chern-Simons theory in the limit of small noncommutative parameter θ, corresponding to high density. Actually, the use of noncommuting spatial coordinates, x_1, x_2, i.e. $[x_1, x_2] = i\theta$, implies a generalized uncertainty relation that controls the effective size of electrons and thus modifies the density of the fluid [16].

Afterwards, Polychronakos extended the theory to describe a finite droplet of fluid, and obtained the $U(N)$ matrix gauge theory called Chern-Simons matrix model [15]. From the quantization of this theory, one obtains the important result that the ground states are exactly given by the Laughlin wave functions [19][20]. However, the Chern-Simons matrix model cannot describe the more general Jain states, and its full quantization encounters some problems that limit its applicability as a theory of the FQHE [21][13][22]. In the last part of the second chapter we present two generalizations of the Chern-Simons matrix model [24][21], where the classical ground states correspond to droplet solutions that generalize the Laughlin fluid. However, these approaches were not able to obtain the quantum states with Jain filling fractions, $\nu = \frac{n}{2kn+1}$.

Chapter three is devoted to our first work [13] in which we propose the generalization given by the Maxwell Chern-Simons matrix theory. This theory contains an additional coupling $g > 0$ that controls matrix noncommutativity. We show that the $g = 0$ theory corresponds to a matrix generalization of the Landau levels, where the physical gauge invariant states are matrix analogs of the expected Laughlin and Jain states and their quasi-hole excitations. In the $g \to \infty$ limit, the Maxwell Chern-Simon matrix theory reduces to the ordinary theory of electrons in Landau levels plus a $O(1/r^2)$ two-dimensional interaction generalizing the one-dimensional Calogero model [47]. Although this interaction is different from the $O(1/r)$ Coulomb potential, the $g = \infty$ theory provides a realistic effective model of the FQHE [6][2]. In our work [13], we conjectured that the matrix ground states found at $g = 0$, corresponding to the Laughlin and Jain series, have a smooth $g \to \infty$ physical limit (no phase transition at finite g values). The connection between these two limits will give a better understanding of the $g = \infty$ theory (that cannot be exactly solved) following the deformation of $g = 0$ states.

In chapter four we present our second work [22] in which we study the semiclassical limit of the Maxwell Chern-Simons matrix theory. Solving the classical equations of motion at $g = 0$, we find semiclassical ground states matching the matrix Jain states presented in chapter three. These states correspond to droplets of incompressible fluid with piece-wise constant density, that is similar to that of the phenomenological Jain wave functions ($g = \infty$) [10]. This result supports the Maxwell Chern-Simons matrix theory as a valid effective theory of the Fractional Hall effect.

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Chapter 1

Introduction to the Quantum Hall Effect

1.1 Landau levels

In this section we review the physics of Landau Levels [23][25]. Consider spin-polarized, planar electrons of mass m and electric charge e in an external, uniform, magnetic field $\mathbf{B} > 0$ (units $\hbar = 1$, $c = 1$). The one-particle Hamiltonian is given by:

$$H = -\frac{1}{2m}(\nabla - ie\mathbf{A})^2. \tag{1.1}$$

We work in the symmetric gauge $A_i = \frac{\mathbf{B}}{2}\epsilon_{ij}x^j,\ \ i,j = 1,2$, for the external vector potential. The fundamental scale set by the external magnetic field is the magnetic length, $\ell = \sqrt{2/e\mathbf{B}}$.

We use holomorphic spatial coordinates $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$, that are natural in the QHE [2] [25]. By introducing two commuting sets of harmonic oscillator operators,

$$\begin{aligned} d &= \frac{z}{2\ell} + \ell\overline{\partial}\ , \qquad d^\dagger = \frac{\overline{z}}{2\ell} - \ell\partial\ , \qquad \left[d, d^\dagger\right] = 1, \\ c &= \frac{\overline{z}}{2\ell} + \ell\partial\ , \qquad d = \frac{z}{2\ell} - \ell\overline{\partial}\ , \qquad \left[c, c^\dagger\right] = 1, \end{aligned} \tag{1.2}$$

the Hamiltonian (1.1) and the canonical angular momentum $J = -ix^i\epsilon^{ij}\partial_j$ can be rewritten as,

$$\begin{aligned} H &= \omega\ (d^\dagger d + \frac{1}{2}), \\ J &= c^\dagger c - d^\dagger d, \end{aligned} \tag{1.3}$$

where $\omega = \frac{e\mathbf{B}}{m}$ is the cyclotron frequency and $\partial = \frac{\partial}{\partial z}$ and $\overline{\partial} = \frac{\partial}{\partial \bar{z}}$ in (1.2). Since the operators c and d commute, the spectrum consists of infinitely degenerate levels of energy $\epsilon_n = \omega n$: these are called the Landau levels. The degenerate states correspond to circular orbitals of the semiclassical motion; they can be characterized by their angular momentum eigenvalue l.

It is easy to see that in completely filled Landau levels, the Hall resistivity is given by $R_H = \nu^{-1}\frac{h}{e^2}$ where $\nu = \frac{N}{N_\Phi}$ is the filling fraction with N the number of electrons and N_Φ the number of quantum fluxes. Because there is one quantum flux per orbital, ν is an integer or a fraction.

Fig.1.1(a) correspond to the $\nu = n$ case in which n Landau levels are filled with one electron per orbital. The system is incompressible due to the exclusion principle with a gap given by the cyclotron frequency. Thus the simple theory of free electrons in Landau levels is sufficient to describe this main physical property of the IQHE. On the other hand, if there are empty orbitals like in fig.1.1(b) for the case $\nu = 1/3$, the free-electron states are compressible; in contrast with the experimental observations: the FQHE requires a model of interacting electrons.

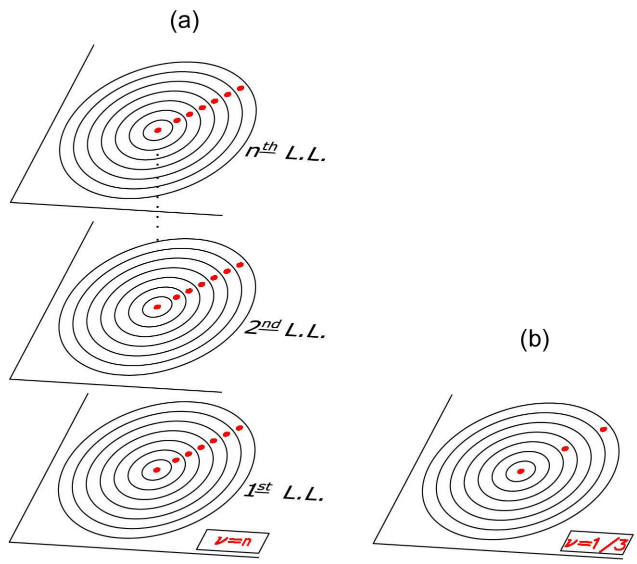

Figure 1.1: Graphical representation of the Landau levels.

1.2 Review of the Laughlin theory

In a remarkable paper [2] Laughlin constructed a class of wave functions given by:

$$\Psi_m(z_1, z_2, ..., z_N) = \prod_{i<j}^{N} (z_i - z_j)^m e^{-\frac{1}{2}\sum_i^N |z_i|^2} , \tag{1.4}$$

with N the number of electrons and m is an integer parameter. Hereafter we chose the magnetic length $\ell = 1$. The Laughlin wave function (1.4) describes spinless electrons in the lowest Landau level: to satisfy the antisymmetry requirement for fermions, m must be odd, $m = 2k + 1$, with k an integer.

To determine the filling fraction of the system, Laughlin used the following analogy with a two-dimensional plasma:

$$\begin{aligned} Z_{plasma} &= \parallel \Psi_{\frac{1}{m}} \parallel^2 = \int \prod_i d^2 z_i e^{-\beta H_{plasma}} , \\ H_{plasma} &= m \sum_i \mid z_i \mid^2 - m^2 \sum_{i<j} log \mid z_i - z_j \mid^2, \end{aligned} \tag{1.5}$$

where H_{plasma} is the Hamiltonian of a one-component classical plasma of charge m interacting with a logarithmic potential and $\beta = \frac{1}{m}$. With this analogy Laughlin was able to show that the density of this state is constant and to calculate the energy of excitations. He found that the parameter m is related to the electron density, and hence, the filling fraction is $\nu = \frac{1}{m} = \frac{1}{2k+1}$.

From (1.4) it is clear that Laughlin's wavefunction vanishes as $(z_i - z_j)^m$ when any two particles i and j approach each other. Thus there is only a small amplitude probability for the particles to be near each other, and the expectation value Coulomb energy is lowered. The Laughlin wavefunction is actually very close to an exact ground state for several short-range repulsive interactions [2] [1].

Laughlin also proposed the form of the excitations: they are vortex solutions called quasi-holes and given by,

$$\Psi_{q-h} = (\eta; z_1, ..., z_N) = \prod_{i=1}^{N} (\eta - z_i) \prod_{i<j} (z_i - z_j)^{2k+1} e^{-\frac{1}{2\ell^2}\sum |z_i|^2}, \tag{1.6}$$

with η the position of the vortex (Fig.1.2).

The quasi-holes have fractional charges and fractional statistics. If we consider the wave function for two quasi-holes,

$$\Psi_{2q-h}(\eta_1, \eta_2; z_1, ..., z_N) = (\eta_1 - \eta_2)^{\frac{1}{2k+1}} \prod_i (\eta_1 - z_i) \prod_i (\eta_2 - z_i) \Psi_{\frac{1}{2k+2}}, \tag{1.7}$$

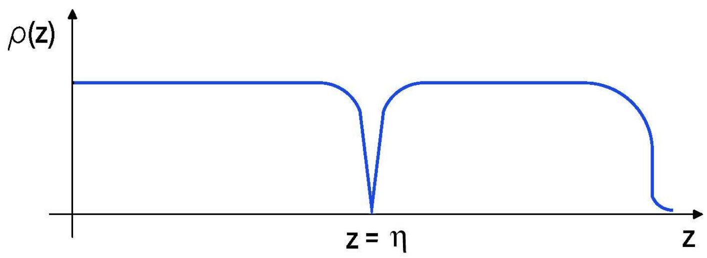

Figure 1.2: Graphical representation of a quasi-hole.

we find a term which depends on their positions $(\eta_1 - \eta_2)$, raised to a fractional power. If we rotate one quasi-hole around the other, like in fig.1.3, we obtain:

$$\Psi(\eta_1 - \eta_2 \longrightarrow e^{i\pi}(\eta_1 - \eta_2)) = e^{i\frac{\pi}{2k+1}}\Psi(\eta_1, \eta_2). \tag{1.8}$$

Therefore, the wave function acquires a non-trivial phase under exchanges of excitations. This implies that quasi-holes have "fractional statistics", $\frac{\theta}{\pi} = \frac{1}{2k+1}$.

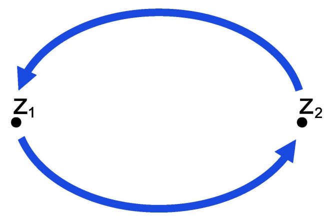

Figure 1.3: Exchange of two quasi-holes at positions z_1, z_2.

To calculate the charge of the quasi-hole one can use the plasma analogy (1.5):

$$\parallel \Psi_{q-h} \parallel^2 = \int \prod d^2 z_i e^{\frac{-1}{m}\left(m\sum_i |z_i|^2 - m^2 \sum_{i<j} log|z_i - z_j|^2 - m\sum_i log|z_i - \eta|\right)}. \tag{1.9}$$

Comparing (1.9) with (1.5) we observe that the particles feel the presence of a charge $\frac{1}{m}$ at the point $z = \eta$, that corresponds to the quasi-hole charge $Q_{q-h} = \frac{1}{m}$.

1.3 The Jain interpretation of the Fractional Quantum Hall Effect

In addition to the primary filling fraction $\nu = \frac{1}{2k+1}$, numerous other fractions have been observed [1]. The more stable ones are the fillings given by $\nu = \frac{n}{2nk+1}$ with n a positive integer. There is a conjecture, given by Jain, to explain the FQHE for these more general filling fractions that is in very good agreement with experiments. Jain explains the FQHE assuming that an even number of quantum fluxes of the external magnetic field are attached to the electrons producing new particles that he called "composite fermions"[1]. He considers fluids with filling fraction $\nu^{-1} = \frac{\Phi}{N\Phi_0} = \frac{N_\Phi}{N} = \frac{1}{n} + 2k$. When $2k$ quantum fluxes are attached to each electron, the same number of fluxes are removed from the external magnetic field, and therefore the filling fraction of the system of composite fermions is given by $\nu^{\star-1} = \frac{N_\Phi - 2kN}{N} = \nu^{-1} - 2k = \frac{1}{n}$ corresponding to an IQHE. As a consequence, the external magnetic field felt by the new particles is:

$$\mathbf{B}^\star = \mathbf{B} - \Delta\mathbf{B} \quad \text{with } \Delta\mathbf{B} = k2\pi\rho_0, \tag{1.10}$$

and ρ_0 the density of electrons. This reduction in the magnetic field is observed experimentally [1]. The incompressibility of the FQHE was explained by Jain as due to the equivalence between the system of electrons with $\nu = \frac{n}{2nk+1}$ and the IQHE of composite fermions with $\nu^\star = n$. So from the point of view of the composite fermions the Laughlin fluid ($\nu = \frac{1}{2k+1}$), consists of an IQHE ($\nu^\star = 1$) of electrons with $2k$ quantum fluxes attached to each electron. This is clear from the Laughlin wave function (1.4) where the factor $\Pi_{i<j}^N (z_i - z_j)^{2k}$ can be shown to describe the $2k$ quantum fluxes and the rest is the IQHE function with filling fraction equal to one. In the general case of $\nu = \frac{n}{2nk+1}$ the wave function proposed by Jain on the basis of his conjectured equivalence is:

$$\Psi_{\frac{n}{2nk+1}}(z_1, ..., z_N) = \prod_{i<j}^N (z_i - z_j)^{2k} \varphi_{\frac{1}{n}}(z_1, ..., z_N), \tag{1.11}$$

with $\prod_{i<j}^N (z_i - z_j)^{2k}$ describing the $2k$ attached quantum fluxes and $\varphi_{\frac{1}{n}}(z_1, ..., z_N)$ being the Landau levels wave function with $\nu = n$ completely filled levels. This wave functions has been confirmed by numerical results of exact diagonalization of the microscopic Hamiltonian with Coulomb and other short-range interactions [10]

Excitations of the Jain states (1.11) also include quasi-holes and quasi-particles, that are local fluctuations in the density. For instance a quasi-particle in the origin

[1] An even number of quantum fluxes does not change the statistics of the new particles.

for the Jain state with $\nu = \frac{n}{2nk+1}$ is given by (fig.1.4),

$$\Psi_{q-p.\frac{n}{2nk+1}}(z_1, ..., z_N) = \prod_{i<j}^{N}(z_i - z_j)^{2k}\varphi'_{\frac{1}{n}}(z_1, ..., z_N), \tag{1.12}$$

where $\varphi'_{\frac{1}{n}}(z_1, ..., z_N)$ corresponds to the wave function with n Landau levels filled and one electron in the first orbital of the $(n+1)$ landau level.

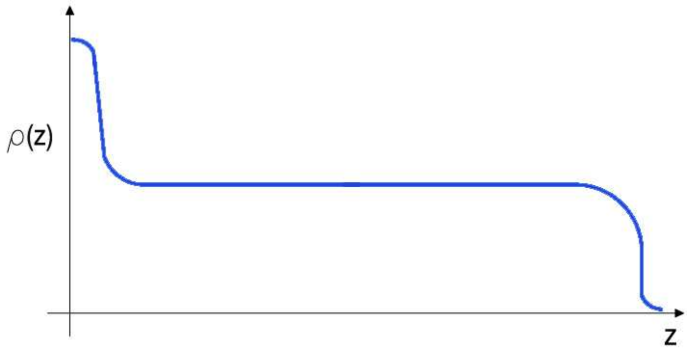

Figure 1.4: Graphical representation of a quasi-particle.

Many experiments confirm the existence of weakly interacting excitations feeling the reduced magnetic field [11][6], i.e. behaving as Jain's composite fermions.

1.4 Fermionic Chern-Simons field theory for the FQHE

Several effective field theories have been proposed to describe the FQHE and to prove the Laughlin and Jain wave functions; in particular the theory of non-relativistic fermions coupled to the Chern-Simons interaction, developed by Fradkin and Lopez [7][8] and others [26][9]. This theory reproduces the Laughlin and Jain results at the mean field level and fluctuations are obtained by perturbative expansion.

In Fradkin and Lopez theory, (2+1) dimensional electrons acquire a flux proportional to their charge through the interaction with the Chern-Simons "statistical" gauge field a_μ. The action is given by,

$$\begin{aligned} S_\theta &= \int d^3z \left(\Psi^*(z) \mid iD_0 + \mu \mid \Psi(z) - \frac{1}{2M} \mid \vec{D}\Psi(z) \mid^2 + \frac{\theta}{2}\epsilon_{\mu\nu\lambda}a^\mu\partial^\nu a^\lambda \right) \\ & -\frac{1}{2}\int d^3z \int d^3z'(\mid \Psi(z) \mid^2 - \bar{\rho})V(\mid \vec{z} - \vec{z}' \mid)(\mid \Psi(z') \mid^2 - \bar{\rho}) \ , \end{aligned} \tag{1.13}$$

where $\bar{\rho}$ is the average particle density, $\Psi(z)$ is a second quantized Fermi field, μ is the chemical potential, θ the Chern-Simons parameter and D_μ is the covariant derivative which couples the fermions to both the external electromagnetic field A_μ and to the statistical gauge field:

$$D_\mu = \partial_\mu + i\frac{e}{c}A_\mu + ia_\mu \ . \tag{1.14}$$

Fradkin and Lopez showed that the theory possesses ground states in the mean field approximation that reproduce the Jain construction. The equation of motion for the Chern-Simons field $a_\mu(z,t)$ are, $F_{\mu\nu} = \epsilon_{\mu\nu\lambda}J^\lambda$, where J^λ is the matter current. In particular, the variation w.r.t. $a_0(z)$ yields the Gauss law for this theory[2]:

$$j_0(\vec{z}) = \theta B(\vec{z}) = \theta\epsilon^{ij}\partial_i a_j(\vec{z}) \ . \tag{1.15}$$

At the quantum level, (1.15) is an operator constraint which selects the physical space of states. For arbitrary values of the Chern-Simons coupling constant θ, the physical states are charge-flux composites: every particle with charge 1 carries a magnetic flux equal to $\frac{1}{\theta}$. The wave functions for these composite particles should exhibit an Aharonov-Bohm effect which leads to fractional statistics [28].

A system of fermions coupled to a Chern-Simons gauge field with coupling constant θ behaves like a system of anyons with statistical angle $\delta = \frac{1}{2\theta}$, measured with respect to the Fermi statistics [27]. If $\theta = \frac{1}{2\pi}\frac{1}{2s}$, where s is an even integer, then $\delta = 2\pi s$ and the system still represents fermions. Fradkin and Lopez presented a detailed proof of the physical equivalence of two theories of particles coupled to a Chern-Simons gauge field with coupling constants θ and θ_0 such that $\frac{1}{\theta_0} = \frac{1}{\theta} + 2\pi \times 2s$, where s is an arbitrary integer. In particular, a theory of interacting fermions is always equivalent to a family of theories of interacting fermions coupled to a Chern-Simons gauge field with coupling constant θ such that $\frac{1}{\theta} = 2\pi \times 2s$.

1.4.1 Semiclassical limit and the Jain ground states

In this section we show that the semiclassical limit of the theory described by the action S_θ of Eq. (1.13), with the choice $\frac{1}{\theta} = 2\pi \times 2s$, yields the same physics as the Jain state.

The partition function of the theory is:

$$Z = \int D\Psi^* D\Psi Da_\mu \, e^{iS_\theta} \ . \tag{1.16}$$

[2] In Chapter 2 and 3, we will see that there is a similar Gauss law in the matrix models that implies an effective flux attached to electrons.

Before treating this path integral in the semiclassical approximation, the fermions are integrated out. At this point the resulting bosonic theory is studied in the saddle point (mean field) approximation.

Using the constraint (1.15), the charge density $j_0(x)$ can be replaced by $\theta B(x)$ in the pair-interaction term of the action, at all points of space-time x. Thus, we can write the pair-interaction term of Eq. (1.13) in the form:

$$\begin{aligned} S_\theta &= \int d^3z \left(\Psi^*(z) \mid iD_0 + \mu \mid \Psi(z) - \frac{1}{2M} \mid \vec{D}\Psi(z) \mid^2 + \frac{\theta}{4}\epsilon_{\mu\nu\lambda} a^\mu F^{\nu\lambda} \right) \\ &\quad -\frac{1}{2}\int d^3z \int d^3z' (\theta B(z) - \bar{\rho}) V(\mid \vec{z} - \vec{z}' \mid)(\theta B(z') - \bar{\rho}) \ . \end{aligned} \tag{1.17}$$

Integrating out the Fermi fields the resulting partition function can be written in terms of an effective action S_{eff} given by:

$$\begin{aligned} &S_{eff} = -iTr\left(log\left[iD_0 + \mu + \frac{1}{2m}\vec{D}^2 \right] \right) + \ S_{CS}(a_\mu - \tilde{A}_\mu) \\ &-\frac{1}{2}\int d^3z \int d^3z' \left[\theta(B(z) - \tilde{B}(z)) - \bar{\rho} \right]) V(z - z') \left[\theta(B(z') - \tilde{B}(z)) - \bar{\rho} \right] , \end{aligned} \tag{1.18}$$

where D_0 and $\vec{D}$ are the covariant derivatives of Eq. (1.13) and S_{CS} is the Chern-Simons action. The field $\tilde{A}_\mu$ represents a small fluctuating electromagnetic field, with vanishing average everywhere, which will be used to probe the system. The electromagnetic currents will be calculated as first derivatives of Z with respect to $\widetilde{A}_\mu$.

The Saddle Point Equations, or classical equations of motion are:

$$\frac{\delta S_{eff}}{\delta a_\mu} = 0. \tag{1.19}$$

By varying S_{eff} with respect to $a_\mu(z)$ we find:

$$\begin{aligned} \langle j_0(z) \rangle_F &= -\theta[\langle B(z) \rangle - \langle \tilde{B}(z) \rangle] \ , \\ \langle j_k(z) \rangle_F &= \theta_{\epsilon_{kl}}(\langle E_l(z) \rangle) - \langle \tilde{E}_l \rangle) \\ &\quad -\theta\epsilon_{kl}\partial_l^{(z)} \int d^3z' V(\mid \vec{z} - \vec{z}' \mid)(\theta(\langle B - \tilde{B} \rangle(z') - \bar{\rho}) \ , \end{aligned} \tag{1.20}$$

where $\langle j_\mu(z) \rangle_F$ represents the expectation value of the charge and current of the equivalent fermion problem.

These equations have many possible solutions which include uniform (liquid) states, Wigner crystals, and non-uniform states with vortex-like configurations. We will consider solutions with uniform particle density $\langle j_\mu(z) \rangle_F = \bar{\rho}$, i.e. the liquid phase solution, and no currents in the ground state. If the external electromagnetic fluctuation is assumed to have zero average, the only possible solutions of this type are:

$$\begin{aligned} \langle B \rangle &= \frac{\bar{\rho}}{\theta} \,, \\ \langle E \rangle &= 0 \,. \end{aligned} \tag{1.21}$$

Eq. (1.21) shows that, for a translational invariant ground state, the effect of the statistical gauge fields at the level of the saddle-point approximation is to change the effective flux experienced by the fermions. The total effective field is $B_{eff} = B + \langle B \rangle = B - \frac{\bar{\rho}}{\theta}$, in agreement with Jain's argument (1.10).

The uniform effective magnetic field B_{eff} which solves equation(1.21), defines a new set of effective Landau levels. Each level has a degeneracy equal to the total number of effective flux quanta N_{eff} and the separation between levels is the effective cyclotron frequency $\omega_{eff} = \frac{e|B_{eff}|}{Mc}$. Similarly, there is an effective cyclotron radius ℓ^{eff}. It is easy to see that the uniform saddle-point state, which satisfies (1.21), has a gap only if the effective field B_{eff} experienced by the N fermions is such that the fermions fill exactly an integer number p of the effective Landau levels. This is the Jain's point of view: the FQHE is an IQHE of a system of electrons dressed by an even number of flux quanta. However, this condition cannot be met for arbitrary values of the filling fraction ν at fixed field (or at fixed density). Let N_{Φ}^{eff} denote the effective number of flux quanta piercing the surface after screening. It is given by:

$$\pm 2\pi N_{\Phi}^{eff} = 2\pi N_{\Phi} - \frac{\bar{\rho}}{\theta} L^2, \tag{1.22}$$

where the sign stands for the case of an effective field parallel or antiparallel to B. Thus, the effective cyclotron frequency ω_c^{eff} is reduced from its free electron value of $\frac{eB}{Mc}$ down to $\omega_c^{eff} = \omega_c(1 - \frac{\nu}{2\pi\theta})$. The effective cyclotron radius is given by $\ell_{eff} = (\frac{\ell}{1-\frac{\nu}{2\pi\theta}})$ which is larger than the non-interacting value. Therefore, even though the bare Landau levels may be separated by a sizable Landau gap $\hbar\omega c$, the effective Landau levels have the smaller gap $\hbar\omega_c^{eff}$.

Substituting the value of θ in (1.22) we obtain,

$$\pm 2\pi N_{\Phi}^{eff} = 2\pi N_{\Phi} - 2\pi 2sN, \tag{1.23}$$

where 2s is an even integer. The spectrum supported by this state has an energy gap if the N fermions fill exactly p of the Landau levels created by the effective field B_{eff}. In other words, the effective filling fraction is $\nu_{eff} = \frac{N}{N_{eff}}$. Using (1.23), the allowed filling fractions and the gap are given by:

$$\nu = \frac{p}{2sp \pm 1} \,, \qquad \hbar\omega_c^{eff} = \frac{\hbar\omega_c}{2sp \pm 1}, \tag{1.24}$$

that are the filling fractions and the gap of the Jain fluid.

Fluctuations around the Saddle point give excitations of the ground states corresponding to the quasi-holes and quasi-particles of the Laughlin and Jain theory. Unfortunately, the gap for quasi-particles has not the desired dependence ($O(\sqrt{\mathbf{B}})$).

Now we begin the study of matrix models apply to the QHE. The next chapter is devoted to the Chern-Simons matrix model, proposed by Susskind in 2001 as another effective theory for the FQHE.

Chapter 2

Chern-Simons Matrix model

2.1 Semiclassical incompressible fluid and noncommutative Chern-Simons theory

In an interesting paper [14] Susskind conjectured that the noncommutative Chern-Simons field theory in two dimensions could describe the Laughlin incompressible fluids in the QHE [2][1]. This conjecture was inspired by the fact that the semiclassical limit of this theory describes incompressible fluids in high magnetic field with Laughlin's filling fractions ($\nu = \frac{1}{k+1}$, k an integer) and its quasi-hole excitations[14][29].

Susskind started from N two dimensional, non-interacting spinless electrons in a high magnetic field $\mathbf{B}$, such that the kinetic energy is frozen (lowest Landau level [30]),

$$L = \frac{e\mathbf{B}}{2} \sum_{\alpha=1}^{N} \epsilon^{ab} \dot{X}^a_\alpha(t) X^b_\alpha(t) \ , \tag{2.1}$$

where $X^a_\alpha(t)$ are the coordinates of the electrons, with $a = 1, 2$ and $\alpha = 1, ..., N$.

If the system behaves like a fluid, we can pass to the continuum description, as follows:

$$X_\alpha(t) \to \vec{X}(\vec{y}, t) \ , \tag{2.2}$$

where $\vec{y}$ is the co-moving system, i.e. the coordinates fixed to the fluid [29][31]. In terms of (2.2) we can write (2.1) as,

$$L = \frac{e\mathbf{B}}{2} \int \ d^2y \ \rho_0 \varepsilon^{ab} \dot{X}_a(y,t) X_b(y,t) \ , \tag{2.3}$$

with ρ_0 the density in the $\vec{y}$ system. The Lagrangian (2.3) has a symmetry of diffeomorphism that preserves the area (w_∞ symmetry)[25][16]. This symmetry implies the conservation of the Jacobian of the system:

$$\frac{\partial J}{\partial t} = 0 \quad \text{with} \quad J = \frac{1}{2}\epsilon_{ab}\left\{X^a, X^b\right\}, \tag{2.4}$$

where $\{,\}$ are Poisson brackets. Without loss of generality, we can consider as initial conditions,

$$\vec{X}(\vec{y}, t=0) = \vec{y}, \tag{2.5}$$

that together with (2.4) imply uniform density of the fluid in the $\vec{X}$ coordinates:

$$\frac{\partial J}{\partial t} = 0 \rightarrow J(t) = J(t=0) = 1 \ , \qquad \rho(y,t) = \mid J \mid \rho_0 = \rho_0 \ . \tag{2.6}$$

Susskind rewrites (2.3) introducing the conserved charge (2.6) explicitly, by means of the Lagrange multiplier A_0,

$$L = \frac{e\mathbf{B}}{2}\int d^2y\rho_0\left[\epsilon^{ab}\left(\dot{X}_a - \frac{1}{2\pi\rho_0}\{X_a, A_0\}\right)X_b + \theta A_0\right] \ , \tag{2.7}$$

where the new parameter θ is related to the density as follows:

$$\theta = \frac{1}{2\pi\rho_0} \ . \tag{2.8}$$

Considering perturbations of $X_a(y,t)$ by means of new fields $A^b(y,t), b = 1,2$, i.e.

$$X^a(y,t) = y^a + \theta\epsilon^{ab}A^b(y,t), \tag{2.9}$$

the Lagrangian (2.7) can be written as:

$$L = \frac{1}{4\pi\nu}\int d^2y\epsilon^{\mu\nu\rho}(\partial_\mu A_\nu A_\rho + \frac{\theta}{3}\left\{A_\mu, A_\nu\right\}A_\rho) \ , \tag{2.10}$$

with $\nu = \frac{1}{e\mathbf{B}\theta}$ the filling fraction. The Lagrangian (2.10) coincides for small θ with that of the noncommutative Chern-Simons theory given by [32]-[42]:

$$L_{NCCS} = \frac{1}{4\pi\nu}\int \ d^2y \ \varepsilon^{\mu\nu\rho}(\partial_\mu A_\nu * A_\rho - \frac{2i}{3}A_\mu * A_\nu * A_\rho), \tag{2.11}$$

where $*$ is the Moyal product (appendix A.1) of noncommutative geometry, defined as,

$$(g * f)(x) = e^{i\frac{\theta}{2}\epsilon_{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}}f(x_1)g(x_2)\mid_{x_1=x_2=x} \ . \tag{2.12}$$

Susskind conjectured that the Laughlin electrons could be described by the full noncommutative theory (2.11), beyond the fluid approximation, so as to take into account the granularity of the particles. Actually, in noncommutative spaces there is a minimal unit of area, θ, which can be identified with the size of the electrons in agreement with (2.8).

Due to the fact that every noncommutative theory is equivalent to a matrix theory with matrices of infinite order, Susskind conjecture implies the corresponding matrix effective theory [43] for the FQHE. The matrix theory equivalent to (2.11) is (see appendix A.2):

$$S_{Susskind} = \int dt \, \frac{B}{2} \mathrm{Tr} \left[\, \varepsilon_{ij} \; X_i(t) \; D_t \; X_j(t) + \; 2\theta \; A_0(t) \right], \tag{2.13}$$

where now $X_1(t)$, $X_2(t)$ and $A_0(t)$ are $N \times N$ matrices, with $N = \infty$, and the covariant derivative is $D_t X_j = \dot{X}_j - i \left[A_0, X_j \right]$.

Another route to obtain (2.13) which emphasizes the discrete particle aspects of the fluid is given by a matrix regularization proposed by Goldstone and Hoppe [44]. Considering (2.3) as a membrane in the y's coordinates, we can use the mapping given by Goldstone and Hoppe (appendix A.3) to pass from (2.3) to (2.13), in which functions on the membrane surface are mapped to matrices of order N. The equivalence is given by the dictionary (appendix A.3):

$$X_a(\vec{y}, t) \leftrightarrow \frac{2}{N} X_a(t) \; , \qquad \{X_a(\vec{y}, t), A_0(y, t)\} \leftrightarrow \frac{-iN}{2} \left[X_a(t), A_0(t) \right] \; ,$$

$$\frac{1}{4\pi} \int d^2 y \leftrightarrow \frac{1}{N} Tr \; , \tag{2.14}$$

where $X_a(t)$ are $N \times N$ Hermitian matrices. Applying (2.14) to (2.3) we obtain the matrix theory given by (2.13).

The Gauss law constraint of the theory, obtained by the variation of (2.13) with respect to A_0, reads:

$$[X_1, X_2] = i\theta \; \mathbf{I}, \tag{2.15}$$

where $\mathbf{I}$ is the identity matrix. In matrix theories, the noncommutative parameter appears as a constant "charge background" that requires the physical states to possess non-trivial gauge transformations.

The trace of (2.15) is satisfied only in terms of infinite-dimensional matrices X_a (and thus A_0). This implies infinite degrees of freedom and thus Susskind's theory applies to an infinite system. Instead, the FQHE is a system with a boundary and a finite number of particles. Polychronakos modified the Susskind's theory to introduce these features [15].

2.2 The Chern-Simons finite matrix model

We now describe quantum Hall states of finite extent consisting of N electrons. The matrices X_a can be represented by $N \times N$ matrices. In the previous section we saw that Susskind theory is inconsistent for finite matrices, and a modified action must be written. The action proposed by Polychronakos is:

$$S = \int dt \frac{\mathbf{B}}{2} Tr \left\{ \epsilon_{ab}(\dot{X}_a + i[A_0, X_a])X_b + 2\theta A_0 - \omega X_a^2 \right\} + \int dt \psi^\dagger (i\dot{\psi} - A_0\psi). \tag{2.16}$$

He adds two new terms to Susskind's action (2.13). The first term is a harmonic oscillator potential for the matrices that confine the eigenvalues, i.e. keep the particles localized in the plane. The second term is a complex N-vector that transforms in the fundamental of the gauge group $U(N)$,

$$\psi \to U\psi \ . \tag{2.17}$$

The Gauss law is now given by:

$$G \equiv \ -i \ \mathbf{B}[X_1, X_2] + \psi\psi^\dagger - \mathbf{B}\theta\mathbf{I} = 0. \tag{2.18}$$

Observe that the trace of (2.18) implies,

$$\psi^\dagger\psi = N\mathbf{B}\theta, \tag{2.19}$$

that can be realized with finite dimensional matrices. The equation of motion for ψ in the $A_0 = 0$ gauge implies $\dot{\psi} = 0$: it is an auxiliary field with trivial dynamics. Thus, we can take the constant value of $\psi \equiv \psi(t = 0)$ to be $\psi = \sqrt{N\mathbf{B}\theta} \mid v\rangle$, with $\mid v\rangle$ a vector of unit length.

The Chern-Simons theory (2.16) has the $U(N)$ symmetry:

$$\begin{aligned} X_a &\to U X_a U^\dagger \ , \qquad \psi \to U\psi, \\ A_0 &\to U A_0 U^\dagger - iU\frac{dU^\dagger}{dt} \ . \end{aligned} \tag{2.20}$$

Under a gauge transformation (2.20), the action (2.16) change by the winding number of the group element U,

$$S \to S - i\mathbf{B}\theta \int dt Tr \left[U^\dagger \dot{U} \right], \tag{2.21}$$

and gauge invariance is satisfied if $\mathbf{B}\theta = k$ is an integer [45].

Polychronakos showed that in the gauge with X_1 diagonal, one can solve the Gauss law at the classical level, obtaining:

$$\begin{aligned} (X_1)_{mn} &= x_n\delta_{mn}, \quad (X_2)_{mn} = y_n\delta_{mn} - \frac{i\theta}{x_m - x_n}(1-\delta_{mn}), \\ \psi_i &= \sqrt{k}, \quad i = 1, ..., N \ . \end{aligned} \tag{2.22}$$

In this gauge, (2.16) reduces to the Calogero model action. Substitution of (2.22) in the Chern-Simons Hamiltonian, $H = \frac{\omega\mathbf{B}}{2}Tr(X_a^2)$, yields:

$$H = \sum_{n=1}^{N}(\frac{\omega}{2\mathbf{B}}p_n^2 + \frac{\mathbf{B}\omega}{2}x_n^2) + \sum_{n\neq m}\frac{\nu^{-2}}{(x_n - x_n)^2}, \tag{2.23}$$

where $p_n = -\mathbf{B}y_n$. Since y_n and p_n are canonically conjugate, they can be interpreted as the phase space coordinates of a system of N particles; identifying $\frac{\mathbf{B}}{\omega}$ with the mass of the particles, we obtain the Calogero model of N particles on the line with two-body repulsive potential parameterized by the coupling constant ν^{-1} taking integer values. The Calogero model is related to the Fractional Quantum Hall effect: It is an integrable model, and its space of states is isomorphic to the excitations of the Laughlin state at filling fractions $\nu = \frac{1}{k+1}$ [48]-[53],[2][54][55]. However the Hilbert spaces of the two problems are different, because the one-dimensional norm of the Calogero model, is different from the two-dimensional measure of the lowest Landau level [55][56][47].

2.2.1 Classical solutions

In this section we study the classical solutions of the Chern-Simons matrix model. We work in holomorphic coordinates $X = X_1 + iX_2$ and $\bar{X} = X_1 - iX_2$, with the bar denoting the Hermitian conjugate of classical matrices. In terms of these variables the Hamiltonian of the theory can be written as:

$$H = \frac{\omega\mathbf{B}}{2}TrX\bar{X} + Tr\ \Lambda(-\frac{\mathbf{B}}{2}[\bar{X}, X] + \psi\psi^\dagger - B\theta), \tag{2.24}$$

where we introduced the Gauss law constraint by means of the Lagrange multiplier Λ.

The equations of motion are given by:

$$\begin{aligned} \dot{X}_{ba} &= \frac{\partial H}{\partial \Pi_{ba}} = \omega X_{ba} + \frac{2}{\mathbf{B}}[\Lambda, X]_{ba}, \\ \dot{\Pi}_{ba} &= -\frac{\partial H}{\partial X_{ab}} = \frac{\mathbf{B}\omega}{2}\bar{X}_{ba} + [\bar{X}, \Lambda]_{ba}\ , \end{aligned} \tag{2.25}$$

with the canonical conjugate momentum $\Pi_{ab} = \frac{B}{2}\bar{X}_{ba}$. The minimum of energy must satisfy the Gauss law (2.18), and the equations of motion (2.25),

$$\begin{aligned} G \equiv & -\frac{\mathbf{B}}{2}[\bar{X}, X] + \psi\psi^{\dagger} - B\theta = 0, \\ [\Lambda, \bar{X}]_{ba} = & \frac{\omega\mathbf{B}}{2}\bar{X}_{ba}. \end{aligned} \tag{2.26}$$

These are the commutation relations for a (truncated) quantum harmonic oscillator, with Λ playing the role of the Hamiltonian. The solutions of (2.26) are [15]:

$$\begin{aligned} \bar{X} &= \sqrt{2\theta}\sum_{n=0}^{N-1}\sqrt{n}\mid n >< n-1\mid \ , \qquad \Lambda = \omega\sum_{n=0}^{N-1} n \mid n >< n \mid \ , \\ \psi &= \sqrt{kN} \ \mid N-1> \ . \end{aligned} \tag{2.27}$$

For instance, for $N = 3$ the matrices $\bar{X}$ and Λ have the form,

$$\bar{X} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{\theta} & 0 & 0 \\ 0 & \sqrt{2\theta} & 0 \end{pmatrix} \ , \quad \Lambda = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \ . \tag{2.28}$$

Solution (2.27) corresponds to a circular quantum Hall droplet of radius $\sqrt{2N\theta}$ [15]. The radius-squared matrix coordinate R^2 is diagonal, and given by [1] (setting $\mathbf{B} = \mathbf{2}$):

$$R^2 \ = \overline{X}X \ = \mathrm{diag}\,(0, k, 2k, \ldots, (N-1)k) \ . \tag{2.29}$$

From the distribution of the eigenvalues in (2.29) it is clear that this solution implies a constant density. A good definition of the density in matrix models is given in terms of the gauge invariant eigenvalues of R^2,

$$\rho(r^2) = \sum_{i=0}^{N-1}\delta(r^2 - \sigma_i), \qquad \sigma_i \in Spec(R^2). \tag{2.30}$$

For semiclassical fluids, this becomes a piecewise continuous function, in the limit $N \to \infty$, that describes two-dimensional rotation-invariant distributions ($\rho(r) = \rho(r^2)/\pi$). A discrete approximation suitable for the continuum limit is [22]:

$$\rho(r^2) = \sum_i \frac{n_i}{\sigma_{i+1} - \sigma_i}\delta_{r^2,\sigma_i}, \tag{2.31}$$

involving the Kronecker delta and the ordered set of distinct eigenvalues, $\sigma_i < \sigma_j$, $i < j$, with multiplicities n_i.

In fig.2.1, we use the definition (2.31), and plot the density of the state (2.27) considering $k = 4$ and $N = 400$. In the large N limit, the filling fraction is the

[1] In this equation, we resolve the ordinary ambiguity of R^2 by matching it eigenvalues to those of the angular momentum [15][13][22].

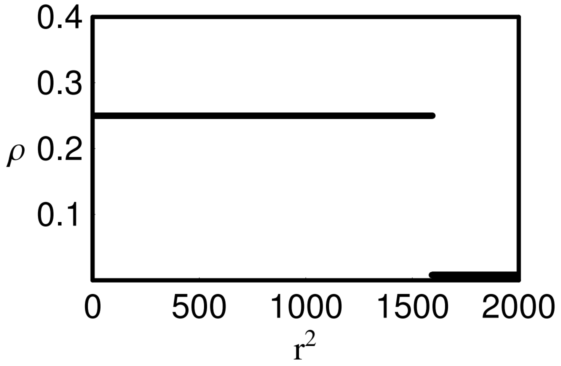

Figure 2.1: Plot of the density of the ground state (2.27) for $\nu = \frac{1}{4}$ and $N = 400$

Laughlin value; according to the identification of the θ parameter (2.8),

$$\nu = \frac{2\pi\rho_0}{e\mathbf{B}} = \frac{1}{k} \ , \quad \rho_0 = \frac{1}{2\pi\theta} \ . \tag{2.32}$$

Polychronakos theory does not contain quasi-particle excitations, only quasi-hole are presents [18]. For example, a quasi-hole in the origin is given by,

$$\bar{X} = \sqrt{2\theta}\left(\sqrt{q} \mid 0\rangle\langle N-1 \mid + \sum_{n=1}^{N-1} \sqrt{n+q} \mid n\rangle\langle n-1 \mid\right), \quad q > 0, \tag{2.33}$$

where q is proportional to the charge of the quasi-hole. In matrix form for $N = 3$, this reads:

$$\bar{X} = \begin{pmatrix} 0 & 0 & \sqrt{q} \\ \sqrt{1+q} & 0 & 0 \\ 0 & \sqrt{2+q} & 0 \end{pmatrix} . \tag{2.34}$$

Fig.2.2 is a plot of the density of a quasi-hole in the origin for $k = 4$, $N = 400$ and $q = 30$.

2.2.2 Covariant quantization

Before quantizing the Chern-Simons matrix model, we express (2.16) in terms of holomorphic matrices, in the $A_0 = 0$ gauge:

$$\begin{aligned} S &= \int dt \left(\frac{\mathbf{B}}{2i}\sum_{nm}\dot{X}_{nm}\bar{X}_{mn} - i\sum_{n}\dot{\psi}_n\psi_n^\dagger - \frac{\mathbf{B}\omega}{2}\sum_{nm}\bar{X}_{nm}X_{mn}\right) , \\ G &= -\frac{\mathbf{B}}{2}[\bar{X}, X] + \psi\psi^\dagger - B\theta = 0. \end{aligned} \tag{2.35}$$

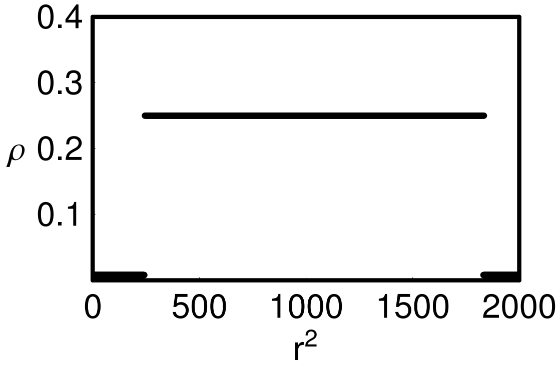

Figure 2.2: Plot of the density for the quasi-hole state (2.33) for $k = 4$, $q = 60$ and $N = 400$.

The form of the action (2.35) is that of $(N^2 + N)$ particles in the lowest Landau level with coordinates X_{nm} and ψ_n.

The canonical commutation relations are given by [17]:

$$\begin{aligned} [[\bar{X_{ij}}, X_{kl}]] &= \frac{2}{\mathbf{B}}\delta_{jk}\delta_{il} \ , \\ [[\bar{\psi_i}, \psi_j]] &= \delta_{ij} \ . \end{aligned} \tag{2.36}$$

The double brackets are used to denote the quantum mechanical commutators between matrix elements. We use the standar polarization in quantum mechanics, i.e. the canonical conjugate momentum becomes:

$$\bar{X}_{nm} \to \frac{2}{\mathbf{B}}\frac{\partial}{\partial X_{mn}}, \qquad \bar{\psi}_n \to \frac{\partial}{\partial \psi_n} \ . \tag{2.37}$$

In terms of (2.37) the (normal ordered) Gauss law constraint applied to a general state $\Psi(X, \psi)$ reads:

$$G_{ij}\Psi(X,\psi) = \left[\sum_l \left(X_{il}\frac{\partial}{\partial X_{jl}} - X_{lj}\frac{\partial}{\partial X_{lj}}\right) - k\delta_{ij} + \psi_i \frac{\partial}{\partial \psi_j}\right]\Psi(X,\psi) = 0 \ ; \tag{2.38}$$

the coherent state $\Psi(X, \psi) = e^{-Tr(\bar{X}X)/2 - \psi^\dagger\psi/2}\Phi(X, \psi)$ with $\Phi(X, \psi)$ a polynomial in the matrix X and the vector ψ [57]. Expression (2.38) implies that the physical states are $U(N)$ singlets.

The total angular momentum applied to the wave function is given by,

$$\sum_{ab}(X_{ab}\frac{\partial}{\partial X_{ab}})\Phi(X,\psi) = \mathcal{J}\Phi(X,\psi), \tag{2.39}$$

where $\mathcal{J}$ is the total number of X matrices occurring in $\Phi(X, \psi)$.

Following the reference [20], we quantize the theory in the gauge in which the matrix X is diagonal, i.e.:

$$\begin{aligned} X &= V^{-1}\Lambda V, \qquad \Lambda = diag(\lambda_1, ..., \lambda_N), \\ \bar{X} &= V^{-1}\tilde{\Lambda} V, \qquad \psi = V^{-1}\phi, \qquad \psi = \tilde{\phi} V \ . \end{aligned} \tag{2.40}$$

Actually, a complex matrix can be diagonalized by a GL(N,C) transformation, up to the zero-measure set of matrices with degenerate eigenvalues. Invariance of the commutation relations (2.36) implies:

$$\begin{aligned} \frac{\partial}{\partial X_{ij}} &= V_{ni}V^{-1}_{jm}\frac{\partial}{\partial \Lambda_{nm}}, \qquad \frac{\partial}{\partial \Lambda_{nm}} = \frac{\partial}{\partial \lambda_n}\delta_{nm} + \frac{1-\delta_{nm}}{\lambda_n - \lambda_m}\left(\frac{\partial}{\partial \nu_{nm}} + \phi_m \frac{\partial}{\partial \phi_n}\right), \\ \frac{\partial}{\partial \psi_j} &= V_{nj}\frac{\partial}{\partial \phi_n} \ , \end{aligned} \tag{2.41}$$

where $\frac{\partial}{\nu_{mn}}$ satisfy $\left[\left[\frac{\partial}{\nu_{mn}}, d\nu_{ij}\right]\right] = \delta_{mi}\delta_{nj}$, with $d\nu = dVV^{-1}$. By substitution (2.41) in the Gauss law constraint (2.38), one finds:

$$\begin{aligned} G_{ij} &= V^{-1}_{im}V_{nj}G^V_{nm}, \qquad G^V_{nm}\Phi(\Lambda, V, \phi) = 0, \\ G^V_{nm} &= \begin{cases} -\frac{\partial}{\partial \nu_{nm}} & n \neq m \\ \phi_n \frac{\partial}{\partial \phi_n} - k & n = m \end{cases} . \end{aligned} \tag{2.42}$$

We have that:

- physical states depend on V only through quantities like the determinant of V,
- the dependence in ϕ doesn't affect the physics, because all physical states contain the same homogeneous polynomial of degree k given by $\prod_{n=1}^{N}(\phi_n)^k$.

In conclusion, (2.42) reduces the degrees of freedom of the theory to N complex eigenvalues λ_n that can be interpreted as coordinates of electrons in the lowest Landau level, as will be clear in the next section.

The general solution (without gauge fixing), of the Gauss law constraint has been found in Refs. [19]. As said before, the physical states are singlets of $U(N)$ made by X_{ab} and ψ_a and with the number of vectors ψ's equal to Nk. A basis is given by:

$$\begin{aligned} \Phi(X, \phi) &= \Phi_{\{n^1_1, ..., n^1_N\}} ... \Phi_{\{n^k_1, ..., n^k_N\}} \qquad \text{with} \\ \Phi_{\{n^j_1, ..., n^j_N\}} &= \epsilon^{i_1 ... i_N}(X^{n^j_1}\psi)_{i_1} ... (X^{n^j_N}\psi)_{i_N}, \quad 0 \leq n^j_1 < n^j_2 < ... < n^j_N. \end{aligned} \tag{2.43}$$

and it is easy to see that the ground state of the theory is [20]:

$$\Phi_{k-gs} = \left[\epsilon^{i_1 \dots i_N} \psi_{i_1} (X\psi)_{i_2} \dots (X^{N-1}\psi)_{i_N}\right]^k , \qquad (2.44)$$

corresponding to the lowest value of the angular momentum (2.39).

In Ref.[19] is presented an equivalent basis, in which the states are factorized into the ground state (2.44) and the "bosonic" powers of X, $\mathrm{Tr}\,(X^{m_i})$, with positive integers $\{m_1, \dots, m_k\}$ unrestricted, i.e.

$$\Phi\,(X, \psi) \;=\; \sum_{\{m_k\}} \mathrm{Tr}\,(X^{m_1}) \cdots \mathrm{Tr}\,(X^{m_k})\;\; \Phi_{k-gs}\;. \qquad (2.45)$$

Let us now perform the change of variables given by (2.40) on the ground state (2.44):

$$\begin{aligned} \Psi_{k-gs}(\Lambda, V, \psi) &= \left[\epsilon^{i_1 \dots i_N} (V^{-1}\phi)_{i_1} (V^{-1}\Lambda\phi)_{i_2} \dots (V^{-1}\Lambda^{N-1}\phi)_{i_N}\right]^k \\ &= \left[(detV)^{-1} det(\lambda_j^{i-1}\phi_j)\right]^k \\ &= (detV)^{-k} \prod_{1 \le n \le m \le N} (\lambda_n - \lambda_m)^k \left(\prod_i \phi_i\right)^k . \end{aligned} \qquad (2.46)$$

We obtain the Laughlin wave function as ground state of the Chern-Simons theory, with the coordinates of the electrons identified with the eigenvalues of X. The dependence on ϕ and V is the same for all the physical states as predicted by (2.42). The filling fraction can be computed from $\mathcal{J} = \frac{N(N-1)}{2\nu}$ with $\nu = \frac{1}{k}$, as in the previous classical solution (2.32). This is a very important result of the Chern-Simons matrix theory; that of reproducing the Laughlin wave function from gauge invariance of the states in presence of the non-trivial background θ, i.e. k.

Now, let us discuss the excitations of the ground state (2.44). Multiplying the wave function by polynomials of $\mathrm{Tr}(X^r)$ as in (2.45), we find states with $\Delta\mathcal{J} = r$. These are the basis of holomorphic excitations over the Laughlin state. For $r = O(1)$, their energy given by the boundary potential, $\Delta E = \omega\Delta\mathcal{J} = O(r\,\mathbf{B}/N)$ is very small: they are the degenerate edge excitations of the droplet of fluid described by conformal field theories [25][59][60][61].

More interesting hereafter are the analogues of the quasi-hole and quasi-particle excitations of the Laughlin state, that are gapful localized density deformations. The quasi-hole is realized by moving one electron from the interior of the Fermi surface to the edge, causing $\Delta\mathcal{J} = O(N)$ and thus a finite gap $\Delta E = O(\mathbf{B})$. Its realization in the matrix theory is for example given by the state $\Phi_{\{n_1,\dots,n_N\}}$ in Eq.(2.43), with $\{n_1, n_2, \cdots, n_M\} = \{1, 2, \cdots, N\}$. On the other hand, the quasi-particle excitation cannot be realized in the Chern-Simons matrix model [13][15][22].

2.2.3 Path integral quantization

In this section we study the path integral quantization and obtain the norm of quantum states [20]. The path integral is given by:

$$\begin{aligned} \langle f \mid i \rangle &= \int \mathfrak{D}X(t)\mathfrak{D}\bar{X}(t)\mathfrak{D}\psi(t)\mathfrak{D}\psi^\dagger(t) \\ & e^{\int dt\left(\frac{\mathbf{B}}{2}Tr(\bar{X}\dot{X})+\psi^\dagger\dot{\psi}-iH\right)}\prod_t \delta(G(t))FP \ , \end{aligned} \tag{2.47}$$

where G is the Gauss-law condition (2.35) and FP is the Faddeev-Popov term for the gauge fixing. We will work in the gauge (2.40). Thus, we express (2.47) in terms of the variables ϕ, V and Λ. The measure of integration can be written as,

$$\mathfrak{D}X = \prod_i \mathfrak{D}X_{ii}\prod_{i\neq j}\mathfrak{D}X_{ij}. \tag{2.48}$$

We change variables from the matrix elements $X_{ij},\ i,j=1,...,N$ to the eigenvalues $\lambda_i, i=1,...,N$ and elements of the matrix V. To find the Jacobian of the change of coordinates we express the differential of volume $dX=\prod_{ij}dX_{ij}$ in the new variables,

$$\begin{aligned} dX_{ij} &= d\left(V^{-1}\Lambda V\right)_{ij} = dV^{-1}_{ik}\Lambda_k V_{kj} + V^{-1}_{ik}d\Lambda_k V_{kj} + V^{-1}_{ik}\Lambda_k dV_{kj} \\ &= V^{-1}_{ik}\left(d\lambda_k\delta_{kl} + d\nu_{kl}(\lambda_k-\lambda_l)\right)V_{kj} \ , \end{aligned} \tag{2.49}$$

where $d\nu = dVV^{-1}$. Expression (2.49) leads to the following result for the measure (2.48):

$$\mathfrak{D}X = \prod_{i\neq j}(\lambda_i-\lambda_j)\prod_i \mathfrak{D}\lambda_i\prod_{i,j}\mathfrak{D}\nu_{ij} = (-1)^{\frac{1}{2}N(N-1)}\Delta(\lambda)^2\prod\mathfrak{D}\lambda\prod\mathfrak{D}\nu \ , \tag{2.50}$$

with $\Delta(\lambda)=\prod_{i<j}(\lambda_i-\lambda_j)$ the Vandermonde determinant. The constant $(-1)^{\frac{1}{2}N(N-1)}$ can be discarded. Using (2.50) the Faddeev-Popov term can be written as:

$$1 = \int \mathfrak{D}\nu\mathfrak{D}\Lambda\delta(VXV^{-1}-\Lambda)\Delta(\lambda)^2. \tag{2.51}$$

The Gauss law condition in the new variables implies,

$$\begin{aligned} \delta(G) &= \prod_{i,j=1}^N \delta\left(\frac{\mathbf{B}}{2}(\lambda_i-\lambda_j)\tilde{\Lambda}_{ij} - k\delta_{ij} + \phi_i\tilde{\phi}_j\right) \\ &= \Delta(\lambda)^{-2}\prod_{i\neq j}\delta\left(\tilde{\Lambda}_{ij} + \frac{2}{\mathbf{B}}\frac{\phi_i\tilde{\phi}_j}{\lambda_i-\lambda_j}\right)\prod_i \delta(\phi_i\tilde{\phi}_i - k). \end{aligned} \tag{2.52}$$

As consequence $\tilde{\Lambda}_{ij} = \tilde{\lambda}\delta_{ij} - \frac{2}{\mathbf{B}}\frac{1-\delta_{ij}}{\lambda_i-\lambda_j}\phi_i\tilde{\phi}_j$, where the unconstrained diagonal elements, $\tilde{\lambda}_i$, become the canonical conjugate variables of the eigenvalues, λ_i.

Actually, in the new coordinates and using the constraint (2.52), the action becomes:

$$
\begin{aligned}
\frac{\mathbf{B}}{2}Tr(\bar{X}\dot{X})+\psi^{\dagger}\dot{\psi} &= \frac{\mathbf{B}}{2}Tr\left(\tilde{\Lambda}\dot{\Lambda}-\left[\Lambda,\tilde{\Lambda}\right]\dot{V}V^{-1}\right)+\tilde{\phi}\dot{\phi}-\tilde{\phi}\dot{V}V^{-1}\phi \\
&= \sum_n\left(\frac{\mathbf{B}}{2}\tilde{\lambda}_n\dot{\lambda}_n+\tilde{\phi}_n\dot{\phi}_n\right)-kTr(V^{-1}\dot{V}), \qquad (2.53)
\end{aligned}
$$

and finally in the gauge (2.40) the path integral is given by (setting $\mathbf{B}=2$),

$$
\begin{aligned}
\langle f \mid i\rangle &= \int\prod_n \mathfrak{D}\tilde{\lambda_n}(t)\mathfrak{D}\lambda_n(t)\mathfrak{D}\tilde{\phi_n}(t)\mathfrak{D}\phi_n(t)\prod_{i,t}\delta\left(\tilde{\phi}_i(t)\phi_i(t)-k\right) \\
& \quad exp\left(\int dt\left[\sum_n\tilde{\lambda}_n\dot{\lambda}_n+\tilde{\phi}_n\dot{\phi}_n-H\left(\Lambda,\tilde{\Lambda}\right)\right]\right)\Bigg|_{\tilde{\Lambda}_{ij}=\tilde{\lambda}_{ij}\delta_{ij}-(1-\delta_{ij})\frac{\phi_i\tilde{\phi}_j}{\lambda_i-\lambda_j}} .
\end{aligned}
\qquad (2.54)
$$

From (2.54) is clear that we obtain an action that corresponds to particles in the lowest Landau level with coordinates $\{\lambda_n,\tilde{\lambda}_n\}$.

However the variable $\tilde{\lambda}_n$ is not the complex conjugate to λ_n and this implies a change in the norm of the physical state with respect to that of the electrons in the lowest Landau levels. The norm in the Fock space of holomorphic functions is defined by [57]:

$$
\langle\Psi_1\mid\Psi_2\rangle=\int\mathfrak{D}X\mathfrak{D}\bar{X}\mathfrak{D}\psi\mathfrak{D}\psi^{\dagger}e^{-Tr\bar{X}X-\psi^{\dagger}\psi}\delta(G)FP\overline{\Psi_1(X,\psi)}\Psi_2(X,\psi). \qquad (2.55)
$$

Now using (2.52) and (2.50) we obtain the norm in terms of the new variables,

$$
\begin{aligned}
\langle\Psi_1\mid\Psi_2\rangle &= \int\prod_n d\tilde{\lambda}_n d\lambda_n d\tilde{\phi}_n d\phi_n e^{-\sum_n(\tilde{\lambda}_n\lambda_n+\tilde{\phi}_n\phi_n)} \\
& \quad \bar{\Phi}_1(\tilde{\Lambda},\tilde{\phi})\Phi_2(\Lambda,\phi)\mid_{\tilde{\phi}_i\phi_i=k,\ \tilde{\Lambda}_{ij}=\tilde{\lambda}_{ij}\delta_{ij}-(1-\delta_{ij})\frac{\phi_i\tilde{\phi}_j}{\lambda_i-\lambda_j}} . \qquad (2.56)
\end{aligned}
$$

Let us analyze the norm of the ground state for $N=2$,

$$
\langle\Psi_{k-gs}\mid\Psi_{k-gs}\rangle_{N=2}=\int d\tilde{\lambda}_1d\lambda_1d\tilde{\lambda}_2d\lambda_2e^{-\tilde{\lambda}_1\lambda_1-\tilde{\lambda}_2\lambda_2}\left(\tilde{\lambda}_1-\tilde{\lambda}_2+\frac{2k}{\lambda_1-\lambda_2}\right)^k(\lambda_1-\lambda_2)^k. \qquad (2.57)
$$

If we compare (2.57) with the norm of the Laughlin wave function in the quantum Hall effect,

$$
\langle\Psi_{k-gs}\mid\Psi_{k-gs}\rangle_{N=2}=\int d\bar{z}_1dz_1d\bar{z}_2dz_2e^{-\bar{z}_1z_1-\bar{z}_2z_2}(\bar{z}_1-\bar{z}_2)^k(z_1-z_2)^k, \qquad (2.58)
$$

it is clear that the ground state properties of the matrix theory and of the Laughlin state agree at long distances but differ microscopically.

In conclusion in this section we have shown that the Chern-Simons matrix model has the Laughlin wave function as ground state. Nevertheless, the theory presents some difficulties that limit its applicability as a effective theory of the FQHE [21]:

- The Chern-Simons matrix model does not possess quasi-particle excitations, only quasi-holes can be realized [15].
- The Jain states with the filling fractions, $\nu = n/(nk+1)$, $n = 2, 3, \ldots$, cannot be described.
- Even if the Laughlin wave function is obtained, the measure of integration differs from that of electrons in the lowest Landau level, owing to the noncommutativity of matrices. As shown before, the ground state properties of the matrix theory and of the Laughlin state only agree at long distances.
- Owing to the inherent noncommutativity, it is also difficult to match matrix observables with electron quantities of the quantum Hall effect [58].

Many authors modified the Chern-Simons matrix model, with the scope of introduce quasi-particle excitations and obtain ground states with other filling fractrions. In the next section we review two generalizations of the Polychronakos model that allow to change the effective area of the electrons [24], [21] and to introduce quasi-particle excitations.

2.3 Chern-Simons model with many boundaries

Since the area of the electrons in the Chern-Simons matrix model, is given by the Gauss law,

$$i\mathbf{B}\left[X_1, X_2\right] - \psi\psi^\dagger + \mathbf{B}\theta\mathbf{I} = 0, \tag{2.59}$$

Polychronakos and Morariu [24] introduced a modification involving more boundary fields $\psi_j, j = 1, ..., N$. In this manner, they could obtain fluids with densities different from the Laughlin ones. They showed that the multi-boundary term is equivalent to a Wilson line in a generic representation of the gauge group.

2.3.1 Wilson line action

Polychronakos and Morariu considered the additional term:

$$S_g = \int dtTr\left[i\lambda g^{-1}(\partial_t + iA_0)g\right], \tag{2.60}$$

where g is valued in the $U(N)$ group and λ is an arbitrary hermitian matrix. Without loss of generality λ can be taken to be diagonal $\lambda = diag(\lambda_1, ..., \lambda_N)$ and at quantum level all the λ_i's must be integers, to maintain gauge invariance.

The action (2.60) is equivalent to:

$$S = \int dtTr\left[ipg^{-1}\dot{g} - gpg^{-1}A_0\right], \tag{2.61}$$

together with the constraint,

$$p - \lambda = 0. \tag{2.62}$$

Quantization of (2.61) with the constraint (2.62) is studied in [62] and [63]: the physical Hilbert space is finite dimensional and provides an irreducible representation of $U(N)$. It is the representation whose lowest weight is given by λ.

In the path integral formulation, the action S_g is equal to a Wilson line in the irreducible representation mentioned above, i.e.,

$$\int dge^{iS_g} = \mathbf{P}e^{i\int dtA_0^a(t)t_a}, \tag{2.63}$$

where $\mathbf{P}$ denotes path ordering.

2.3.2 Connection with boundary fields

Under some assumptions [24], the action S_g is equivalent to the action:

$$S_\psi = \int dt\sum_{j=1}^{n}\left[\psi_j^\dagger(i\partial_t - A_0)\psi_j\right] + \sum_{j=n+1}^{N}\left[\psi_j^\dagger(-i\partial_t + A_0)\psi_j\right], \tag{2.64}$$

where ψ_j is a multiplet of boundary fields, with n of them transforming under the fundamental representation of the gauge group and $(N-n)$ transforming under the anti-fundamental representation. The equivalence holds by fixing the initial conditions:

$$\psi_j^\dagger\psi_k = \mid\lambda_j\mid\delta_{jk}, \tag{2.65}$$

and choosing the basis where the matrix generator G_ψ,

$$G_\psi = \sum_{j=1}^{n} \psi_j \psi_j^\dagger - \sum_{j=n+1}^{N} \psi_j \psi_j^\dagger, \tag{2.66}$$

of the Gauss law is diagonalized. Expression (2.66) is a Hermitian matrix which projects in the space spanned by ψ_j, so it can be diagonalized by a $U(N)$ transformation in this space. This implies a redefinition of the ψ_j's such that they remain orthogonal to each other, and satisfy (2.65). It was found that this classical equivalence between S_ψ and S_g, is also verified at quantum level [24].

2.3.3 Applications

The presence of many boundary vectors makes it possible to change the effective area of the particles. In this section we consider for example the solution corresponding to two layers of fluid with the same density. Consider the case of N fields in the anti-fundamental representation given by:

$$\begin{aligned} \psi_i &= \sqrt{p} \mid i\rangle \ , \qquad i = 0, 1, ..., \frac{N}{2} - 2, \frac{N}{2}, \frac{N}{2} + 1, ..., N-2 \ , \\ \psi_i &= \sqrt{\left(\frac{N}{2}(k-p) + p\right)} \mid i\rangle \ , \qquad i = \frac{N}{2} - 1, N-1, \end{aligned} \tag{2.67}$$

with p a positive number. In terms of holomorphic coordinates, $X = X_1 + iX_2$ and $\bar{X} = X_1 - iX_2$, the Gauss law constraint can be written as:

$$\frac{\mathbf{B}}{2}\left[\bar{X}, X\right] + \sum_i \psi_i \psi_i^\dagger + \mathbf{B}\theta = 0. \tag{2.68}$$

The classical minimum of energy satisfying the constraint (2.68) is given by:

$$\bar{X} = \sum_{n=0}^{\frac{N}{2}-2} \sqrt{(2\theta - 2\frac{p}{\mathbf{B}})(n+1)} \mid n+1\rangle\langle n \mid + \sum_{n=\frac{N}{2}}^{N-1} \sqrt{(2\theta - 2\frac{p}{\mathbf{B}})(n - \frac{N}{2} + 1)} \mid n+1\rangle\langle n \mid . \tag{2.69}$$

It has a block diagonal form, as one can see considering for instance $N = 6$,

$$\bar{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2\theta - 2\frac{p}{\mathbf{B}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2(2\theta - 2\frac{p}{\mathbf{B}})} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2\theta - 2\frac{p}{\mathbf{B}}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2(2\theta - 2\frac{p}{\mathbf{B}})} & 0 \end{pmatrix}. \tag{2.70}$$

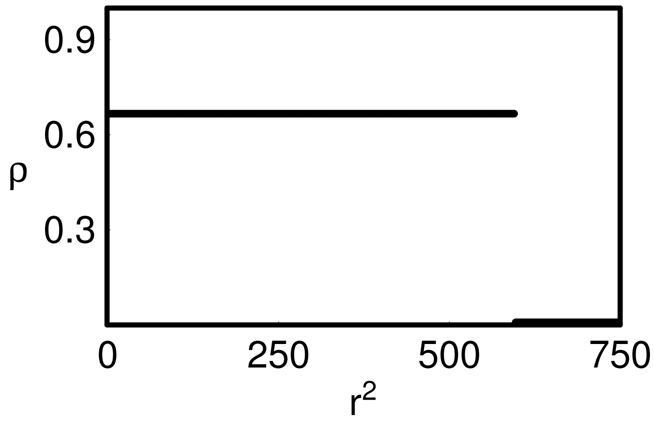

Figure 2.3: Plot of the density for the two-layer state (2.69) with $N = 400$, $k = 4$ and $p = 1$.

Solution (2.69) describes a droplet of fluid with classical filling fraction $\nu = \frac{2}{k-p}$.

The multi-boundary generalization of the Chern-Simons matrix model admits densities different from the Laughlin ones, but does not reproduce naturally the Jain composite fermion theory. Moreover, the other problems outlined at the end of Section 2.2 remain present.

2.4 Chern-Simons model with a scalar field

Another modification of the Polychronakos matrix model was given in ref.[21], that introduced a noncommutative scalar field ϕ, coupled to the boundary vector field ψ.

Following ref.[21], we define the holomorphic coordinates, $Z = \frac{1}{\sqrt{2\theta}}(X_1 + iX_2)$ and $Z^\dagger = \frac{1}{\sqrt{2\theta}}(X_1 - iX_2)$. In terms of these variables, the Susskind Lagrangian takes the form:

$$L_{CS} = \frac{i\kappa}{2} Tr\left(Z^\dagger D_0 Z - Z D_0 Z^\dagger\right) + \kappa Tr A_0. \tag{2.71}$$

The new term involving the scalar field is,

$$L_s = Tr\left\{\phi^\dagger i D_0 \phi - \frac{1}{2m} D_i\phi (D_i\phi)^\dagger + \frac{\lambda}{2\theta}(\phi^\dagger\phi)^2\right\}, \tag{2.72}$$

with ϕ a matrix field in the fundamental representation, with $U(N)$ gauge transformation $\phi \to U\phi$. The covariant derivatives $D_\mu = \partial_\mu + i\hat{a}_\mu$ acting on ϕ are:

$$\begin{aligned} D_0\phi &= \partial_o\phi + iA_0\phi, \\ D_i\phi &= \frac{i}{\theta}\epsilon_{ij}\left[\hat{x}_j, \phi\right] + i\hat{a}_i\phi = \frac{i}{\theta}\epsilon_{ij}\left[X_j\phi - \phi\hat{x}_j\right], \end{aligned} \tag{2.73}$$

where $\hat{x}_i$ are noncommutative coordinates $[\hat{x}_i, \hat{x}_j] = i\epsilon_{ij}\theta$, and we express the partial derivative as $\partial_i = \frac{i}{\theta}\epsilon_{ij}[\hat{x}^j, \ ]$. In (2.73) the matrices X_i are defined as:

$$X_i = \hat{x}_i - \theta\epsilon_{ij}\hat{a}_j, \tag{2.74}$$

with the noncommutative gauge potential $\hat{a}_i, i = 1, 2$ parameterizing the deviation from the ground states solution $[X_1, X_2] = i\theta$.

Defining the current,

$$J_i = \frac{-i}{2m}\left[(D_j\phi)\phi^\dagger - \phi(D_j\phi)^\dagger\right], \tag{2.75}$$

the Hamiltonian can be written as:

$$H = Tr\frac{1}{2m}(D_1\phi \pm iD_2\phi)(D_1\phi \pm iD_2\phi)^\dagger \pm \frac{\epsilon_{ij}}{2}Tr(\left[Z, Z^\dagger\right] - 1 \mp \lambda m\phi\phi^\dagger)\phi\phi^\dagger. \tag{2.76}$$

Taking $\lambda m\kappa = \pm 1$ so that the last term vanish due to the constraint,

$$\left[Z, Z^\dagger\right] = 1 - \frac{1}{\kappa}\phi\phi^\dagger, \tag{2.77}$$

the Hamiltonian reduces to the first term. If we consider states of zero energy, we obtain the set of BPS equations:

$$\begin{aligned} \hat{a}_0 &= \frac{1}{2\kappa\theta}\phi\phi^\dagger \ , \\ D_1\phi \pm iD_2\phi &= 0 \ , \\ \left[Z, Z^\dagger\right] &= 1 - \frac{1}{k}\phi\phi^\dagger. \end{aligned} \tag{2.78}$$

Any solution of (2.78) is also a solution to the full time-independent equations of motion corresponding to $L_{CS} + L_s$. We are interested in matrices of finite order N but we have assumed $[\hat{x}^1, \hat{x}^2] = i\theta$ that is satisfied by infinite matrices only. Nevertheless, the Lagrangian (2.72) can be modified [21], to include solutions of finite order satisfying the BPS equations (2.78).

Equations (2.78) has multi-layer solutions, but it doesn't presents quasi-hole solutions. This can be remedied by adding a Polychronakos type boundary field, coupling to the scalar field ϕ, as follows:

$$L_\psi = \psi^\dagger iD_0\psi - \frac{\lambda}{2\theta}\psi^\dagger\phi\phi^\dagger\psi, \tag{2.79}$$

that modifies the Gauss law constraint,

$$\left[Z, Z^\dagger\right] = 1 - \frac{1}{k}\phi\phi^\dagger - \frac{1}{k}\psi\psi^\dagger. \tag{2.80}$$

The coupling term between ψ and ϕ in (2.79) was introduced to allow the Hamiltonian to have a BPS form almost identical to (2.76), but with the Gauss law (2.77) replaced by (2.80).

2.4.1 Applications

As a first example we consider the classical solution corresponding to a quasi-hole of charge +1 in the origin of the fluid. Such solution is given by:

$$\begin{aligned} Z &= \sum_{n=2}^{N-1} \sqrt{n-1} \mid n-1\rangle\langle n \mid , \\ \psi &= \sqrt{(N-1)k} \mid N-1\rangle , \\ \phi &= \sqrt{k} \mid 0\rangle\langle 0 \mid . \end{aligned} \tag{2.81}$$

It is easy to see that (2.81) satisfy the BPS solution (2.78).

As second example we give the solution of a quasi-particle of charge proportional to q (q>0) in the origin of the fluid:

$$\begin{aligned} Z &= \sqrt{q} \mid N-1\rangle\langle 1 \mid + \sum_{n=1}^{N-1} \sqrt{n-1} \mid n-1\rangle\langle n \mid , \\ \psi &= \sqrt{Nk} \mid N-1\rangle , \\ \phi &= \sqrt{k} \mid 0\rangle\langle 0 \mid . \end{aligned} \tag{2.82}$$

The two modifications of the Chern-Simons matrix model, reviewed in the previous sections describe the physics of the QHE only at classical level. Besides this, these generalizations could not obtain fluids with the more general fillings $\nu = \frac{n}{kn+1}$.

Chapter 3

U(N) Maxwell Chern-Simons matrix gauge theory

In this chapter we propose and analyze our generalization of the Polychronakos theory that is based on the Maxwell-Chern-Simons matrix theory [13]. This includes an additional kinetic term quadratic in time derivatives and the potential $V = -g\mathrm{Tr}\,[X_1, X_2]^2$, parameterized by the positive coupling constant g. All the terms in the action are fixed by the gauge principle because they are obtained by dimensional reduction of the three-dimensional Maxwell-Chern-Simons theory. The matrix theory has been discussed in the literature of string theory as the low-energy effective theory of a stack of N D0-branes [64][65] on certain higher-brane configurations [67]; in particular, D0-branes have been proposed as fundamental degrees of freedom in string theory [68][69][70] (appendix A.4).

We start by discussing the canonical analysis of the Maxwell-Chern-Simons matrix theory [72][73], in presence of the uniform background θ and the "boundary" term (2.16). The theory involves three time-dependent $N \times N$ Hermitean matrices, $X_i(t)$, $i = 1, 2$ and $A_0(t)$, and the auxiliary complex vector $\psi(t)$: it is defined by the action,

$$
\begin{aligned}
S \;&=\; \int dt\; \mathrm{Tr}\left[\frac{m}{2}\left(D_t\, X_i\right)^2 \;+\; \frac{\mathbf{B}}{2}\varepsilon_{ij}\; X_i\; D_t\; X_j \;+\; \frac{g}{2}\left[X_1, X_2\right]^2 \;+\; \mathbf{B}\theta\; A_0\right] \\
&\quad -i \int \psi^{\dagger}\; D_t\psi \;. \qquad (3.1)
\end{aligned}
$$

The form of the covariant derivatives is: $D_t X_i = \dot{X}_i - i\,[A_0, X_i]$ and $D_t\psi = \dot{\psi} - iA_0\psi$. Under U(N) gauge transformations: $X_i \;\to\; UX_iU^{\dagger}$, $A_0 \;\to\; U\,(A_0 - id/dt)\,U^{\dagger}$, and $\psi \;\to\; U\psi$, the action changes by a total derivative, such that invariance under large gauge transformations requires the quantization, $\mathbf{B}\theta = k \in \mathbb{Z}$, as in the case of the Chern-Simons model 2.20. Hereafter we set $m = 1$ and measure dimensionful constants accordingly.

The canonical momenta are given by the following Hermitian matrices:

$$\Pi_i \equiv \frac{\delta S}{\delta \dot{X}_i^T} \;=\; D_t X_i - \frac{\mathbf{B}}{2}\varepsilon_{ij} X_j \ , \tag{3.2}$$

and $\chi = \delta S/\delta\dot{\psi} = -i\psi^\dagger$. After Legendre transformation on these variables, one finds the Hamiltonian:

$$H \;=\; \mathrm{Tr}\left[\frac{1}{2}\left(\Pi_i + \frac{\mathbf{B}}{2}\,\varepsilon_{ij}\,X_j\right)^2 \;-\; \frac{g}{2}\,[X_1, X_2]^2\right] \ . \tag{3.3}$$

The variation of S w.r.t. the non-dynamical field A_0 gives the Gauss-law constraint; its expression in term of coordinate and momenta reads:

$$G \;=\; 0\ , \qquad\qquad G \;=\; i\ [X_1, \Pi_1] \;+\; i\ [X_2, \Pi_2] \;-\; \mathbf{B}\theta\ \mathrm{I} \;+\; \psi\otimes\psi^\dagger\ , \tag{3.4}$$

where I is the identity matrix. At the quantum level, the operator G generates U(N) gauge transformations of X_i and ψ, and requires the physical states to be U(N) singlets subjected to the additional condition (3.5) counting the number of ψ_a components.

By taking the trace of G, one fixes the norm of the auxiliary vector ψ,

$$\mathrm{Tr}\ G \;=\; 0 \quad\longrightarrow\quad \|\psi\|^2 = \mathbf{B}\theta N = kN\ . \tag{3.5}$$

The auxiliary vector is necessary to represent the Gauss law on finite-dimensional matrices, as in the Chern-Simons model (see section 2.2).

3.1 Covariant quantization

We now quantize all the $2N^2$ matrix degrees of freedom X^i_{ab} and later impose the Gauss law as a differential condition on wave functions. The Hamiltonian (3.3) for $g = 0$ is quadratic and easily solvable: the sum over matrix indices decomposes into N^2 identical terms that are copies of the Hamiltonian of Landau levels [25]. To see this, introduce the matrix:

$$A = \frac{1}{2\ell}\,(X_1 + i\ X_2) \;+\; \frac{i\ell}{2}\,(\Pi_1 + i\ \Pi_2)\ , \tag{3.6}$$

and its adjoint $A^\dagger$, involving the “magnetic length” $\ell = \sqrt{2/\mathbf{B}}$.

The quantum commutation relations following from (3.2) are,

$$\left[\left[X^i_{ab}, \Pi^j_{cd}\right]\right] \;=\; i\,\delta^{ij}\,\delta_{ad}\,\delta_{bc}\;, \qquad\qquad \left[\left[\psi^\dagger_a, \psi_b\right]\right] \;=\; \delta_{ab}\;. \tag{3.7}$$

The canonical commutators imply the following relations of N^2 harmonic oscillators:

$$\left[\left[A_{ab}, A^\dagger_{cd}\right]\right] \;=\; \delta_{ad}\,\delta_{bc} \;, \qquad\qquad [[A_{ab}, A_{cd}]] \;=\; 0\;. \tag{3.8}$$

Note that $A^\dagger$ is the adjoint of A both as a matrix and a quantum operator. The Hamiltonian can be expressed in term of A and $A^\dagger$ as follows:

$$H \;=\; \mathbf{B}\;\mathrm{Tr}\left(A^\dagger\;A\right) \;+\; \frac{\mathbf{B}}{2}N^2 \;-\frac{g}{2}\mathrm{Tr}\left[X_1, X_2\right]^2\;. \tag{3.9}$$

In the term $\mathrm{Tr}(A^\dagger A) \;=\; \sum_{ab} A^\dagger_{ab}A_{ba}$ one recognizes N^2 copies of the Landau level Hamiltonian corresponding to N^2 two-dimensional "particles" with phase-space coordinates, $\{\Pi^i_{ab}, X^i_{ab}\}$, $a, b = 1, \ldots, N$, $i = 1, 2$.

The one-particle state are also characterized by another set of independent oscillators corresponding to angular momentum excitations that are degenerate in energy and thus occur within each Landau level. To find them, introduce the matrix,

$$B = \frac{1}{2\ell}\left(X_1 - i\;X_2\right) \;+\; \frac{i\ell}{2}\left(\Pi_1 - i\;\Pi_2\right)\;, \tag{3.10}$$

and its adjoint $B^\dagger$. They obey:

$$\left[\left[B_{ab}, B^\dagger_{cd}\right]\right] \;=\; \delta_{ad}\,\delta_{bc} \;, \qquad\qquad [[B_{ab}, B_{cd}]] \;=\; 0\;, \tag{3.11}$$

and commute with all the $A_{ab}, A^\dagger_{ab}$.

The total angular momentum of the N^2 "particles" can be written in the U(N) invariant form

$$J \;=\; \mathrm{Tr}\left(X_1\;\Pi_2 \;-\; X_2\;\Pi_1\right) \;=\; \mathrm{Tr}\left(B^\dagger B \;-\; A^\dagger A\right)\;. \tag{3.12}$$

Therefore, the B oscillators count the angular momentum excitations of the particles within each Landau level. In conclusion, the $g = 0$ theory exactly describes N^2 free particles in the Landau levels. In section 3.2 we shall discuss the effect of gauge symmetry that selects the subset of multi-particle states obeying the Gauss law, $G = 0$ (3.4).

3.1.1 Projection to the lowest Landau level and Chern-Simons matrix model

For large values of the magnetic field $\mathbf{B}$, one often considers the reduction of the theory to the states in lowest Landau level that have vanishing energy (3.9), i.e. obey

$A_{ab} = 0 \ \ \forall a, b$. All higher levels can be projected out by imposing the constraints $A = A^\dagger = 0$, that can be written classically (cf. (3.6)):

$$\Pi_2 = \frac{\mathbf{B}}{2} X_1 \ , \qquad \Pi_1 = -\frac{\mathbf{B}}{2} X_2 \ , \tag{3.13}$$

corresponding to vanishing kinetic term in the Hamiltonian (3.3). In this projection, two of the four phase space coordinates per particle are put to zero: if we choose them to be Π_1, Π_2, the remaining variables X_1, X_2 become canonically conjugate. This can also be seen from the action (3.1), because the kinetic term $m\,(D_i X)^2$ vanishes and one is left with the Chern-Simons term implying the identification of one coordinate with a momentum [30][25] (see section 2.2.2).

Upon eliminating Π_1, Π_2, the Gauss law (3.4) becomes:

$$G \ = \ -i\mathbf{B} \ [X_1, X_2] - \mathbf{B}\theta + \psi \otimes \psi^\dagger \ ; \tag{3.14}$$

namely, it reduces to the noncommutativity condition of the Chern-Simon matrix model (2.18), with action (2.16). Moreover, the potential term in the Hamiltonian (3.3) becomes a constant on all physical states verifying $G = 0$: one finds, using the normalization (3.5),

$$-\ \mathrm{Tr}\,[X_1, X_2]^2 \ = \ \mathrm{Tr}\left(\theta\ \mathrm{I} - \frac{1}{\mathbf{B}}\psi \otimes \psi^\dagger\right)^2 \ = \ \theta^2\ N(N-1)\ . \tag{3.15}$$

In conclusion, the Hamiltonian (3.3) reduces to a constant, i.e. it vanishes. This shows that the Maxwell-Chern-Simons matrix theory projected to the lowest Landau level is equivalent to the previously studied Chern-Simons matrix model [15].

3.2 Physical states at $g = 0$ and the Jain composite-fermion correspondence

In this section we are going to solve the Gauss law condition (3.4) and find the gauge invariant states. In the lowest Landau level, these states are the same as those of the Chern-Simons matrix model(see section 2.2.2); later we discuss the general physical states (sections 3.2.2,3.2.3). We introduce the complex matrices,

$$\begin{aligned} X &= X_1 + i\ X_2\ , \qquad & \overline{X} = X_1 - i\ X_2\ , \\ \Pi &= \frac{1}{2}\left(\Pi_1 - i\ \Pi_2\right)\ , \qquad & \overline{\Pi} = \frac{1}{2}\left(\Pi_1 + i\ \Pi_2\right)\ , \end{aligned} \tag{3.16}$$

and use the bar for denoting the Hermitean conjugate of classical matrices, keeping the dagger for the quantum adjoint. We set the magnetic length to one, i.e. $\mathbf{B} = 2$. The wave functions of the Maxwell-Chern-Simons theory take the form,

$$\Psi \;=\; e^{-\mathrm{Tr}\left(\overline{X}X\right)/2-\overline{\psi}\psi/2}\;\Phi(X,\overline{X},\psi)\;. \tag{3.17}$$

For energy and angular momentum eigenstates, the function Φ in (3.17) is a polynomial in the matrices $X, \overline{X}$ and the auxiliary field ψ. The integration measure reads:

$$\langle\Psi_1|\Psi_2\rangle \;=\; \int \mathcal{D}X\mathcal{D}\overline{X}\;\mathcal{D}\psi\mathcal{D}\overline{\psi}\;e^{-\mathrm{Tr}\overline{X}X-\overline{\psi}\psi}\;\Phi_1^*(X,\overline{X},\psi)\;\Phi_2(X,\overline{X},\psi)\;. \tag{3.18}$$

The operators A_{ab}, B_{ab} and $\psi_a, \psi_b^\dagger$ characterizing the Hilbert space (cf. section 3.1) become differential operators acting on wave functions:

$$\begin{aligned}
A_{ab} \;&=\; \frac{1}{2}\,X_{ab} + i\,\overline{\Pi}_{ab} \;=\; \frac{X_{ab}}{2} + \frac{\partial}{\partial\overline{X}_{ba}}\;, \qquad\quad A^\dagger_{ab} \;=\; \frac{\overline{X}_{ab}}{2} - \frac{\partial}{\partial X_{ba}}\;,\\
B_{ab} \;&=\; \frac{\overline{X}_{ab}}{2} + \frac{\partial}{\partial X_{ba}}\;, \qquad B^\dagger_{ab} = \frac{X_{ab}}{2} - \frac{\partial}{\partial\overline{X}_{ba}}\;, \qquad \psi^\dagger_a \;=\; \frac{\partial}{\partial\psi_a}\;.
\end{aligned} \tag{3.19}$$

Correspondingly, the Gauss law condition (3.4) becomes:

$$\begin{aligned}
&G_{ab}\;\Psi_{\mathrm{phys}}(X,\psi) \;=\; 0\;,\\
&G_{ab} = \sum_c \left(X_{ac}\,\frac{\partial}{\partial X_{bc}} - X_{cb}\,\frac{\partial}{\partial X_{ca}} + \overline{X}_{ac}\,\frac{\partial}{\partial\overline{X}_{bc}} - \overline{X}_{cb}\,\frac{\partial}{\partial\overline{X}_{ca}} \right) - k\;\delta_{ab} + \psi_a\,\frac{\partial}{\partial\psi_b}\;.
\end{aligned} \tag{3.20}$$

This operator acting on wave functions performs an infinitesimal gauge transformation of its variables: $X, \overline{X}, \psi$. Note that the expression of G_{ab} in (3.20) was normal ordered for this to obey the U(N) algebra [15].

The action of the angular momentum (3.12) on the polynomial part of wave functions is,

$$\sum_{ab}\left(X_{ab}\,\frac{\partial}{\partial X_{ab}} - \overline{X}_{ab}\,\frac{\partial}{\partial\overline{X}_{ab}} \right)\;\Phi(X,\overline{X},\psi) \;=\; \mathcal{J}\;\Phi(X,\overline{X},\psi)\;. \tag{3.21}$$

The eigenvalue $\mathcal{J}$ is just the total number of X matrices occurring in Φ minus that of $\overline{X}$. For states with constant density[1], the angular momentum measures the extension of the "droplet of fluid", such that we can associate a corresponding filling fraction ν by the formula (see section 2.2),

$$\nu \;=\; \lim_{N\to\infty}\;\frac{N(N-1)}{2\mathcal{J}}\;. \tag{3.22}$$

[1] See Refs.[15][20] for the definition of the density in the matrix theory.

In a physical system of finite size, one can control the density of the droplet, i.e. the angular momentum, by adding a confining potential V_C to the Hamiltonian:

$$H \ \rightarrow \ H \ + \ V_C \ = \ H \ + \ \omega \ \mathrm{Tr}\left(B^\dagger \ B\right) \ . \tag{3.23}$$

and tune its strength ω. This potential is diagonal on all states and becomes quadratic in the lowest Landau level, $V_C \rightarrow \omega \mathrm{Tr}(\overline{X}X)$ (see section 2.2). Typical values for ω will be of order $\mathbf{B}/N$, that do not destroy the Landau-level structure but give a small slope to each level.

3.2.1 The Jain composite-fermion transformation

Jain conjectured (see chapter 1) that a system of electrons with inverse filling fraction parametrized by:

$$\frac{1}{\nu} = \frac{\mathbf{B}}{2\pi\rho_o} = \frac{1}{m} + k \ , \qquad m = 1, 2, 3, \cdots \ , \tag{3.24}$$

can be mapped into a system of weakly interacting "composite fermions" at effective filling ν^*,

$$\frac{1}{\nu} \ \rightarrow \ \frac{1}{\nu^*} \ = \ \frac{1}{m}, \tag{3.25}$$

by removing (or "attaching") k quantum units of flux per particle (k even). From (3.24), the remaining effective magnetic field felt by the composite fermions is:

$$\mathbf{B} \ \rightarrow \ \mathbf{B}^* \ = \ B - \Delta\mathbf{B} \ , \qquad \Delta\mathbf{B} = k \ 2\pi\rho_o \ . \tag{3.26}$$

The relation between excluded magnetic field $\Delta\mathbf{B}$ and density is the key point of Jain's argument [10] [5]. The Lopez-Fradkin theory of the fractional Hall effect (See section 1.4) implements this relation as the equation of motion for the added Chern-Simons interaction.

Here we would like to stress that the Chern-Simon matrix model provides another realization of the Jain composite-fermion transformation (3.25,3.26) for $m = 1$. For $k = 0$, the matrix theory reduced to the eigenvalues λ_a is equivalent to a system of free fermions in the lowest Landau level, i.e. to $\nu^* = 1$ [75][20][76]. In the presence of the θ background, the noncommutativity of matrix coordinates (3.14) forces the electrons to acquire a finite area of order θ, by the uncertainty principle, leading to the (semiclassical) density $\rho_o = 1/2\pi\theta$ (2.8) [14]. Using this formula of the density and the quantization of $\mathbf{B}\theta$, we re-obtain the Jain relation (3.26),

$$\mathbf{B}\theta \ = k \ \in \ \mathbb{Z} \ \ \rightarrow \ \ \mathbf{B} \ = \ k \ 2\pi\rho_o \ . \tag{3.27}$$

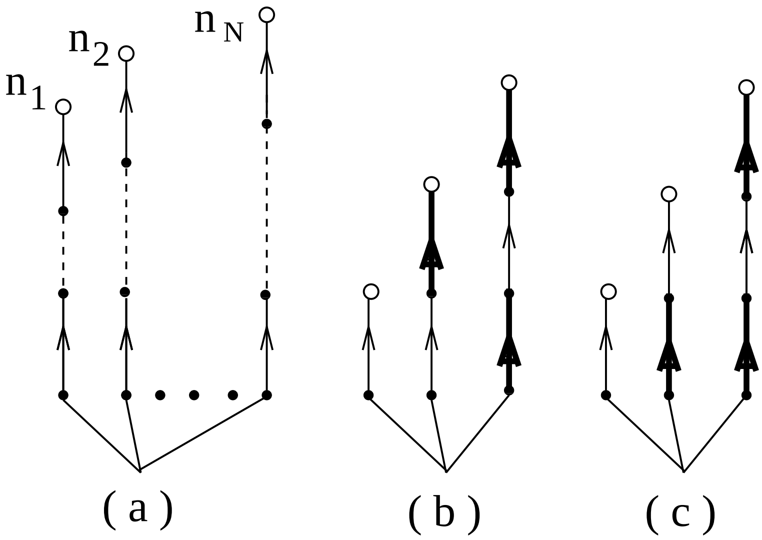

Figure 3.1: Graphical representation of gauge invariant states: (a) general states in the lowest Landau level (cf. Eq.(2.43)); (b) and (c) examples in the second and third one for $N = 3$.

Given that noncommutativity is expressed by the Gauss law of the matrix theory, we understand that the mechanism for realizing the Jain transformation is analogous to that of the Lopez-Fradkin theory, but it is expressed in terms of different variables.

The results of the Chern-Simons matrix theory were however limited, because the (matrix analogues of) Jain states for $m = 2, 3, \ldots$ could not be found. In the following, we shall find them in Maxwell-Chern-Simons matrix theory.

3.2.2 General gauge-invariant states and their degeneracy

Consider first the case $k = 1$. The states in the lowest Landau level, i.e. the polynomials $\Phi_{\{n_1,\ldots,n_N\}}(X, \psi)$ in Eq. (2.43), can be represented graphically as "bushes", as shown in Fig.(1a). The matrices X_{ab} are depicted as oriented segments with indices at their ends and index summation amounts to joining segments into lines, as customary in gauge theories. The lines are the "stems" of the bush ending with one ψ_a, represented by an open dot, and the epsilon tensor is the N-vertex located at the root of the bush. Bushes have N stems of different lengths: $n_1 < n_2 < \cdots < n_N$. The position i_ℓ of one X on the ℓ-th stem, $1 \leq i_\ell \leq n_\ell$, is called the "height" of X on the stem.

The general solutions of the $k = 1$ Gauss law (3.4,3.20) will be Φ polynomials involving both X and $\overline{X}$: given that they transform in the same way under the gauge group (cf. 3.20), the polynomials will again have the form of bushes whose stems are arbitrary words of X and $\overline{X}$. Angular momentum and energy eigenstates are linear combinations of bushes with given number $\mathcal{J} = N_X - N_{\overline{X}}$. From the commutation relations (3.8,3.11), energy and momentum eigenstates can be easily obtained by applying the $A^\dagger_{ab}$ and $B^\dagger_{ab}$ operators (3.19) to the empty ground state $\Psi_o = \exp\left(-\mathrm{Tr}\overline{X}X/2 - \overline{\psi}\psi/2\right)$. Their energy $E = \mathbf{B}N_A$ and momentum $\mathcal{J} = N_B - N_A$ are expressed in terms of the number of $A^\dagger$ and $B^\dagger$ operators, N_A and N_B respectively. The polynomial part Φ of the wave function is thus expressed in the following variables:

$$\Psi = e^{-\mathrm{Tr}\ \overline{X}X/2-\overline{\psi}\psi/2}\ \Phi(\overline{B},\overline{A},\psi)\ , \qquad E = \mathbf{B}\ N_A\ , \quad J = N_B - N_A\ , \tag{3.28}$$

where $\overline{B} = X - \partial/\partial\overline{X}$ and $\overline{A} = \overline{X} - \partial/\partial X$ commute among themselves, $[[\overline{A}_{ab}, \overline{B}_{cd}]] = 0$, and can be treated as c-number matrices. Since their U(N) transformations are the same as those of $X, \overline{X}$, they can be equivalently used to build the gauge invariant bush states. Examples of these general states are drawn in Fig.(3.1b, 3.1c) for $N = 3$: the variable $\overline{B}_{ab}$, replacing X_{ab} in the lowest Landau level, is represented by a thin segment, while $\overline{A}_{ab}$ is depicted in bold. Upon expanding $\overline{A}, \overline{B}$ in coordinates and derivatives acting inside Φ, one obtains in general a sum of $(X, \overline{X})$-bushes as anticipated.

The form of the general $k = 1$ gauge-invariant states suggests a pseudo-fermionic Fock-space representation involving N "gauge-invariant particles", as it follows:

- Each stem in the bush is considered as a "one-particle state" with quantum numbers, n_{Ai}, n_{Bi}, characterizing individual energies and momenta that are additive over the N particles, $N_A = \sum_{i=1}^N n_{Ai}$, $N_B = \sum_{i=1}^N n_{Bi}$.
- Since two stems cannot be equal, one should build a Fermi sea of N such one-particle states.
- The one-particle states form again Landau levels with energies $\varepsilon_i = \mathbf{B}n_{Ai}$, but there are additional degeneracies at fixed momentum with respect to the ordinary system; actually, in each stem, all possible words of $\overline{A}$ and $\overline{B}$ of given length yield independent states, owing to matrix noncommutativity (assuming large values of N).

Such "gauge invariant Landau levels" are shown in Fig.(3.2), together with their degeneracies, $(n_A + n_B)!/n_A!n_B!$, given by the number words of two letters with multiplicities n_A and n_B. These gauge invariant states should not be confused with the Landau levels discussed in section 3.1, that are relative to the states of the N^2 gauge variant "particles" with $X_{ab}, \overline{X}_{ab}$ coordinates. The analysis of some examples shows that the gauge invariant states are many-body superpositions of the former N^2 states that are neither bosonic nor fermionic and thus rather difficult to picture. Instead, the interpretation in terms of N fermionic "gauge-invariant particles" is rather simple and also convenient for the physical limit $g = \infty$ of commuting matrices (to be discussed in section 3.3). Finally, the gauge invariant states solution of the $k > 1$ Gauss law are given by tensoring k copies of the structures just described, in complete analogy with the lowest-level solutions (2.45). Thus there are k Fermi seas to be filled with N "gauge-invariant particles" each.

In the following, we are going to introduce a set of projections of the $g = 0$ Maxwell-Chern-Simons theory that will reduce the huge degeneracy of matrix states.

Degeneracies are better accounted for in a finite system, so we first modify the Hamiltonian to this effect. For example, the quadratic confining potential V_C (3.23) permits degenerate states that have equal energy and angular momentum – this also occurs in the ordinary Landau levels. The problem can be solved by using finite-box boundary conditions, that can be simulated by modifying the confining potential V_C in the Hamiltonian (3.23) as follows:

$$V_C \;=\; \omega \,\mathrm{Tr}\left(B^\dagger\; B\right) \;+\; \omega_n \,\mathrm{Tr}\left(B^{\dagger n}\; B^n\right) \;, \tag{3.29}$$

for a given value of n. The added operator $\mathrm{Tr}\left(B^{\dagger n}\; B^n\right)$ commutes with the $g = 0$ Hamiltonian and angular momentum and has the following spectrum: when acting on stems, each B_{ba} is a derivative that erases one $\overline{B}_{ab}$ matrix and fixes the indices at the loose ends of the stem to a and b respectively. Next, further $(n-1)$ derivatives act, with index summations, and finally the length-n strand $\overline{B}^n_{ab}$ is added to complete a new bush without cut strands. On stems with $n_B \geq n$, this operator has a diagonal action with eigenvalue $O(N^{n-1})$; on other strands, it is non-diagonal with $O(1)$ coefficients. Therefore, in the limit of large N and in the physical regime $n_A \ll n_B$, the confining potential (3.29) effectively realizes the finite-box condition $n_{Bi} \leq n$ for all Landau levels.

In a finite system of size n, the degeneracy of the k-th "gauge invariant Landau level" is $O(n^k/k!)$ and the total degeneracy grows exponentially, $O(\exp(n))$, for large energy. In presence of a quadratic confining potential, it would grow exponentially with the energy and give rise to a Hagedorn transition at finite temperature. Here one rediscovers a known property of matrix theories that makes them more similar to string theories than to field theories of ordinary matter [68][67].

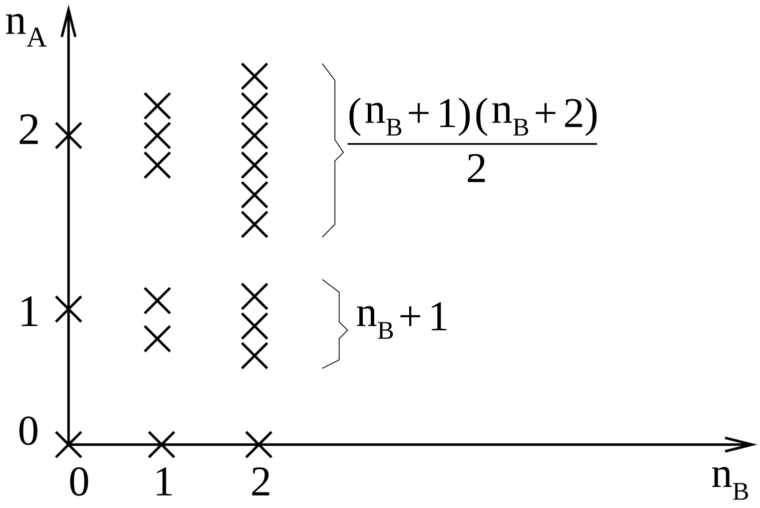

Figure 3.2: Pseudo-fermionic Fock space representation of gauge invariant states for $k = 1$.

In our physical setting, we should consider this feature as a pathology of the $g = 0$ theory that should be cured in some way. Actually, for $g > 0$ the potential term in the Hamiltonian, $V = (g/8)\mathrm{Tr}\left[X, \overline{X}\right]^2$, tends to eliminate the degeneracy due to matrix noncommutativity, as it follows. Consider a pair of degenerate states at $g = 0$, that differ for one matrix commutation, such those shown in Fig.(3.2), and call their sum and difference Ψ_+ and Ψ_-, respectively. For large values of g, the state Ψ_+ can have a finite energy, while Ψ_- will acquire a growing energy, corresponding to the freezing of the degrees of freedom of the commutator.

Therefore, the Maxwell-Chern-Simons matrix theory for large values of g possesses degeneracies that are consistent with ordinary two-dimensional matter; indeed, in section 3.3 we shall show that the theory at $g = \infty$ reduces to the ordinary quantum Hall effect with $O(1/r^2)$ interparticle interactions. In conclusion, the matrix degeneracies at $g = 0$ can be dealt with by the theory itself by switching the V potential on.

This fact is however not particularly useful from the practical point of view, because we do not presently know how to compute the spectrum of the theory for $g > 0$. Precisely as in the original problem of the quantum Hall effect, the free theory is highly degenerate and the degeneracy is broken by interaction (potential). The introduction of matrix variables would not appear as a great improvement towards solving this problem, given that their degeneracies are actually larger.

In spite of this, we shall find that matrix states do capture some features of the quantum Hall dynamics. In the following we shall introduce truncations of the $g = 0$ matrix theory that will eliminate most degeneracies and will naturally select non-degenerate ground states that are in one-to-one relation with the Jain hierarchical states [5] (see section 1.3).

3.2.3 The Jain ground states by projection

We consider the lowest Landau level condition $A_{ab}\Psi = 0$, $\forall a, b$, that singles out the Laughlin wave functions as the unique ground states at filling fractions $\nu = 1/(k+1)$. Although apparently not gauge invariant, it follows from the gauge invariant condition of vanishing energy, because the Hamiltonian, $H = 2\sum_{ab} A^\dagger_{ab} A_{ab}$, is a sum of positive operators that should all individually vanish.

Consider now the weaker condition, $(A_{ab})^2 \Psi = 0$, $\forall\ a, b$, allowing the N^2 gauge variant "particles" to populate the first and second Landau level. On the polynomial wave function, this projection reads:

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^2 \Phi(\overline{A}, \overline{B}, \psi) \ = \ 0 \ , \qquad \forall\ a, b \ . \tag{3.30}$$

The solutions are polynomials that are at most linear in each gauge-variant component $\overline{A}_{ab}$: one can think to an expansion of Φ in powers of $\overline{A}_{ab}$ that must stop at finite order as in the case of Grassmann variables. The condition $(A_{ab})^2 \Psi = 0$ is not manifestly gauge invariant. Nevertheless, when acting on the bush states described in the previous section (cf. Fig(3.1)), this condition respects gauge invariance (see appendix A.6 for a discussion of this point).

In the following we shall study the truncated matrix theories that are defined by the projections: $(A_{ab})^m \Psi = 0$, for m taking the successive values $2, 3, 4, \ldots$; their wave functions contain the N^2 gauge variant "particles" filling the lowest m Landau levels.

We first discuss the theory with second level projection $A^2 = 0$: we outline the solutions of condition (3.30) leaving the details to appendix A.5. Let us try to insert one or more $\overline{A}$ at points on the bush and represent them as bold segments, as in Fig.(3.1). The differential operator (3.30) acts by sequentially erasing pairs of bold lines on the bush in any order, each time detaching two branches and leaving four free extrema with indices fixed to either a or b, with no summation on them. For example, when acting on a pair of $\overline{A}$ located on the same stem, it yields a non-vanishing result: this limits to one $\overline{A}$ per stem. Cancelations can occur for pairs of $\overline{A}$ on different stems, owing to the antisymmetry of the epsilon tensor, as it follows:

$$\left(A^b_a\right)^2 \Phi = \cdots\ +\ \varepsilon^{\dots i \dots j \dots} \left(\cdots M_{ia}\ N_{ja} \cdots V^b\ W^b\right)\ +\ \cdots\ , \qquad (a, b \text{ fixed}), \quad (3.31)$$

that vanishes whenever $M = N$. The analysis of appendix A.5 shows that there cannot be further cancelations involving linear combinations of different bushes. Therefore, the general solution of (3.30) is a bush involving one $\overline{A}$ per stem (max N matrices in total), all of them located at the same height on the stems, as follows:

$$\Phi^{(II)}_{\{n_1,\dots,n_\ell;p;n_{\ell+1},\dots,n_M\}} \ =\ \varepsilon^{i_1\dots i_N} \prod_{k=1}^{\ell} \left(\overline{B}^{n_k}\psi\right)_{i_k} \prod_{k=\ell+1}^{N} \left(\overline{B}^p\overline{A}\ \overline{B}^{n_k}\psi\right)_{i_k}\ ,$$
$$0 \le n_1 < \cdots < n_\ell\ , \quad 0 \le n_{\ell+1} < \cdots < n_N\ . \quad (3.32)$$

These states can be related to Slater determinants of the ordinary Landau levels: assuming diagonal expressions for both $\overline{B}$ and $\overline{A}$, the matrix states become Slater determinants of N electron one-particle states [5][78]. This relation is surjective in general, because states differing by matrix orderings get identified; however, for states of form of Eq. (3.32), the matrix degeneracy is limited to the p dependence. This shows how the projection $A^2 = 0$ works in reducing degeneracies.

Let us analyze the possible matrix states in the $A^2 = 0$ theory with finite-box conditions, referring to Figs.(3.1, 3.2) for examples. The most compact state corresponds to *homogeneous* filling all the allowed states in the first and second Landau levels with $N/2$ "gauge invariant particles" each; it reads:

$$\Phi_{1/2,\ gs} = \varepsilon^{i_1\dots i_N} \prod_{k=1}^{N/2} \left(\overline{B}^{k-1}\psi\right)_{i_k} \prod_{k=1}^{N/2} \left(\overline{A}\ \overline{B}^{k-1}\psi\right)_{i_{N/2+k}}\ , \qquad (3.33)$$

with angular momentum $\mathcal{J} = N(N-4)/4$. One easily sees that this state is non-degenerate for boundary conditions enforcing maximal packing, $n_{Bi} \le N/2$, due to the vanishing of the p parameter in (3.32). Assuming homogeneity of its density, we can assign it the filling fraction $\nu^* = 2$ using (3.22).

Let us now discuss the states in the $A^2 = 0$ theory for generic k values. Gauge invariant states should be products of k bushes, as in (2.45): they survive the projection (3.30), provided that the two derivatives always vanish when distributed over all bushes. Given the product state with one bush of type (3.33), obeying $A^2\ \Phi_{1/2,\ gs} = 0$,

$$\Phi_{k+1/2,\ gs} \ =\ \Phi_{k-1,\ gs}\ \Phi_{1/2,\ gs}\ , \qquad (3.34)$$

the other factor involving $k-1$ bushes should satisfy $A\ \Phi_{k-1,\ gs} = 0$ and actually be the Laughlin state (2.43). The state (3.34) is also non-degenerate with appropriate tuning of the boundary potential. From the $\mathcal{J}$ value, one can assign the filling fraction[2], $1/\nu = k + 1/2$, to this state.

We thus find the important result that the projected Maxwell-Chern-Simons theory possesses non-degenerate ground states that are the matrix analogues of the Jain states obtained by composite-fermion transformation at $\nu^* = 2$, Eqs. (3.24,3.25). The matrix states (3.34,3.33) would actually be equal to Jain's wave functions, if the $\overline{A}, \overline{B}$ matrices were diagonal: the ψ dependence would factorize and the matrix states reduce to the Slater determinants of Jain's wave functions (before their projection to the lowest Landau level) [5][78]. Indeed, the diagonal limit can be obtained as follows. We note that the derivatives present in the expressions (3.28) of $\overline{A}$ and $\overline{B}$ vanish when acting on the states (3.33) due to antisymmetry of the epsilon tensor: in the expression of these states we can replace, $\overline{B} \to X, \overline{A} \to \overline{X}$. Therefore, the Jain and matrix states become identical in the limit of diagonal $X, \overline{X}$, that is realized for $g \to \infty$ as discussed in section 3.3.

The correspondence extends to the whole Jain series: the other $\nu^* = m$ non-degenerate ground states are respectively obtained in the theories with $A^m = 0$ projections. Before discussing the generalization, let us analyze the other allowed states by the $A^2 = 0$ projection. They are obtained by relaxing the boundary conditions for (3.33), i.e. by reducing the density of the system, allowing for lower fillings of the "gauge invariant Fermi sea". The non-degenerate Laughlin ground state and its quasi-hole are clearly allowed states in the lowest level (cf. section 2.2.2). The quasi-particle over the Laughlin state is obtained by having one particle in the second Landau level, leading to the form (3.32) involving one $\overline{A}$ only, i.e. $\ell = N - 1, p = 0, n_N = 0$,

$$\begin{aligned} \Phi_{k,\ 1qp} &= \Phi_{k-1,\ gs}\ \Phi^{(II)}_{1,\ 1qp}\ , \\ \Phi^{(II)}_{1,\ 1qp} &= \varepsilon^{i_1 \ldots i_N}\ \left(\overline{A}\psi\right)_{i_N} \prod_{k=1}^{N-1} \left(\overline{B}^{k-1}\psi\right)_{i_k} \ . \end{aligned} \tag{3.35}$$

This is a quasi-particle in the inner part of the Laughlin fluid, it is non-degenerate and has the gap $\Delta E_{1qp} = \mathbf{B}$ (disregarding the confining potential) and $\Delta\mathcal{J} = -N$. Other quasi-particles are density rings that can be degenerate due to the free p parameter in (3.32). Multi quasi-particle states are obtained by inserting more than one $\overline{A}$ in $\Phi^{(II)}$, on different stems of the bush, according to (3.32): $\Phi_{k,\ \ell qp} = \Phi_{k-1,\ gs}\ \Phi^{(II)}_{1,\ \ell qp}$. Their energy is linear in the number of quasi-particles. We thus find that the projected $g = 0$ Maxwell-Chern-Simons matrix theory reproduces the Jain composite-fermion correspondence also for quasi-particle excitations [5], but with additional degeneracies.

[2] Keeping in mind the contribution of 1 from the Vandermonde of the integration measure.

Let us not proceed to find the states in the $g = 0$ theory with higher projections. In the $A^3 = 0$ theory, the $k = 1$ bushes may have two $\overline{A}$ matrices per stem at most, obeying the following rules (proofs are given in Appendix A.5):

- If the bush has only one $\overline{A}$ per stem, i.e. for second-level fillings, the $\overline{A}$'s can stay on the stems at two values of the height, i.e. can form two bands.
- If there are stems with both one and two $\overline{A}$'s, then the $\overline{A}$'s can form two bands, with the extra condition for single-$\overline{A}$ stems that their $\overline{A}$'s should stay on the lowest band.

The first rule implies that the earlier $\nu^* = 2$ homogeneous state (3.33) becomes degenerate in the $A^3 = 0$ theory at the same density. On the other hand, the $A^3 = 0$ theory admits a maximal density state with $N/3$ gauge-invariant particles per level, that is unique due to the second rule:

$$\Phi_{1/3,\ gs} = \varepsilon^{i_1 \dots i_N} \prod_{k=1}^{N/3} \left[\left(\overline{B}^{k-1} \psi \right)_{i_k} \left(\overline{A}\, \overline{B}^{k-1} \psi \right)_{i_{k+N/3}} \left(\overline{A}^2\, \overline{B}^{k-1} \psi \right)_{i_{k+2N/3}} \right] . \qquad (3.36)$$

This state corresponds to filling fraction $\nu^* = 3$. Next, the product states,

$$\Phi_{k+1/3,\ gs} \ = \ \Phi_{k-1,\ gs}\, \Phi_{1/3,\ gs} \ , \qquad (3.37)$$

obeys the $A^3 = 0$ condition for $k > 1$: these ground states realize the Jain composite-fermion construction for $\nu^* = 3$ and have the expected filling fraction $\nu = m/(mk+1)$ for $m = 3$.

The pattern repeats itself in the $A^4 = 0$ theory (see appendix A.5): there are three $\overline{A}$'s per stem at most, that can form up to three bands; however, if single and/or double-$\overline{A}$ stems are present together with the three-$\overline{A}$ stems, the $\overline{A}$'s of the former stems should stay on the lowest bands. Therefore, the maximal density state is again unique, having form analogous to (3.36) and filling $\nu^* = 4$.

In conclusion, the $A^m = 0$ projected theory possesses the following non-degenerate ground states with Jain fillings $\nu = m/(mk + 1)$:

$$\Phi_{k+1/m,\ gs} \ = \ \Phi_{k-1,\ gs}\, \Phi_{1/m,\ gs} \ , \qquad (3.38)$$

where

$$\Phi_{1/m,\ gs} = \varepsilon^{i_1 \dots i_N} \prod_{k=1}^{N/m} \left[\left(\overline{B}^{k-1} \psi \right)_{i_k} \left(\overline{A}\, \overline{B}^{k-1} \psi \right)_{i_{k+N/m}} \cdots \left(\overline{A}^m\, \overline{B}^{k-1} \psi \right)_{i_{k+(m-1)N/m}} \right] . \qquad (3.39)$$

In the $A^m = 0$ theory, the lower density states that were non-degenerate in the $A^k = 0$ theories, $k < m$, become degenerate. Nevertheless, there are non-degenerate quasi-particles of the $(m-1)$ Jain state just below.

In conclusion, we have found that the ground states with homogeneous fillings of the properly projected Maxwell-Chern-Simons matrix theory reproduce the Jain pattern of the composite fermion transformation. These matrix states are unique solutions for certain (maximal) values of the density, while Jain states are judiciously chosen ansatzs among many possible multi-particle states of the ordinary Landau levels.

These results indicate that the Jain composite-fermion excitations have some relations with the D0-brane degrees of freedom and their underlying gauge invariance. Both of them have been described as dipoles. According to Jain [5] and Haldane-Pasquier [79], the composite fermion can be considered as the bound state of an electron and a hole (a vortex of the electron fluid): the reduced effective charge would then account for the smaller effective magnetic field $\mathbf{B}^*$ (3.26) felt by these excitations. On the other side, matrix gauge theories, such as the Maxwell-Chern-Simons theory, are equivalent to noncommutative theories whose fundamental degrees of freedom are dipoles. Clearly, a better understanding of the potential term $\mathrm{Tr}[X,\overline{X}]^2$ in our matrix theory is necessary to clarify the dipole description.

We finally remark that the matrix coordinates are less noncommutative on the Jain states then on the Laughlin ones. Indeed, the general form of the Gauss law (3.4) can be rewritten in terms of $X, \overline{X}, A, \overline{A}$ as follows:

$$[X,\overline{X}] \;+\; \frac{2}{B}\,[\overline{X},A] \;+\; \frac{2}{B}\,[\overline{A},X] \;=\; 2\left(\theta - \frac{1}{B}\psi\otimes\overline{\psi}\right)\;. \tag{3.40}$$

On the Laughlin states belonging to the lowest Landau level, this reduces to the coordinates noncommutativity (2.18), because $A = \overline{A} = 0$; on states populating higher levels, there are other terms contributing to noncommutativity besides the matrix coordinates. In section 3.3, we shall discuss the theory in the opposite $g = \infty$ limit, where $[X,\overline{X}] = 0$, and thus non-commutativity is entirely realized between coordinates and momenta.

3.2.4 Generalized Jain's hierarchical states

In the $A^m = 0$ projected theories with $m \geq 3$, there are other solutions of the Gauss law for $k > 1$ besides the Jain states (3.38). Any combination of the $k = 1$ solutions

(3.39) is possible, as follows:

$$\begin{aligned}\Phi_{\frac{1}{p_1}+\cdots+\frac{1}{p_k},\ gs} &= \prod_{i=1}^{k} \Phi_{\frac{1}{p_i},\ gs}\ , \\ \frac{1}{\nu} &= 1\ +\ \sum_{i=1}^{k} \frac{1}{p_i}\ .\end{aligned} \tag{3.41}$$

In this equation, we also wrote the associated filling fractions using Eq.(3.22), i.e. assuming homogeneous densities. The states (3.41) obey the condition $A^q = 0$ with $q = 1 + \sum_{i=1}^{k}(p_i - 1)$. The Jain mapping to a single set of $\nu^* = q$ effective Landau levels does not hold for these generalized states. Actually, analogous states were considered by Jain as well [5], and disregarded as unlikely further iterations of the composite-fermion transformation. In the matrix theory, we seek for arguments to disregard them as well.

Let us compare the generalized (3.41) and standard (3.38) Jain states at fixed values of the background k (keeping in mind that the physical values are $k = 2, 4$). The energy of the generalized states is additive in the $\nu^* = p_i$, $k = 1$, blocks and reads:

$$E_{\frac{1}{p_1}+\cdots+\frac{1}{p_k},\ gs} = \frac{\mathbf{B}N}{2} \sum_{i=1}^{k} (p_i - 1)\ +\ V_C\ . \tag{3.42}$$

The analysis of some examples of fillings and energies makes it clear that these additional solutions have in general higher energies for the same filling or are more compact for the same energy than the standard Jain states (3.38) (see Table 3.1). States of higher energies are clearly irrelevant at low temperatures. Furthermore, higher-density states strongly deviate from the semiclassical incompressible fluid value $\nu = 1/(k + 1)$ for background $\mathbf{B}\theta = k$, that is specific of the Laughlin factors [14]. This fact indicates that they might not be incompressible fluids with uniform densities. Further discussion of this point is postponed to section 3.3.

3.3 $g \to \infty$ limit and electron theory

In this section we switch on the potential $V = -(g/2)\mathrm{Tr}[X_1, X_2]^2$ in the Hamiltonian (3.3) and perform the $g \to \infty$ limit. The potential is a quartic interaction between the matrices that does not commute with the Landau term, $\mathbf{B}\mathrm{Tr}\left(A^\dagger A\right)$: thus, the $g = 0$ eigenstates obtained in the previous section by filling a given number of Landau levels will evolve for $g > 0$ into mixtures of states.

m	$\#p_i = 1$	$\#p_i = 2$	$\#p_i = 3$	$\#p_i = 4$	$1/\nu$	$E/\mathbf{B}$
1	k				$k+1$	0
2	$k-1$	1			$k+1/2$	$N/2$
3	$k-2$	2			k	N
3	$k-1$	0	1		$k+1/3$	N
4	$k-3$	3			$k-1/2$	$3N/2$
4	$k-2$	1	1		$k-1/6$	$3N/2$
4	$k-1$	0	0	1	$k+1/4$	$3N/2$

Table 3.1: Examples of generalized (3.41) and standard (3.38) Jain states for fixed value of k, ordered by Landau level m with corresponding fillings ν and energies E (disregarding the confining potential). Note that the experimentally relevant values are $k = 2, 4$ [5].

At the classical level, the V potential suppresses the matrix degrees of freedom different from the eigenvalues, and projects them out for $g \to \infty$. This can be seen by using the Ginibre decomposition of complex matrices [80], which reads: $X = \overline{U}(\Lambda + R)U$, where U is unitary (the gauge degrees of freedom), Λ diagonal (the eigenvalues) and R complex upper triangular (the additional d.o.f.). Inserting this decomposition in the potential, we find for $N = 2$:

$$V = \frac{g}{8}\mathrm{Tr}\left[X, \overline{X}\right]^2 = \frac{g}{4}|r|^4 + \frac{g}{4}|\overline{r}\left(\lambda_1 - \lambda_2\right)|^2 , \qquad X = \begin{pmatrix} \lambda_1 & r \\ 0 & \lambda_2 \end{pmatrix} . \tag{3.43}$$

Thus for large g, the variable r is suppressed. For general N, the potential kills all the $N(N-1)$ real degrees of freedom contained in the R matrix.

Let us now discuss the matrix theory in the $g = \infty$ limit, i.e. for $R = 0$: X and $\overline{X}$ commute among themselves (they are called "normal" matrices [76]) and can be diagonalized by the same unitary transformation:

$$\begin{aligned} & X = \overline{U}\Lambda U , \quad \overline{X} = \overline{U}\overline{\Lambda}U , \qquad \Lambda = diag\ (\lambda_a) , \\ & \left[X, \overline{X}\right] = 0 . \end{aligned} \tag{3.44}$$

In the $g = \infty$ limit, we analyze the theory following a different strategy from that of section 3: we fix the gauge invariance, solve the Gauss law at the classical level and then quantize the remaining variables, which are the complex eigenvalues λ_a and their conjugate momenta p_a, following the analysis of Refs. [72][73]. We take the diagonal gauge for the matrix coordinates and decompose the momenta $\Pi, \overline{\Pi}$, in diagonal and off-diagonal matrices, respectively called p and Γ:

$$X = \Lambda \ , \qquad \Pi = p + \Gamma \ , \quad \overline{\Pi} = \overline{p} + \overline{\Gamma} \ . \tag{3.45}$$

The Gauss law constraint (3.4) can be rewritten:

$$\begin{aligned} [X, \Pi] + \left[\overline{X}, \overline{\Pi}\right] &= -i \, \mathbf{B}\theta + i \, \psi \otimes \overline{\psi} \ , \\ (\lambda_a - \lambda_b) \, \Gamma_{ab} + \left(\overline{\lambda}_a - \overline{\lambda}_b\right) \overline{\Gamma}_{ab} &= -i \left(k \ \delta_{ab} - \psi_a \ \overline{\psi}_b\right) \ . \end{aligned} \tag{3.46}$$

The second of (3.46) implies $|\psi_a|^2 = k$ for any value of $a = b$. We can further fix the remaining $U(1)^N$ gauge freedom by choosing $\psi_a = \sqrt{k}$, $\forall a$, such that the r.h.s. of Eq. (3.46) becomes proportional to $(1 - \delta_{ab})$.

Therefore the Gauss law completely determines the off-diagonal momenta: their rotation invariant form is,

$$\Gamma_{ab} \ = \ \frac{ik}{2} \ \frac{\overline{\lambda}_a - \overline{\lambda}_b}{|\lambda_a - \lambda_b|^2} \ , \qquad\qquad a \neq b \ . \tag{3.47}$$

By inserting this back into the Hamiltonian (3.3), we find that diagonal and off-diagonal terms decouple and we obtain,

$$\begin{aligned} H &= 2 \ \mathrm{Tr} \left[\left(\frac{\overline{X}}{2} - i \ \Pi \right) \left(\frac{X}{2} + i \ \overline{\Pi} \right) \right] \\ &= 2 \sum_{a=1}^{N} \left(\frac{\overline{\lambda}_a}{2} - i p_a \right) \left(\frac{\lambda_a}{2} + i \overline{p}_a \right) \ + \ \frac{k^2}{2} \sum_{a \neq b = 1}^{N} \frac{1}{|\lambda_a - \lambda_b|^2} \ . \end{aligned} \tag{3.48}$$

The same result is obtained starting from the Lagrangian (3.1) and solving for A_0 in the gauge $X = \Lambda$ at $g = \infty$ [73].

Therefore, the theory reduced to the eigenvalues corresponds to the ordinary Landau problem for N electrons plus an induced two-dimensional Calogero interaction. Note also that the matrix measure of integration (3.18) reduces to the ordinary expression after incorporating one Vandermonde factor $\Delta(\lambda)$ in the wave functions [76]. The occurrence of the Calogero interaction is a rather common feature of matrix theories reduced to eigenvalues: the induced interaction is analog to the centrifugal potential appearing in the radial Schroedinger equation. In the present case, the interaction is two-dimensional, owing to the presence of two Hermitean matrices, and thus it is rather different from the exactly solvable one-dimensional case [15][54].

We conclude that the Maxwell-Chern-Simons matrix theory in the $g = \infty$ limit makes contact with the physical problem of the fractional quantum Hall effect: the only difference is that the Coulomb repulsion e^2/r is replaced by the Calogero interaction k^2/r^2. Numerical results [2][81] [5][78] indicate that quantum Hall incompressible fluid states are rather independent of the detailed form of the repulsive potential at short distance, for large $\mathbf{B}$. In particular, the Calogero potential does not have the long-range tail of the Coulomb interaction and is closer to the class of much-used Haldane short-range potentials [81]. Although the physics of incompressible fluids is universal, the form of the potential might affect the detailed quantitative predictions of the theory for some quantities such as the gap: this issue is postponed to the future.

Some remarks are in order:

- The physical condition imposed by the Gauss law (3.46) is still that outlined in section 3.2.1: it forces the electrons to stay apart by locking their density to the value of the background parameter k. The solution of this constraint is however rather different at the two points $g = 0$ and $g = \infty$: for $g = 0$, it is the geometric, or kinematic, condition of noncommutativity (2.18), while at $g = \infty$ this is a dynamical condition set by a repulsive potential with appropriate strength.
- Such dynamical condition is far more complicate to solve, and it allows many more excited states than the kinematic condition; there are many more available states in the lowest Landau level at $g = \infty$ than in the $g = 0$ matrix theory.
- Note also that the $g = \infty$ theory is not, by itself, less difficult than the ab-initio quantum Hall problem: the gap is non-perturbative and there are no small parameters. The advantage of embedding the problem into the matrix theory is that of making contact with the solvable $g = 0$ limit, as we discuss in the next section.

3.4 Conjecture on the phase diagram

In Figure (3.3) we illustrate the phase diagram of the Maxwell-Chern-Simons matrix theory as a function of its parameters $\mathbf{B}/m$ and g. The quantized background charge $\mathbf{B}\theta = k$ is held fixed over the diagram together with the parameters ω, ω_n in the confining potential (3.29).

The axes $g = 0$ and $g = \infty$ have been discussed in sections 3.2 and 3.3, respectively. For $g = 0$, the theory is solvable and displays a set of states that are in one-to-one relation with the Laughlin and Jain ground states with filling fractions $\nu = m/(mk+1)$. These non-degenerate states can be selected by choosing the appropriate projection $A^m = 0$ and the value of k, and by tuning ω, ω_n. For $g = \infty$, we found

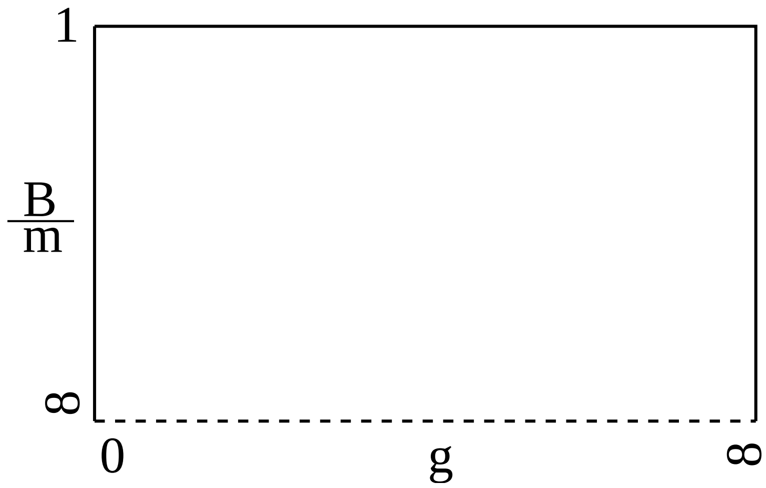

Figure 3.3: Phase diagram of the Maxwell-Chern-Simons matrix theory. The axes $g = 0$ and $g = \infty$ have been discussed in sections 3 and 4, respectively. The Chern-Simons matrix model sits at the left down corner.

that the theory describes the real fractional Hall effect, but we do not know how to solve the Calogero interaction and find the ground states.

Let us consider the evolution of one Jain state as g is switched on, while keeping the other parameters fixed. Given that the potential Tr $\left[X, \overline{X}\right]^2$ does not commute with the $g = 0$ Landau Hamiltonian, this state will mix with other ones. If it remains non-degenerate as g grows up to infinity, we can say that the matrix theory remains in the same universality class and that the qualitative features found at $g = 0$ remain valid in the physical limit $g = \infty$. In the case of level crossing at some finite value $g = g^*$, the two regimes of the theory are unrelated.

Unfortunately, we do not presently have a method of solution of the $g \neq 0$ Hamiltonian. Nevertheless, we would like to conjecture that the Laughlin and Jain states at $g = 0$ do remain non-degenerate. Namely, that there is no phase transition at finite g values when the theory is tuned on such ground states at $g \sim 0$ (by appropriate choices of m, k, ω, ω_n).

Our conjecture is indirectly supported by the numerical results by Jain and others [81][5] [78], through the following classical argument. These authors found that the Laughlin and Jain states in the quantum Hall effect are very close to the exact numerical ground states for a variety of short-range potentials, including the Calogero one realized at $g = \infty$. Now, consider the $g > 0$ evolution of the Jain matrix ground states: the effect of the potential can be seen, at the classical level, as that of eliminating the additional matrix d.o.f. and make both $X, \overline{X}$ matrices diagonal (up to a gauge transformation, see section 3.3). In this case, the Jain matrix states become Slater determinants of Hall states (cf. section 3.2.3) and exactly reduce to the expressions introduced by Jain [5]. Therefore, it is rather reasonable to expect that the evolution the $g = 0$ matrix states will bring them into the diagonalized, i.e. original Jain states at $g = \infty$, up to small deformations.

On the contrary, other states such as those of the generalized Jain hierarchy (see section 3.5), that have no counterpart in the $g = \infty$ theory, are likely to become degenerate at finite g.

In conclusion, our conjecture of smooth evolution of matrix Jain states is supported by the numerical analysis of the Jain composite-fermion theory. Further support is given by the form of the semiclassical density of $g = 0$ matrix states as discussed in section 4.2.

Let us finally remark that, the limit $\mathbf{B} \to \infty$ cannot be taken at $g = 0$, because quasi-particle excitations and Jain states in the matrix theory have energies of $O(\mathbf{B})$ and would be projected out. Instead, the limit $\mathbf{B} = \infty$ can surely be taken in the $g = \infty$ physical theory (holding $k = \mathbf{B}\theta$ fixed), because the fractional quantum Hall states are known to remain stable. This implies that the two limits are ordered: the correct sequence is $\lim_{\mathbf{B}\to\infty} \lim_{g\to\infty} \Psi$, and the opposite choice is cut out in the phase diagram of Fig.3.3.

In summary, in this chapter we have generalized the Susskind-Polychronakos proposal of noncommutative Chern-Simons theory and matrix models. We have found:

- A description of the expected Jain states and their quasi-particle excitations within a matrix generalization of the Landau levels.
- An interesting phase diagram, parametrized by the additional coupling g, with a manifestly physical limit for the matrix theory at $g = \infty$.

Chapter 4

Semiclassical Droplet States in Maxwell Chern-Simons matrix theory

In this chapter, we present our second work [22]: we find the gauge invariant form of the projection $A^m \approx 0$, introduced in chapter 3, and its semiclassical physical meaning in terms of single-particle occupancy (section 4.1). Next, we study the matrix Jain states in the semiclassical approximation, by analytically solving the classical equations of motion, further constrained by the Gauss law and the semiclassical version of the $A^m \approx 0$ condition (section 4.2). The ground states are found to be two-step droplets of incompressible fluid with piecewise constant density; this is the same density shape of the phenomenological Jain states before projection to the lowest Landau level [5] (where the density of incompressible fluids becomes strictly constant).

The fact that the matrix Jain states at $g = 0$ already have the expected droplet density of physical $g = \infty$ states, supports our earlier claim that these ground states could remain stable while varying $0 < g < \infty$ (see chapter 3). Other ground states corresponding to generalized Jain constructions with different filling fraction, although possible in the $g = 0$ theory, are found not to possess piecewise constant density. We argue that the modulated density shape is a signal of ground-state instability at finite g values, since the corresponding phenomenological Jain states ($g = \infty$) are known to be unstable [5]. We complete our study of semiclassical solutions by describing the quasi-holes excitations above the matrix Jain states.

4.1 Properties of the projection $A^m \approx 0$

In this section we discuss the physical meaning of the projection:

$$(A_{ab})^m \ \Psi\left(\overline{A},\overline{B}\right) \ = \ 0 \ , \qquad \forall a, b, \tag{4.1}$$

that limits the degeneracy of matrix quantum states at $g = 0$. Although the operator $(A_{ab})^m$ is not gauge invariant, its kernel restricted to gauge invariant states yields a gauge invariant condition[1], as explicitly seen in the previous chapter. Therefore, there should exist a manifestly gauge invariant expression for this condition, that is found in this section.

A simple example is useful to clarify the following discussion. In a two dimensional quantum mechanical problem with rotation invariance ($O(2)$ global symmetry), we consider the condition:

$$P_m\,\Phi \ \equiv \ \left(\frac{\partial}{\partial x}\right)^m \ \Phi\left(r^2\right) \ = \ 0 \ , \qquad r^2 = x^2 + y^2 \ , \tag{4.2}$$

where Φ is a reduced (polynomial) wave function. The condition is not $O(2)$ invariant but its kernel acting on rotation invariant functions does: indeed, it limits the order of the polynomial to $O(r^{m-1})$. This example suggests two remarks:

- The condition (4.2) can have many different forms, that correspond to points on its orbit in the “gauge” $O(2)$ group: for example, an equivalent form is $(\partial/\partial y)^m \Phi = 0$, corresponding to a $\pi/2$ rotation. All these conditions are equally satisfied.
- A manifestly gauge-invariant expression can be obtained by integrating over the gauge orbit, as follows:

$$P_m \ \longrightarrow \ P_m^{g.i.} \ = \ \int_0^{2\pi} d\theta \ \left(\cos\theta\,\frac{\partial}{\partial x} \ + \ \sin\theta\,\frac{\partial}{\partial y}\right)^m \ . \tag{4.3}$$

However, this vanish for m odd: the average looses information because the operator $(\partial/\partial x)^m$ is not positive definite. Clearly, it can be made positive (and gauge invariant) by contracting with another gauge-dependent term to obtain powers of the dilatation operator $D^m = (x^i\partial/\partial x^i)^m$.

We are now going to follow analogous steps for the condition $A^m \approx 0$. First we find an equivalent, more general form. Consider an infinitesimal $SU(N)$ gauge transformation $U = 1 + i\varepsilon T$: the Hermitean matrix T can be expressed by the matrices $E^{(ij)}$

[1] A formal proof of this statement is given in Appendix A.6

with a single non-vanishing component, $E^{(ij)}_{ab} = \delta^i_a \delta^j_b$, in symmetric or antisymmetric combinations, $T = E^{(ij)} + E^{(ji)}$ or $T = i(E^{(ij)} - E^{(ji)})$. Upon performing the gauge transformation, the $m = 2$ constraint (4.1), $(U^\dagger A U)^2_{ab}$, acquires an additional $O(\varepsilon)$ term that should also vanish on the wave functions obeying, $(A_{ab})^2 \Psi = 0$:

$$0 \;\approx\; A_{ab}\, \left[E^{ij}, A\right]_{ab} \;=\; A_{ab} \left(\delta_{ai}\, A_{jb} - A_{ai}\, \delta_{jb}\right) \,, \qquad \forall\, i \neq j \,, \quad \forall\, a, b. \tag{4.4}$$

We now analyse the various cases:

- I. If $a = b$ and $i = a$ or $j = a$, we obtain the conditions,

$$0 \;\approx\; A_{aa}\, A_{ja} \;\approx\; A_{aa}\, A_{ai}, \qquad \forall\, i, j \neq a \,.$$

- II. If $a \neq b$, we obtain,

$$\begin{array}{lll} \text{1. for } \; i = a \text{ and } j \neq b & \longrightarrow & 0 \;\approx\; A_{ab}\, A_{jb} \,, \quad \forall\, j \neq a, b \,, \\ \text{2. for } \; i \neq a \text{ and } j = b & \longrightarrow & 0 \;\approx\; A_{ab}\, A_{ai} \,, \quad \forall\;\, i \neq a, b \,, \\ \text{3. for } \; i = a \text{ and } j = b & \longrightarrow & 0 \;\approx\; A_{ab} \left(A_{bb} - A_{aa}\right) \,. \end{array}$$

  Note that each term in the linear combination of case II.3 independently vanishes by case I.

These conditions can be summarized as follows:

$$\begin{aligned} A_{ab}\, A_{a'b}\, \Psi &= 0 \,, \qquad \forall\, a, a', b, \\ A_{ab}\, A_{ab'}\, \Psi &= 0 \,, \qquad \forall\, a, b, b'. \end{aligned} \tag{4.5}$$

They are more general than the original expression (4.1) for $m = 2$, corresponding to $a = a'$ or $b = b'$. Of course, iteration to $O(\varepsilon^2)$ of the gauge transformation produce further identities: these involve linear combinations of A^2 terms that are not particularly useful; for example, one such condition is: $A_{ab}A_{jc} + A_{jb}A_{ac} \approx 0$.

The $O(\varepsilon^2)$ analysis is necessary to obtain the generalized constraint for $m = 3$: the $O(\varepsilon)$ expression is similarly, $A_{ab}\, A_{ab}\, A_{ab'} \approx 0$, and its further transformation yields,

$$0 \;\approx\; 2\, A_{ab}\, \left[E^{ij}, A\right]_{ab}\, A_{ab'} \;+\; A_{ab}\, A_{ab}\, \left[E^{ij}, A\right]_{ab'} \,.$$

This expression contains the $m = 3$ constraint analogous to (4.5):

$$\begin{aligned} A_{ab}\, A_{a'b}\, A_{a''b}\, \Psi &= 0 \,, \qquad \forall\, a, a', a'', b, \\ A_{ab}\, A_{ab'}\, A_{ab''}\, \Psi &= 0 \,, \qquad \forall\, a, b, b', b'', \end{aligned} \tag{4.6}$$

together with other relations involving linear combinations of cubic terms.

Following the $O(2)$ example, we can now transform the new expressions (4.5) into positive definite operators. We recall that the lowest Landau level condition corresponds to the vanishing of the total energy, that is a sum of positive terms:

$$H \ = \ \mathrm{Tr}\left(A^\dagger\ A\right) \ = \ \sum_{a,b} A^*_{ab}\ A_{ab} \ \approx \ 0 \qquad \Leftrightarrow \qquad A_{ab} \ \approx \ 0 \ , \ \ \forall\, a, b \ . \tag{4.7}$$

We can construct the following positive definite expressions:

$$Q_2 \ = \ \sum_{a,b,b'} A^\dagger_{b'a}\ A^\dagger_{ba}\ A_{ab}\ A_{ab'} \ , \tag{4.8}$$

$$Q'_2 \ = \ \sum_{a,a',b} A^\dagger_{ba'}\ A^\dagger_{ba}\ A_{ab}\ A_{a'b} \ , \tag{4.9}$$

whose vanishing is equivalent to the $m = 2$ conditions (4.5). These quantities are not yet gauge invariant but are convenient for the physical interpretation. We introduce the (gauge variant) energy operators for one-particle matrix states, that are summed over matrix indices of one row or column of A_{ab}, Z_a or Z'_b, respectively:

$$Z_a \ = \ \sum_b A^\dagger_{ba}\ A_{ab} \ , \qquad Z'_b \ = \ \sum_a A^\dagger_{ba}\ A_{ab} \ . \tag{4.10}$$

Using these energy operators, we can rewrite (4.8,4.9) as follows:

$$Q_2 \ = \ \sum_a Z_a \left(Z_a \ - \ 1\right) \ , \qquad Q'_2 \ = \ \sum_b Z'_b \left(Z'_b \ - \ 1\right) \ . \tag{4.11}$$

In this form, the constraints $Q_2\Psi \ = \ Q'_2\Psi \ = \ 0$ admit the following physical interpretation: there is a gauge choice in which the allowed states contains at most one "particle" in the second Landau level (energy equal to one) for (a, b) indices belonging to each row and column.

The constraint for $m = 3$ (4.6) similarly becomes:

$$Q_3 \ = \ \sum_a Z_a \left(Z_a \ - \ 1\right)\ \left(Z_a \ - \ 2\right) \ , \qquad Q'_3 \ = \ \sum_b Z'_b \left(Z'_b \ - \ 1\right)\ \left(Z'_b \ - \ 2\right) \ . \tag{4.12}$$

This requires that there at most 2 particles in the second Landau level or a single particle in the third level for any set of indices in a row or column. The matrix labels are not gauge invariant, then these occupancies are only verified in specific gauges; nevertheless, the present form of the constraints can be implemented in the semiclassical limit on expectation values, $\langle A_{ab}\rangle$, as explained in the next section.

Next, we obtain the gauge-invariant form of the constraint Q_2, Q_2' by averaging over the gauge group. We define:

$$Q_2^{g.i.} \;=\; \int \mathcal{D}U \; Q_2'(U) \;=\; \sum_b \int \mathcal{D}U \; U^\dagger_{bi} \; A^\dagger_{ia'} \; U^\dagger_{bj} \; A^\dagger_{ja} \; A_{ak} \; U_{kb} \; A_{a'l} \; U_{lb} \; , \tag{4.13}$$

where $\mathcal{D}U$ is the invariant Haar measure [83]. The integrand is positive definite for any U value, because it can be thought of as the norm of a vector: $Q_2(U) \sim \sum_b \; |A \cdot v^{(b)}|^4$, where $v_a^{(b)} = U_{an} \delta^b_n$ is a rotated unit vector. Therefore, we do not loose any information by performing the group average.

Group integrals of products of $U, U^\dagger$ matrices can be found e.g. in ref.[83]: their results can be described as follows. Representing the unitary matrices with upper and lower indices, $U_{ab} \to U_a{}^b$, $(U^\dagger)_{ab} \to (U^\dagger)^a{}_b$, the result of integrating n $(U, U^\dagger)$ pairs is a combination of n delta functions relating the upper indices among themselves times other n deltas connecting the lower indices. The simplest integral is:

$$\int \mathcal{D}U \; (U^\dagger)^a{}_{a'} \; U_{b'}{}^b \;=\; \frac{1}{N} \, \delta^{ab} \; \delta_{a'b'} \; .$$

In the general case of n $(U, U^\dagger)$ pairs, the pairings of upper (lower) indices by delta functions follow patterns given by the permutation of n elements, with specific weight for each conjugacy class of permutations [83]. For $n = 2$, one finds:

$$\int \mathcal{D}U \; (U^\dagger)^a{}_{a'} \; U_{b'}{}^b \; (U^\dagger)^c{}_{c'} \; U_{d'}{}^d \;=$$
$$\frac{1}{N^2-1} \left[\delta^{ab}\delta^{cd}\delta_{a'b'}\delta_{c'd'} \;+\; \delta^{ad}\delta^{cb}\delta_{a'd'}\delta_{c'b'} - \frac{1}{N} \left(\delta^{ab}\delta^{cd}\delta_{a'd'}\delta_{c'b'} \;+\; \delta^{ad}\delta^{cb}\delta_{a'b'}\delta_{c'd'} \right) \right] .$$

In the case of the constraint Q_2' (4.9), all the upper indices are simultaneously taking the same value b; thus, the different delta-function pairings of upper indices take the same unit value. As a result, the pairings of lower indices get averaged over, and reduce to a plain sum over all pair permutations:

$$Q_2^{g.i.} \;\propto\; (\delta_{ki} \; \delta_{lj} \;+\; \delta_{kj} \; \delta_{li}) \; A^\dagger_{ia'} \; A^\dagger_{ja} \; A_{ak} \; A_{a'l} \; . \tag{4.14}$$

Upon commuting the operators to bring summed indices close each other, we finally find the manifestly gauge-invariant form of the $A^2 \approx 0$ constraint (disregarding the normalization):

$$Q_2^{g.i.} \approx 0 \; , \qquad Q_2^{g.i.} \;=\; \mathrm{Tr}\left(A^\dagger A A^\dagger A\right) \;+\; \left(\mathrm{Tr}\; A^\dagger A\right)^2 \;-\; (N+1)\; \mathrm{Tr}\left(A^\dagger A\right) \; . \tag{4.15}$$

The same expression is also obtained by group averaging the other operator Q_2 in (4.8). One can check that the action of the gauge-invariant constraint $Q_2^{g.i.}$ on bush wave functions (cf. section 3.2.2) is completely equivalent to that of the gauge-variant condition $A^2 \approx 0$ [13].

The gauge invariant form of the $m = 3$ constraint can be similarly obtained by group averaging (4.6), leading to:

$$Q_3^{g.i.} = \sum_{\sigma \in \mathcal{S}_3} A^\dagger_{i_1 b} \, A^\dagger_{i_2 b'} \, A^\dagger_{i_3 b''} \, A_{i_{\sigma(3)} b''} \, A_{i_{\sigma(2)} b'} \, A_{i_{\sigma(1)} b} \, . \tag{4.16}$$

The form of this expression corresponding to (4.15) is not particularly illuminating. The gauge-invariant expression (4.16) straightforwardly generalizes to higher m values.

In conclusion, in this section we have found equivalent forms of the projections $A^m \approx 0$ of $g = 0$ matrix states: the first expression (4.11,4.12) in terms of occupation numbers is useful for the semiclassical limit considered in the next section; the second expression (4.15,4.16) is manifestly gauge invariant. In the latter form, the constraint can be added to the Hamiltonian with a large positive coupling constant to realize a softer form of projection, where matrix states violating the constraint are now allowed but possess very high energy. For example, the quasi-particles excitations over the Jain ground states $\nu = m/(mk + 1)$ would be possible.

4.2 Droplet ground state solutions

In this section we study the $g = 0$ Maxwell-Chern-Simons theory in the semiclassical limit: we solve the classical equation of motion including the quantum constraints, first for the ground states and then for the quasi-hole excited states. We shall find the semiclassical states that correspond to the quantum states with homogeneous filling and composite-fermion structure of chapter 3 [13]. The motivations for this semiclassical analysis are twofold: on one side, previous experience [15][68][58][21] [88] with noncommutative field theory has shown that the classical fluid dynamics incorporates some properties of the full quantum theory. From another side, it is know that the Laughlin states in the quantum Hall effect are incompressible fluids that become semiclassical in the thermodynamic limit $N \to \infty$ [25][60]. The semiclassical ground states we find in this section are also incompressible fluids which, we believe, may give rather accurate descriptions of the quantum matrix states for large N values [76].

Let us start by writing the classical equations of motion: the Hamiltonian of the

Maxwell-Chern-Simons theory at $g = 0$ can be written as follows:

$$
\begin{aligned}
H &= 2\,\mathrm{Tr}\left(\overline{A}A\right) + \omega\,\mathrm{Tr}\left(\overline{B}B\right) + \mathrm{Tr}\left[\Lambda\left([\overline{A},A]+[\overline{B},B]-k+\psi\otimes\overline{\psi}\right)\right] \\
&\quad + \sum_a \Gamma_a\left(Z_a-\gamma\right) + \sum_b \Gamma'_b\left(Z'_a-\gamma'\right) , \qquad\qquad \gamma,\gamma' = 0,1,\ldots,m-1 \; .
\end{aligned}
\tag{4.17}
$$

We set $\mathbf{B} = 2$, $\mathbf{B}\theta = k \in \mathbb{Z}$, and included the Gauss law constraint via the Hermitean Lagrange multiplier Λ. The projection $A^m \approx 0$ analyzed in section 4.1 is enforced by adding two other Lagrange multipliers Γ_a, Γ'_b times the energies, $Z_a = \sum_b \overline{A}_{ba}\, A_{ab}$, $Z'_b = \sum_a \overline{A}_{ba}\, A_{ab}$, of single-particle states with matrix indices summed over rows or columns. We replace the nonlinear constraints (4.11,4.12) with linear expressions involving the parameters γ, γ' taking the allowed values of Z_a, Z'_b. Since the constraints are not gauge invariant, we shall assume that we work in a gauge where they take integer values. The gauge-invariant form of the constraint (4.15) found at the end of section 4.1 is also not convenient because it would lead to non-linear equations of motion that cannot be solved analytically. For the same reason, we limit the confining potential (3.29) to the quadratic term: later we shall see how to avoid ground state degeneracies that may arise with this potential.

We vary the Hamiltonian with respect to $\overline{A}, \overline{B}$, canonically equivalent to the original X, Π, and obtain the equations:

$$
i\,\dot{A}_{ab} = 2A_{ab} - [\Lambda, A]_{ab} + A_{ab}\left(\Gamma_a + \Gamma'_b\right) , \tag{4.18}
$$

$$
i\,\dot{B} = -[\Lambda, B] + \omega\, B , \tag{4.19}
$$

$$
G = \left[\,\overline{A}, A\,\right] + \left[\,\overline{B}, B\,\right] - k + \psi\otimes\overline{\psi} = 0 , \tag{4.20}
$$

$$
Z_a = \sum_b \overline{A}_{ba}\, A_{ab} = \gamma , \qquad \gamma = 0,1,\ldots,m-1 \; . \tag{4.21}
$$

$$
Z'_b = \sum_a \overline{A}_{ba}\, A_{ab} = \gamma' , \qquad \gamma' = 0,1,\ldots,m-1 \; . \tag{4.22}
$$

We first discuss ground state solutions corresponding to $\dot{A} = \dot{B} = 0$.

4.2.1 Jain ground states

As we showed in chapter 3, the Maxwell Chern-Simons theory contains Jain-like ground states (3.38), that involve higher Landau levels ($A \neq 0$). Their filling fractions can be written as in composite fermion construction [5],

$$
\frac{1}{\nu} = \frac{1}{\nu^*} + k + 1 , \qquad k \text{ even}, \qquad \nu^* = 2,3,\ldots , \tag{4.23}
$$

and their energy and angular momentum values are recalled in Table 3.1. We first note that these states are characterized by energies $O(N)$ and angular momentum $J = O(N^2)$, thus implying that the matrix A must have elements of $O(1)$ and be much smaller than B. Indeed, the constraints, $Z_a, Z'_b = 0, 1, \ldots, m-1$, limits the squares of A_{ab} matrix elements summed over each row or column to take at most the total value $(m-1)$. Were it not for this constraint, the A, B matrices could be rescaled in the ground state equations (4.18,4.19,4.20) to eliminate the k dependence, leading to solutions with $E = O(kN)$ at least.

We now describe the solution of the ground state equations of motion in the $A^2 \approx 0$ projected theory ($\gamma, \gamma' = 0, 1$ in (4.21)). Under some minor hypotheses, we find a single solution corresponding to the unique quantum state with $\nu^* = 2$ (3.34) [13]. Working in analogy with the Laughlin case (2.27), we shall try a distribution of R^2 eigenvalues leading to a piecewise constant density. We can consider the gauge in which Λ is diagonal, $\Lambda = \mathrm{diag}(\ell_a)$, and assume that ψ has a single non-vanishing component, i.e. the last one, as in (2.27), such that the term, $(k\ I\ - \psi \otimes \overline{\psi})$, is also diagonal. The equation for B (4.19),

$$(\ell_a - \ell_b)\, B_{ab} = \omega\ B_{ab}\ , \tag{4.24}$$

requires that B is a raising operator, i.e. non vanishing on a single diagonal, $B_{ab} \propto \delta_{a,b+n}$; moreover, the Gauss law (4.20) requires, $[\overline{B}, B] \sim k\ I$, apart from $O(1)$ corrections due to $[\overline{A}, A]$. Therefore, B should be non-vanishing on the first diagonal:

$$B_{ac}\ =\ \delta_{a,c+1}\ b_{c+1}\ , \qquad\qquad c = 0, \ldots, N-2\ . \tag{4.25}$$

Eq. (4.24) implies evenly spaced Λ eigenvalues, $\ell_{a+1} - \ell_a = \omega$, and leaves the components b_c undetermined. The equation (4.18) for A reads:

$$A_{ab}\ \neq\ 0 \qquad \longrightarrow \qquad \Gamma_a\ +\ \Gamma'_b = (a-b)\omega\ -\ 2\ , \tag{4.26}$$

that can always be solved for Γ_a, Γ'_b. The constraints, $Z_a, Z'_b = 0, 1$, imply that A_{ab} has one non-vanishing element per row and column, at most, equal to one. If it had exactly one element per row and column, it would be the representation of a permutation, $\sigma \in \mathcal{S}_N$. Therefore, we can write:

$$A_{ab}\ =\ \delta_{a,\sigma(b)}\ a_{b+1}\ , \qquad\qquad a_b = 0, 1\ , \qquad\qquad \sigma\ \in\ \mathcal{S}_N\ . \tag{4.27}$$

We now consider the Gauss law (4.20): all terms in this equation are diagonal matrices, leading to a system of $(N-1)$ scalar equations for the A, B matrix elements $\{a_b, b_b\}$. Note that both matrices $\overline{A}A$ and $\overline{B}B$ are diagonal and thus their elements are positive integers in the semiclassical theory: $b_b^2 \in \mathbb{Z}_+$. After introducing,

$$\beta_b = b_b^2\ , \qquad\qquad \alpha_b = a_b^2\ , \tag{4.28}$$

we obtain the system:

$$\begin{aligned}
\beta_1 &= k - \alpha_1 + \alpha_{\sigma(1)} \ , \\
\beta_2 - \beta_1 &= k - \alpha_2 + \alpha_{\sigma(2)} \ , \\
\ldots &= \ldots\ldots \ , \\
\beta_{N-1} - \beta_{N-2} &= k - \alpha_{N-1} + \alpha_{\sigma(N-1)} \ .
\end{aligned} \tag{4.29}$$

The solution can be found by thinking to the expected shape of the droplet. The quantum state (3.34) is made of k generalized Slater determinants with homogeneous filling of N "one-particle" states[2]. Each one-particle state is expected to give a constant contribution to the density of the droplet: there are $(k-1)$ Laughlin terms and one term with $N/2$ "particles" in the second Landau levels spanning half of the angular momentum range, as confirmed by the quantum numbers, $E = \mathbf{B}N/2$ and $J = (k-1+1/2)\ N^2/2 + O(N)$. The contribution are additive in terms of angular momentum eigenvalues, $J \sim \mathrm{Tr}\ \overline{B}B = \sum_{i=1}^{N-1}\ \beta_i$ (the $O(N)$ contribution of Tr $\overline{A}A$ is subdominant for $N \to \infty$). Therefore, we expect, $\beta_i \sim (k-1)i$, for one half of the range, say $0 < i < N/2$, and $\beta_i \sim (k+1)i$ for the other half. Moreover, β_i should be continuous at $i = N/2$ in order to obey the corresponding equation with $\alpha_i = O(1)$. We take:

$$\begin{aligned}
\beta_i &= (k-1)\ i\ , & 0 < i \le \frac{N}{2}\ , \\
\beta_i &= (k+1)\ i\ -\ N, & \frac{N}{2} < i < N\ .
\end{aligned} \tag{4.30}$$

The solution for A is found by inspection: it has $N/2$ non-vanishing elements equal to one on the diagonal of the lower half sector.

Summarizing, the ansatz semiclassical ground state solution for $\nu^* = 2$ is given by (N even):

$$\begin{aligned}
B &= \sum_{n=1}^{N/2} \sqrt{n(k-1)}\ |\ n\rangle\langle n-1\ |\ +\ \sum_{n=\frac{N}{2}+1}^{N-1} \sqrt{n(k+1)-N}\ |\ n\rangle\langle n-1\ |\ , \\
A &= \sum_{n=0}^{\frac{N}{2}-1} |\ n+\frac{N}{2}\rangle\langle n\ |\ \ .
\end{aligned} \tag{4.31}$$

[2]This Fock-space analogy is meaningful for diagonal matrices, and may not be correct in general: its limitations will be discussed in section 4.2.2 .

In matrix form for $N = 4$, it reads:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{k-1} & 0 & 0 & 0 \\ 0 & \sqrt{2(k-1)} & 0 & 0 \\ 0 & 0 & \sqrt{3k-1} & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \qquad (4.32)$$

This solution has same energy $E = \mathbf{B}N/2$ of the quantum state (3.34) and same angular momentum $J = (k - 1/2)N^2/2 + O(N)$ to leading order (cf. Table 3.1). The matrix $R^2 = (B + \overline{A})(\overline{B} + A)$ contains off-diagonal terms from the mixed products: however, these give subdominant $O(1/\sqrt{N})$ corrections to the eigenvalues as is clear in a simple two-by-two matrix example. Thus, $\mathrm{Spec}(R^2) = \mathrm{Spec}(B\overline{B})(1+O(1/\sqrt{N}))$, confirming the earlier identification of droplet shape with angular momentum spectrum.

In Fig.4.1(a), the density (2.31) of the droplet of fluid is plotted by computing the exact spectrum for $N = 400$: up to finite-N fluctuations, this is a two-step constant density as anticipated. We recall that the same droplet shape is found for the Jain phenomenological states before their projection to the lowest Landau level [5]; the density becomes constant only after projection[3].

In chapter 3, we argued that the matrix ground states at $g = 0$ match one-to-one the phenomenological Jain states that are good ansatz in the physical limit $g = \infty$: the two sets of states become identical in the limit of both $\overline{X}, X$ diagonal, that can be formally reached at $g = \infty$. To establish a relation at the quantum level, we would need to consider the evolution of the matrix ground states as the coupling is varied in between, $0 < g < \infty$, and to check that the gap never vanishes, i.e. that there are no phase transitions in (B, g) plane (cf. Fig.3.3) separating the $g = 0$ and $g = \infty$ regions at these density (i.e. total angular momentum) values (see section 3.4). While this behaviour remains to be proved, it is supported by the result that the matrix ($g = 0$) and phenomenological ($g = \infty$) states have similar densities of incompressible fluids.

We also note, in chapter 2, that the solution (4.31) could also be obtained in the lowest-level theory (Chern-Simons matrix model) by replacing the A matrix with $N/2$ different "boundary" auxiliary fields $\psi \to \psi_\alpha$, $\alpha = 1, \ldots, N/2$. This multi-boundary generalization of Polychronakos' model has been considered in section 2.3: it naturally describes multicomponent droplets, i.e. $1/\nu = n/k$ for n boundary fields. However, the description of Jain states is rather unnatural, because the number of auxiliary

[3] The lowest-level projection in the matrix theory cannot be done at present, lacking an understanding of the $g > 0$ regime: at $g = 0$, it would give a trivial result because the Laughlin state is the unique lowest-level ground state for any k value.

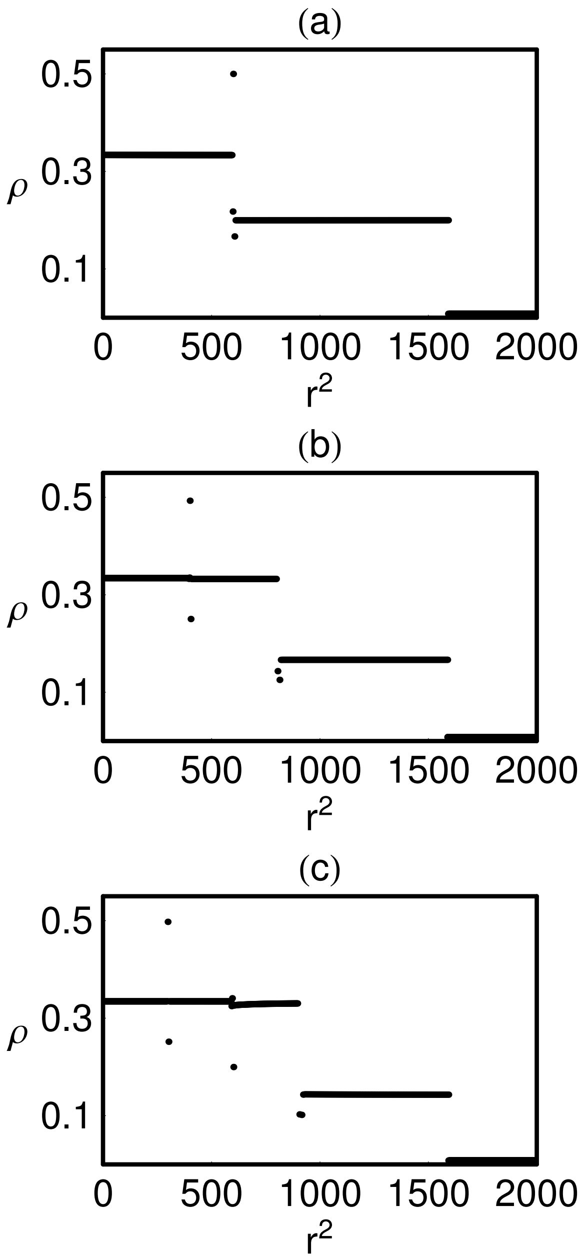

Figure 4.1: Plot of the density for the Jain matrix ground states with $1/\nu_{cl} = 1/\nu^* + k$, for $k = 4$ and $N = 400$: (a) $\nu^* = 2$ (4.31); (b) $\nu^* = 3$ (4.33); and (c) $\nu^* = 4$ (4.35).

fields is macroscopic and should be adjusted for each Jain state; moreover, this theory does not admit the physical limit of commuting matrices.

The solution (4.31) can be easily generalized for the theory with projection $A^3 \approx 0$, possessing a Jain ground state with $\nu^* = 3$: this is found at the specific density that is reached by tuning the boundary potential. The constraint now allows the A_{ab} components (4.27) to take values $a_b = 0, 1, \sqrt{2}$; we assume again a single non-vanishing element per row and column, eq. (4.27), otherwise the commutator, $[\overline{A}, A]$, would have off-diagonal terms that cannot be matched in the Gauss law equation (4.20). Therefore, the equations (4.29) are unchanged. Let us recall that the quantum solution contains $(k-1)$ Laughlin terms and the $\nu^* = 3$ piece that puts three particles in the same angular momentum state, ranging from zero to $N/3$. Thus, the B ansatz contains eigenvalues spaced by $(k-1)$ for 2/3 of the droplet and by $(k+2)$ for 1/3 of it. The matrix A that solves the Gauss law (4.29) involves elements on a diagonal extending for 2/3 of the matrix (N should be a multiple of 3). In conclusion:

$$
\begin{aligned}
B &= \sum_{n=1}^{2N/3} \sqrt{n(k-1)} \mid n\rangle\langle n-1 \mid + \sum_{n=\frac{2N}{3}+1}^{N-1} \sqrt{n(k+2)-2N} \mid n\rangle\langle n-1 \mid , \\
A &= \sum_{n=0}^{\frac{N}{3}-1} \mid n + \frac{N}{3}\rangle\langle n \mid + \sum_{\frac{N}{3}}^{\frac{2N}{3}-1} \sqrt{2} \mid n + \frac{N}{3}\rangle\langle n \mid . \qquad (4.33)
\end{aligned}
$$

In matrix form for $N = 6$:

$$
\begin{aligned}
B &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{k-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2(k-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3(k-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4(k-1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5k-2} & 0 \end{pmatrix} , \\
A &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \end{pmatrix} . \qquad (4.34)
\end{aligned}
$$

The droplet shape plotted in Fig.4.1(b) has again two steps, up to local fluctuations that vanish for $N \to \infty$.

The ansatz solution with $\nu^* = 4$ in the theory $A^4 \approx 0$ again involves a matrix B with two-speed spectrum and a matrix A with elements $a_b = 1, \sqrt{2}, \sqrt{3}$, on the diagonal extending over 3/4 of the matrix (N multiple of 4):

$$\begin{aligned} B &= \sum_{n=1}^{3N/4} \sqrt{n(k-1)} \mid n\rangle\langle n-1 \mid \; + \sum_{n=\frac{3N}{4}+1}^{N-1} \sqrt{n(k+3)-3N} \mid n\rangle\langle n-1 \mid , \\ A &= \sum_{n=0}^{\frac{N}{4}-1} \mid n+\frac{N}{4}\rangle\langle n \mid \; + \sum_{\frac{N}{4}}^{\frac{2N}{4}-1} \sqrt{2} \mid n+\frac{N}{4}\rangle\langle n \mid \; + \sum_{\frac{2N}{4}}^{\frac{3N}{4}-1} \sqrt{3} \mid n+\frac{N}{4}\rangle\langle n \mid \; . \end{aligned} \tag{4.35}$$

In matrix form for $N = 8$:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{k-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2(k-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3(k-1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4(k-1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5(k-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6(k-1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7k-3} & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 \end{pmatrix} . \tag{4.36}$$

The density for $N = 400$ and $k = 4$ is plotted in Fig.4.1(c).

4.2.2 Correspondence of semiclassical and quantum states

Here we provide a simple argument to support the identification of the semiclassical solutions with the quantum states of chapter 3. Consider first the correspondence for the Laughlin states, (2.44) and (2.27). We choose the gauge in which the expectation values of B matrix elements on the quantum state take the classical values (2.27) found in chapter 2, up to subleading corrections for $N \to \infty$. Let us rewrite the $N = 4$ wave function in terms of these non-vanishing terms only[4]:

$$\begin{aligned} \Phi_{k,\,gs} &= \left[\varepsilon^{a_1 a_2 a_3 a_4} \; \psi_{a_1} \; (\overline{B}\psi)_{a_2} \; (\overline{B}^2\psi)_{a_3} \; (\overline{B}^3\psi)_{a_4} \right]^k \\ &\sim \; \left[\varepsilon^{3210} \; \psi_3 \; (\overline{B}_{23}\psi_3) \; (\overline{B}_{12}\overline{B}_{23}\psi_3) \; (\overline{B}_{01}\overline{B}_{12}\overline{B}_{23}\psi_3) \right]^k \; . \end{aligned} \tag{4.37}$$

[4]Although this expansion should hold for $N \to \infty$, we write the $N = 4$ case for simplicity; the expression for general N can be easily inferred.

This "semiclassical wave function" describes "particles" with matrix indices, (01), (12), (23), in angular momentum states that precisely match the occupation numbers $\overline{B}_{ab}B_{ba}$ given by the classical solution (2.29), equal to $(k, 2k, 3k)$, respectively. This is a self-consistent argument for the correspondence of states: in the semiclassical $N \to \infty$ limit, the quantum states match the semiclassical solutions for the leading occupation numbers.

A similar relation holds for the $\nu^* = 2, 3, 4$ Jain states. For $\nu^* = 2$, the quantum wave function is (3.34); we evaluate it on the semiclassical non-vanishing $\overline{A}_{ab}, \overline{B}_{ab}$ values (4.31), given explicitly for $N = 4$:

$$\begin{aligned}\Phi_{k+1/2,\ gs} &= \left[\varepsilon^{a_1a_2a_3a_4}\ \psi_{a_1}\ (\overline{B}\psi)_{a_2}\ (\overline{B}^2\psi)_{a_3}\ (\overline{B}^3\psi)_{a_4}\right]^{k-1} \\ &\quad \times\varepsilon^{a_1a_2a_3a_4}\ \psi_{a_1}\ (\overline{B}\psi)_{a_2}\ (\overline{A}\psi)_{a_3}\ (\overline{A}\overline{B}\psi)_{a_4} \\ &\sim \left[\varepsilon^{3210}\ \psi_3\ (\overline{B}_{23}\psi_3)\ (\overline{B}_{12}\overline{B}_{23}\psi_3)\ (\overline{B}_{01}\overline{B}_{12}\overline{B}_{23}\psi_3)\right]^{k-1} \\ &\quad \times\varepsilon^{3210}\ \psi_3\ (\overline{B}_{23}\psi_3)\ (\overline{A}_{13}\psi_3)\ (\overline{A}_{02}\overline{B}_{23}\psi_3)\ . \end{aligned} \tag{4.38}$$

The "one-particle" occupancies of both energy and angular momentum states given by the wave function again match the expectation values of the corresponding number operators, $\overline{A}_{ab}A_{ba}$ and $\overline{B}_{ab}B_{ba}$, of the classical solution. The correspondence extends to the other $\nu^* = m$ states that have spectrum of occupancies given by (4.33,4.35). This argument support our belief that the large N limit of the matrix theory is semiclassical for the incompressible fluid ground states (piecewise constant density) and their small excitations.

4.2.3 Generalized Jain states

In the analysis of [13], we found other quantum solutions to the constraint $A^m \approx 0$, for $m \geq 3$, besides Jain composite fermion wave functions. They were presented in chapter 3, eq. (3.41) and summarized in Table 3.1: these are analogs of Jain's generalized hierarchical states, made by products of two or more wave functions with higher-level fillings ($p_i > 1$). In the semiclassical analysis, we find that some of these states have corresponding solutions with piecewise constant density, while most of them do not. Besides, we find spurious ground states that are allowed by the simplistic quadratic boundary potential used in (4.17). Let us describe these solutions in turn.

Spurious solutions

There is a variant of the composite-fermion solution for $m = 3, 4, \ldots$, Eqs. (4.31,4.33), where the A matrix elements take the same values, but their positions are permuted. For $m = 3$, this is:

$$
\begin{aligned}
B &= \sum_{n=1}^{N/3} \sqrt{n(k-2)} \mid n\rangle\langle n-1 \mid \; + \sum_{n=\frac{N}{3}+1}^{N-1} \sqrt{n(k+1)-N} \mid n\rangle\langle n-1 \mid , \\
A &= \sum_{n=0}^{\frac{N}{3}-1} \sqrt{2} \mid n + \frac{N}{3}\rangle\langle n \mid \; + \sum_{n=\frac{N}{3}}^{\frac{2N}{3}-1} \mid n \; + \; \frac{N}{3}\rangle\langle n \mid \; .
\end{aligned} \tag{4.39}
$$

The total energy and angular momentum values are the same as those of the $m = 3$ Jain state, $E = \mathbf{B}N$, $\mathcal{J} \sim (k-2/3)N^2/2$ (cf. Table 3.1). The corresponding B matrix again describes a two-step droplet. In order to find the corresponding quantum state, we use the classical-quantum correspondence of the previous section. The single-particle occupation numbers of the classical solution, for e.g. $N = 6$, correspond to those of $(k-2)$ Laughlin factors plus the occupations $(3, 6, 9)$, $(2, 2, 1, 1)$ and (12) for the components $(\overline{B}_{23}, \overline{B}_{34}, \overline{B}_{45})$, $(\overline{A}_{02}\overline{A}_{13}\overline{A}_{24}\overline{A}_{35})$ and ψ_5, respectively. These components should fit into two wave function of the type (3.32) that obey $A^2 \approx 0$ (the product wave function obeying $A^3 \approx 0$). The solution, rewritten in gauge invariant form, is:

$$
\begin{aligned}
\Phi \;=\; & (\Phi_{1,\,gs})^{k-2} \left(\varepsilon^{a_1a_2a_3a_4a_5a_6} \; \psi_{a_1} \; (\overline{B}\psi)_{a_2} \; (\overline{B}^2\psi)_{a_3} \; (\overline{B}^3\psi)_{a_4} \; (\overline{AB}^2\psi)_{a_5} \; (\overline{AB}^3\psi)_{a_6}\right) \\
& \times \left(\varepsilon^{a_1a_2a_3a_4a_5a_6} \; \psi_{a_1} \; (\overline{B}\psi)_{a_2} \; (\overline{A}\psi)_{a_3} \; (\overline{AB}\psi)_{a_4} \; (\overline{AB}^2\psi)_{a_5} \; (\overline{AB}^3\psi)_{a_6}\right) .
\end{aligned} \tag{4.40}
$$

In this state, we recognize that some strands do not have minimal length: thus, this is an excited state for a Hamiltonian with more realistic boundary terms (3.29) realizing finite-box conditions[5]. The expectation value of the higher boundary potential $\langle \mathrm{Tr}\left(\overline{B}^2 \; B^2\right)\rangle$ on this state is actually larger than that of the $m = 3$ Jain state (4.33) with same energy and angular momentum: this confirms our interpretation of the solution (4.39).

[5] The quadratic potential used in (4.17) is known to yield such degeneracies ([25]).

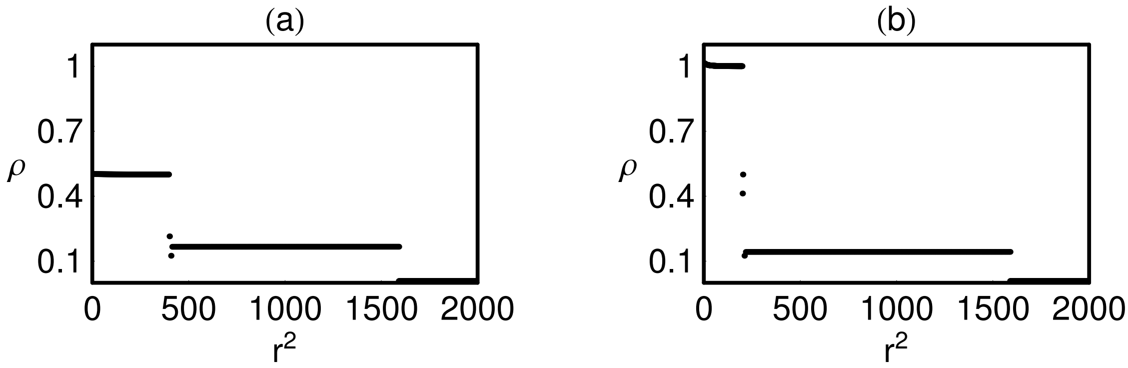

Figure 4.2: Plot of the density for the generalized Jain states: (a) $1/\nu_{cl} = 1/k - 1$ (4.42); (b) $1/\nu_{cl} = 1/k - 3/2$ (4.44), with $k = 4$ and $N = 400$.

Other two-step density states

Among the generalized Jain state in Table 3.1, there are those made by two kinds of terms, as follows:

$$\Phi_{\frac{1}{\nu},\ gs} \ = \ (\Phi_{1,\ gs})^{k-n} \left(\Phi_{\frac{1}{2},\ gs}\right)^{n} \ , \qquad \frac{1}{\nu} \ = \ \frac{n}{2} + (k-n) + 1 \ , \qquad n = 2, 3, \dots \ , \tag{4.41}$$

in the notation of Eq.(3.41). They obey, $A^{n+1} \approx 0$, for $n = 2, 3, \dots$, and violate the composite-fermion transformation (4.23) [5]. In the droplet interpretation of classical solutions of section 4.2.1, we expect a density of R^2 eigenvalues equal to $(k+n)$ for half of the spectrum and $(k-n)$ for the second half. The ansatz has the two-block structure of the solution (4.31), with maximal value $A_{ab} = n$ in agreement with the constraint.

The first non-trivial value is $n = 2$, i.e. $m = 3$ in Table 3.1:

$$\begin{aligned} B \ &= \ \sum_{n=1}^{N/2} \sqrt{(k-2)n} \mid n\rangle\langle n-1 \mid \ + \sum_{n=\frac{N}{2}+1}^{N-1} \sqrt{(k+2)n - 2N} \mid n\rangle\langle n-1 \mid, \\ A \ &= \ \sum_{n=0}^{\frac{N}{2}-1} \sqrt{2} \mid \frac{N}{2} + n\rangle\langle n \mid . \end{aligned} \tag{4.42}$$

In matrix representation for $N = 4$:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{k-2} & 0 & 0 & 0 \\ 0 & \sqrt{2(k-2)} & 0 & 0 \\ 0 & 0 & \sqrt{3k-2} & 0 \end{pmatrix} \ , \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \ . \tag{4.43}$$

The analogous state for $n = 3$, corresponding to $(\Phi_{1/2,\ gs})^3\ (\Phi_{1,\ gs})^{k-3}$, is:

$$
\begin{aligned}
B &= \sum_{n=1}^{N/2} \sqrt{(k-3)n}\mid n\rangle\langle n-1\mid + \sum_{n=\frac{N}{2}+1}^{N-1} \sqrt{(k+3)n-3N}\mid n\rangle\langle n-1\mid, \\
A &= \sum_{n=0}^{\frac{N}{2}-1} \sqrt{3}\mid \frac{N}{2}+n\rangle\langle n\mid,
\end{aligned}
\qquad (4.44)
$$

and in matrix form for $N = 4$:

$$
B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{k-3} & 0 & 0 & 0 \\ 0 & \sqrt{2(k-3)} & 0 & 0 \\ 0 & 0 & \sqrt{3(k-1)} & 0 \end{pmatrix} , \qquad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \end{pmatrix} . \qquad (4.45)
$$

In Fig.4.2, we plot the density for these generalized Jain states: these are droplets with two-step constant density similar to that of composite-fermion states. At present, we do not have strong arguments to dispose of these additional states: this issue will be further discussed at the end of the chapter.

States with many-step density

Other generalized matrix Jain states (3.41) in Table 3.1, for $A^m \approx 0$, $m \geq 4$, are made by the product of three or more different terms. The simplest one for $m = 4$ is $\Phi_\nu = \Phi_1^{k-2}\ \Phi_{1/2}\ \Phi_{1/3}$ with energy $E/\mathbf{B} = 3N/2$ and angular momentum $J \sim (k-1-1/6)N^2/2$. Within the droplet interpretation of classical solutions discussed before, we seek for a three-step solution (N multiple of 6),

$$
\beta_i \sim i(k{-}2)\,, \quad 1 < i < \frac{N}{2}\,; \qquad \beta_i \sim ik\,, \quad \frac{N}{2} < i < \frac{2N}{3}\,; \qquad \beta_i \sim i(k{+}3)\,, \quad \frac{2N}{3} < i < N\,.
$$

However, there is no simple A_{ab} solution with entries $(0, 1, \sqrt{2}, \sqrt{3})$, that fulfills the Gauss law equation for the same energy and angular momentum of the quantum state. We proved this fact for an ansatz with piecewise constant density making up to 6 steps. A four-step solution (see Fig.4.3) can be found with quantum numbers differing macroscopically from the quantum values, $E \sim 1.4\ N, J \sim (k-1-0.14)N^2/2$: in particular, the larger angular momentum identifies it as an excited state. Presumably, the quantum state can be better approximated by allowing a very large number of steps, leading to a modulated (or singular) density profile in the large-N limit. This result indicates that most of the multi-component generalized matrix quantum states found in chapter 3 for projections $A^m \approx 0$, $m \geq 4$, are not semiclassical incompressible fluids.

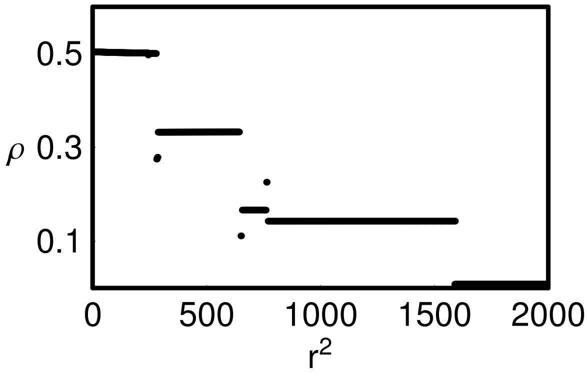

Figure 4.3: Plot of the density of the 4-step excited state in the $A^4 \approx 0$ theory, $1/\nu_{cl} \sim k-1-0.14$, with $k=4$ and $N=400$.

4.2.4 Quasi-holes solutions

As shown in chapter 3, the $g=0$ matrix theory, projected by $A^m \approx 0$, possess quasi-hole excitations above the $\nu^* = m$ Jain ground states. In this section we give the corresponding semiclassical solutions for $\nu^*=2$, corresponding to deformation of the density of solution (4.31) in fig 4.1(a).

The classical equation of motion for A and B, eq. (4.18,4.19), are linear and admit a general solution for excitations:

$$\begin{aligned} A_{ab}(t) &= \mathrm{e}^{-i(\Gamma_a+2)t} \left(\mathrm{e}^{it\Lambda}\ A(0)\ \mathrm{e}^{-it\Lambda}\right)_{ab}\ \mathrm{e}^{-i\left(\Gamma'_b\right)t} \ , \\ B(t) &= \mathrm{e}^{-i\omega t}\ \mathrm{e}^{it\Lambda}\ B(0)\ \mathrm{e}^{-it\Lambda} \ . \end{aligned} \tag{4.46}$$

Therefore, we should only solve the Gauss law (4.20) and the constraint (4.21).

In the two-step fluid density in fig 4.1(a), one can have more than one quasi-hole corresponding to punching either of the two possible fluids. A hole in the complete fluid is obtained by generalizing the quasi-hole of the Laughlin state (2.2), found in ref.[15]: it is a deformation of the B matrix (4.31) that describes a quasi-hole of charge $q>0$ situated at the origin. The matrix A remains unchanged:

$$\begin{aligned} B &= \sum_{n=1}^{N/2} \sqrt{(k-1)(n+q)}\ \mid n\rangle\langle n-1\mid \\ &+ \sum_{n=\frac{N}{2}+1}^{N-1} \sqrt{(k-1)q+n(k+1)-N}\ \mid n\rangle\langle n-1\mid\ +\ \sqrt{(k-1)q}\ \mid 0\rangle\langle N-1\mid\ , \\ A &= \sum_{n=0}^{\frac{N}{2}-1} \mid n+\frac{N}{2}\rangle\langle n\mid \ . \end{aligned} \tag{4.47}$$

In matrix representation:

$$B = \begin{pmatrix} 0 & 0 & 0 & \sqrt{q(k-1)} \\ \sqrt{(1+q)(k-1)} & 0 & 0 & 0 \\ 0 & \sqrt{(2+q)(k-1)} & 0 & 0 \\ 0 & 0 & \sqrt{(3+q)k-1-q} & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.48}$$

The corresponding density is shown in Fig.4.4(a).

A quasi-hole only affecting the upper layer of the $\nu^* = 2$ fluid is shown in Fig.4.4(b). It is given by the solution:

$$\begin{aligned} B &= \sum_{n=0}^{q} \sqrt{(k+1)(n+1)} \mid n+1\rangle\langle n \mid + \sum_{n=q+1}^{\frac{N}{2}+q} \sqrt{(k-1)(\alpha-q-1+n)} \mid n+1\rangle\langle n \mid \\ &+ \sum_{n=\frac{N}{2}+q+1}^{N-2} \sqrt{(k+1)(\beta+n-\frac{N}{2}-q-1)} \mid n+1\rangle\langle n \mid, \\ A &= \sum_{n=0}^{q} \mid n\rangle\langle q+1+n \mid + \sum_{n=0}^{\frac{N}{2}-q-2} \mid \frac{N}{2}+q+1+n\rangle\langle 2q+2+n \mid , \end{aligned} \tag{4.49}$$

with $\alpha = \frac{q+k(2+q)}{k-1}$, $\beta = \frac{2+q+(k-1)\frac{N}{2}}{k+1}$ and q a positive integer. In matrix representation for $N = 8$ and $q = 1$, it reads:

$$\begin{aligned} B &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{(k+1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2(k+1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3k+1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4k} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{5k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{6k-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{7k-1} & 0 \end{pmatrix}, \\ A &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.50}$$

The displacement from the origin of the upper layer corresponds to $\Delta\mathcal{J} = (k+1)(q+1)$.

It is also possible to create a circular depletion in the whole fluid, of size (charge)

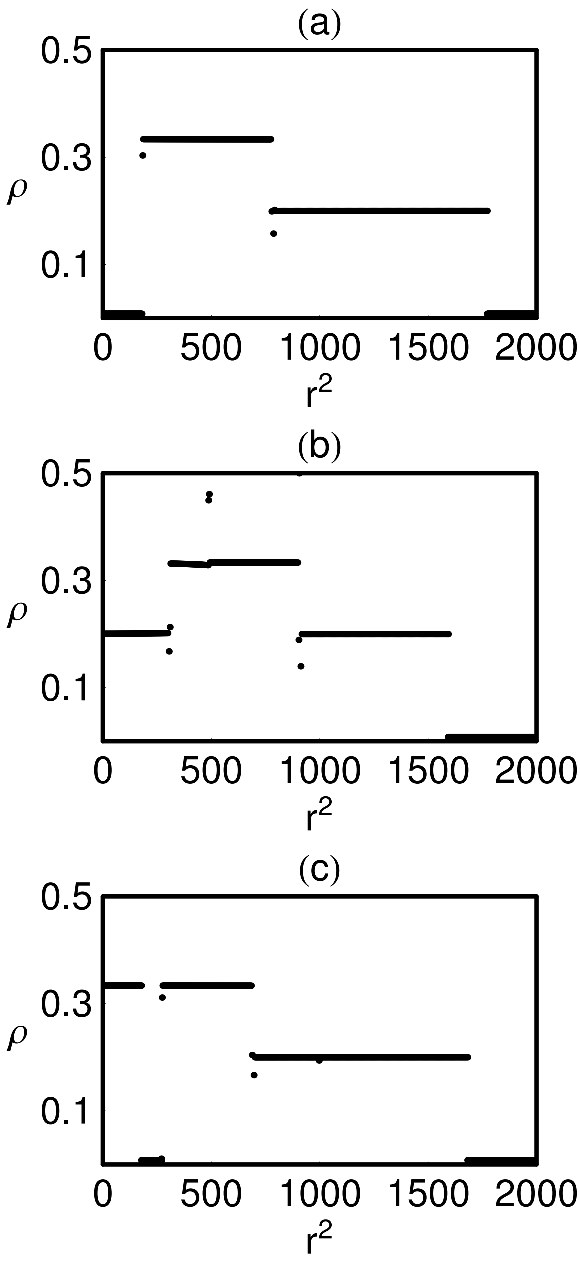

Figure 4.4: Plot of the density of the $\nu^* = 2$ Jain ground state, $1/\nu_{cl} = k + 1/2$, for $k = 4$ and $N = 400$, including: (a) one quasi-hole in the origin (4.47) with $q = 60$; (b) a quasi-hole in the upper layer of the fluid (4.49) with $q = 60$; (c) the quasi-hole out of the origin (4.51) with $q = 30$ and $r = 60$.

$\Delta\mathcal{J} = q$ outside the origin at a distance $\Delta\mathcal{J} = (k-1)(r-1)$ (see fig.4.4(c)):

$$\begin{aligned}
B &= \sum_{n=0}^{r-1} \sqrt{(k-1)(n+1)} \mid n+1\rangle\langle n \mid + \sum_{n=r}^{\frac{N}{2}-2} \sqrt{(k-1)(n+1+q)} \mid n+1\rangle\langle n \mid \\
&+ \sqrt{(k-1)q} \mid r\rangle\langle n-1 \mid + \sqrt{(\frac{N}{2}+q)(k-1)} \mid \frac{N}{2}\rangle\langle \frac{N}{2}-1 \mid \\
&+ \sum_{n=\frac{N}{2}}^{N-2} \sqrt{(k+1)(n+1) - N + (k-1)q} \mid n+1\rangle\langle n \mid, \\
A &= \sum_{n=0}^{\frac{N}{2}-1} \mid n+\frac{N}{2}\rangle\langle n \mid . \qquad (4.51)
\end{aligned}$$

In matrix representation for $N = 6$ and $q = 2$:

$$\begin{aligned}
B &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{(k-1)} & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2(k-1)} & 0 & 0 & 0 & \sqrt{r(k-1)} \\
0 & 0 & \sqrt{(3+r)(k-1)} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{(4+r)k-2-r} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{(5+r)k-1-r} & 0
\end{pmatrix}, \\
A &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}. \qquad (4.52)
\end{aligned}$$

In this case, the solution of the Gauss law is obtained in terms of a two-component auxiliary field ψ, and it holds for $rq \ll N$, namely the depletion would move back to the origin in the scaling limit $N \to \infty$.

In this chapter, after providing better forms for the projection, $A^m \approx 0$, limiting state degeneracy, we have obtained the semiclassical ground states of the theory. They correspond to the quantum states found in the previous chapter [13], that reproduce the Jain composite-fermion construction of phenomenological wave functions. The density of states in the main Jain series, $\nu = m/(mk+1)$, has been found to be that of incompressible fluids: this confirms our expectation that the matrix states at $g = 0$ are not too different from the physical states at $g = \infty$.

The study of the phase diagram of the matrix theory is clearly necessary to make better contact between the nice results ($g = 0$) and the physical regime ($g = \infty$), upon varying the potential $V = g\ \mathrm{Tr}[\overline{X}, X]^2$. We expect that the relevant incompressible fluid states have a smooth evolution for $g > 0$ and we plan to include the quartic potential in the semiclassical analysis by means of a mean-field approximation.

The explicit semiclassical solutions in this paper can also be useful to study the symmetries and algebraic properties of matrix ground states. We would like:

- To make contact with the $SU(m)$ symmetry of the conformal field theories describing the edge excitations of Jain states [25].
- To find a projection of states more refined than $A^m \approx 0$, that could discriminate the hierarchical Jain states from the generalized (unstable) ones. Such an expectation is based on the general belief that the observed Hall states should be uniquely characterized by algebraic conditions and gauge invariance, rather than by detailed dynamics, because they are exceptionally robust and universal.

Chapter 5

Conclusion

In this thesis we have studied the matrix models applied to the physics of the Fractional QHE. In chapter 2, we started with the Susskind's proposal of the Chern-Simons matrix model as a effective theory [14] and found the very interesting result, that the ground state of this theory is the Laughlin wave function. Nevertheless, the Jain states are not contained in the Chern-Simons matrix model, and a larger theory is necessary.

In chapter three we have presented our first work in which we generalized the Chern-Simons matrix model, proposing as a effective theory for the FQHE, the Maxwell Chern-Simons matrix theory. We studied two interesting limits of the theory corresponding to the $g = 0$ and $g = \infty$ regimes. As we have shown (cf. section 3.2.2) the ground states in $g = 0$ are highly degenerate in the energy, due to the noncommutativity between matrices. To solve this problem, we introduced a class of projectors given by $A^m_{ab}; a, b = 1, .., N; m = 0, 1, ..$ and we obtained, for each n, a truncated Fock space with a non-degenerate gauge invariant ground state given by a matricial version of the Jain wave function of filling fraction $\nu = \frac{m}{2km+1}$. It becomes the Jain state for electrons in the case of diagonal matrices. Excitations of the ground states correspond to matrix generalizations of the Jain quasi-holes and quasi-particles. For another hand, in the $g = \infty$ regime the theory can be reduced to the Landau levels plus a $O(1/r^2)$ two-dimensional interaction that is a good effective description of the physics of the FQHE.

A crucial point in our proposal is the conjecture of smooth connection (no phase transition) between the $g = 0$ and $g = \infty$ regimes. This conjecture is supported principally, by two arguments:

- As said before, the $g = 0$ theory presents matricial Jain states that become the Jain wave functions in the limit of diagonal matrices. We have shown in chapter three that this limit corresponds with the $g = \infty$ regime of the Maxwell Chern-Simons theory, that is the physical limit, and contains as ground states almost exactly the Jain wave functions.

- Another argument is the similar behaviors in the $g = 0$ and $g = \infty$ theories. In chapter four we analyzed the semiclassical limit of the $g = 0$ theory and found that the matricial Jain state are made of droplets of incompressible fluid with piece-wise constant density; this is the same density shape of the phenomenological Jain states ($g = \infty$), before the projection to the lowest Landau level.

In conclusion, the Maxwell Chern-Simons matrix theory makes possible to connect an exactly solvable matrix theory ($g = 0$) with the physical theory describing the FQHE ($g = \infty$). The interpolation from $g = 0$ to higher values of g, will permit a better understanding of the physics of the FQHE; in particular its relevant non-perturbative effects.

We hope to continue our analysis of the theory in the region $g > 0$ in the future: we plan to follow the evolution of the $g = 0$ droplet ground states by means of semiclassical and mean field methods.

Author's Publications

A. Cappelli and I. D. Rodriguez, *Jain states in a matrix theory of the quantum Hall effect,* JHEP **0612** (2006) 056, hep-th/0610269.

A. Cappelli and I. D. Rodriguez, *Semiclassical Droplet States in Matrix Quantum Hall Effect,* [hep-th/0711.4982] (submitted to JHEP).

Appendix A

A.1 Moyal star product

The Moyal product is defined as follows:

$$(f * g)(x) = e^{\frac{i}{2}\theta\epsilon^{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}} f(x_1)g(x_2)\mid_{x_1=x_2=x}, \tag{A.1}$$

with θ a real parameter. In the limit of small θ, equation (A.1) implies:

$$(f * g)(x) = f(x)g(x) + \frac{i}{2}\theta\{f, g\}\mid_{x=x_1=x_2} + O(\theta^2), \tag{A.2}$$

where $\{\ ,\ \}$ indicates usual Poisson brackets. Using (A.2) we can calculate the commutator between $f(x)$ and $g(x)$,

$$[f, g] = f * g - g * f = i\theta\{f, g\}\mid_{x=x_1=x_2} + O(\theta^2), \tag{A.3}$$

that corresponds to the Poisson brackets, to first order in θ. In particular, choosing $f = x^1$ and $g = x^2$ we have noncommutative coordinates,

$$\left[x^1, x^2\right] = x^1 * x^2 - x^2 * x^1 = i\theta, \tag{A.4}$$

showing that θ measures noncommutativity.

The Moyal product plays a crucial role in "phase space quantum mechanics" that amounts to associating classical distributions in phase space with quantum mechanical Hermitian operators. This map was first discussed by Wigner in 1932.

A.1.1 Wigner quasi-probability distribution

In Wigner's idea, quantum effects in phase space, are accounted for deformations of the classical coordinates.

The phase space distribution of a quantum state $\mid \psi\rangle$, called Wigner distribution function $u_\psi(x,y)$, is defined by:

$$\begin{aligned} u_\psi(x,y) &= \langle\psi \mid: \delta(\hat{x}-x)\delta(\hat{y}-y) :\mid \psi\rangle \\ &= \left(\frac{1}{2\pi}\right)^2 \int d\alpha d\beta \langle\psi \mid e^{i\alpha(\hat{x}-x)+i\beta(\hat{y}-y)} \mid \psi\rangle \\ &= \frac{1}{2\pi}\int d\beta e^{-i\beta y}\bar{\psi}(x-\frac{\beta\theta}{2})\psi(x+\frac{\beta\theta}{2})\ , \end{aligned} \tag{A.5}$$

where x and y are the phase space coordinates, $[\hat{x},\hat{y}] = i\theta$, and the symbol $:\ :$ indicates the Weyl ordering of the noncommuting operators $\hat{x}$,$\hat{y}$, that is symmetric between them, as better explained later.

The function (A.5) verifies,

$$\int dy\ u_\psi(x,y) = \mid \psi(x) \mid^2, \quad \int dx\ u_\psi(x,y) = \frac{\mid \tilde{\psi}(\frac{y}{\theta}) \mid^2}{2\pi}. \tag{A.6}$$

Namely, it is the probability distribution in one coordinate, once traced the other one. Clearly, the Wigner distribution function cannot be positive definite as a function of both x and y: it is a quasi-probability distribution.

A.1.2 Phase space representation of an operator

The expectation value of an operator is represented by the average with the distribution function as in classical statistical mechanics:

$$\langle\psi \mid: F(\hat{x},\hat{y}) :\mid \psi\rangle = \int dxdy F(x,y)u_\psi(x,y). \tag{A.7}$$

From (A.7) we find that the Wigner operator: $F(\hat{x},\hat{y})$: is given by,

$$\begin{aligned} : F(\hat{x},\hat{y}) : &= \left(\frac{1}{2\pi}\right)^2 \int dxdyd\alpha d\beta F(x,y)e^{i\alpha(\hat{x}-x)+i\beta(\hat{y}-y)} \\ &= \int d\alpha d\beta \tilde{F}(\alpha,\beta)e^{i\alpha\hat{x}+i\beta\hat{y}}, \end{aligned} \tag{A.8}$$

where $\tilde{F}(\alpha,\beta)$ is the Fourier transform of $F(x,y)$.

Expression (A.8) defines the Wigner map that associate a function on the phase space with a quantum operator. It can be written in another way,

$$: F(\hat{x},\hat{y}) := F(-i\frac{\partial}{\partial\alpha}, -i\frac{\partial}{\partial\beta})e^{i\alpha\hat{x}+i\beta\hat{y}} \mid_{\alpha=\beta=0}, \tag{A.9}$$

that makes it clear that : $F(\hat{x},\hat{y})$: corresponds to the classical function $F(x,y)$ evaluated on the operators $\hat{x}$ and $\hat{y}$ in totally symmetric combinations, e.g. $xy \to \frac{1}{2}(\hat{x}\hat{y}+\hat{y}\hat{x})$; $x^2y \to \frac{1}{3}(\hat{x}^2\hat{y}+\hat{x}\hat{y}\hat{x}+\hat{y}\hat{x}^2)$, etc.

A.1.3 Representation of operator algebra

The product between two quantum operators is given by the following convolution of Fourier transforms of the associated classical functions,

$$
\begin{aligned}
:F(\hat{x},\hat{y})::G(\hat{x},\hat{y}): &= \int d\alpha d\beta d\alpha' d\beta' \tilde{F}(\alpha,\beta)\tilde{G}(\alpha',\beta') e^{i\alpha\hat{x}+i\beta\hat{y}} e^{i\alpha'\hat{x}+i\beta'\hat{y}} \\
&= \int d\alpha d\beta d\alpha' d\beta' \tilde{F}(\alpha-\alpha')\tilde{G}(\alpha') e^{\frac{i\theta}{2}(\alpha\beta'-\beta\alpha')} e^{i\alpha\hat{x}+i\beta\hat{y}} \\
&= \int d\alpha d\beta \tilde{H}(\alpha,\beta) e^{i\alpha\hat{x}+i\beta\hat{y}}, \qquad \text{(A.10)}
\end{aligned}
$$

with $\tilde{H}(\alpha,\beta)$ the Fourier transform of $H(x,y)$ given by:

$$
H(x,y) = F(x,y) e^{\frac{i\theta}{2}(\overleftarrow{\partial_x}\overrightarrow{\partial_y}-\overleftarrow{\partial_y}\overrightarrow{\partial_x})} G(x,y) = (F*G)(x,y), \qquad \text{(A.11)}
$$

where $*$ indicates the Moyal product. Thus, the Wigner map allow the study of a quantum mechanical system in phase space, characterized by noncommutative coordinates $[\hat{x},\hat{y}] = i\theta$: the operators on the Hilbert space, $F(\hat{x},\hat{y})$, $G(\hat{x},\hat{y})$, are replaced by functions on the phase space, $F(x,y)$, $G(x,y)$, and their noncommutative product is given by the Moyal form.

A.2 Map of the Chern-Simons matrix model to the noncommutative field theory

The Chern-Simons matrix model is given by the Lagrangian,

$$\mathcal{L} = \frac{\mathbf{B}}{2}\epsilon^{ij}Tr(X_i D_t X_j) + \mathbf{B}\theta A_0, \tag{A.12}$$

where A_0 and $X_i, i = 1, 2$, are $N \times N$ matrices, $\mathbf{B}$ is the magnetic field and $D_t X^i = \dot{X}^i - i\,[A_0, X^i]$ with $i, j = 1, 2$.

We introduce the classical ground states solutions x^i that verify,

$$\left[x^i, x^j\right] = i\theta\epsilon^{ij}. \tag{A.13}$$

Note that (A.13) is valid only for $N \to \infty$ matrices.

We consider small perturbations $A_j(x^i)$ of X^i around the classical solutions x^i,

$$X^i = x^i + \theta\epsilon^{ij}A_j(x^i). \tag{A.14}$$

Any matrix can be expressed in terms of finite sums of products $e^{ipx^1}e^{iqx^2}$; so the $N \times N$ matrices A_i can be thought of as functions of the x^i's.

Replacing (A.14) in (A.12) and dropping total time derivatives we find,

$$\begin{aligned}
\mathcal{L} &= \frac{\mathbf{B}}{2}Tr(-\theta\dot{A}_1(x_1 + \theta A_2) + iA_0\,[x_1 + \theta A_2, x_2 - \theta A_1] - \theta\dot{A}_2(x_2 - \theta A_1) - \\
&\quad iA_0\,[x_2 - \theta A_1, x_1 + \theta A_2]) + \mathbf{B}\theta A_0 \\
&= \frac{\mathbf{B}\theta^2}{2}Tr\left(A_1\dot{A}_2 - \dot{A}_1 A_2 + 2A_0(\partial_2 A_1 - \partial_1 A_2) + 2iA_0\,[A_1, A_2]\right).
\end{aligned} \tag{A.15}$$

In (A.15), we used $[x^i, f(x^1, x^2)] = i\theta\epsilon^{ij}\partial_j f$.

Finally, using the antisymmetric tensor $\epsilon^{\mu\nu\rho}$, expression (A.15) can be written as:

$$\mathcal{L} = \frac{\mathbf{B}\theta^2}{2}\mathrm{Tr}\left(-\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{2i}{3}\epsilon^{\mu\nu\rho}A_\mu A_\nu A_\rho\right). \tag{A.16}$$

Now we take $N \to \infty$ and pass to the continuum limit where the $N \times N$ matrices A_i are mapped into smooth functions $A_i(x_i)$ of the noncommutative coordinates x_i. We identify $\theta Tr \to \frac{1}{2\pi}\int dx_1 dx_2$ where $\theta N = \frac{V}{2\pi} = \frac{1}{2\pi\rho_0}$ with V and ρ_0 the volume and density in the x's coordinates.

In a noncommutative space, as we shown in Appendix A.1, the ordinary product between functions must be replaced by the $*$ product. Thus in the $N \to \infty$

limit the Chern-Simons matrix model is equivalent to the Chern-Simons non-commutative theory,

$$\mathcal{L} = -\frac{\mathbf{B}\theta}{4\pi} \int d^2x \left(\epsilon^{\mu\nu\rho} A_\mu * \partial_\nu A_\rho - \frac{2i}{3} \epsilon^{\mu\nu\rho} A_\mu * A_\nu * A_\rho \right) . \tag{A.17}$$

A.3 Goldstone-Hoppe Matrix regularization

In the paper [44], Goldstone and Hoppe studied a membrane theory in which the membrane surface Σ has the topology of the sphere S^2. They introduced a regularization procedure in which functions on the membrane surface are mapped to $N \times N$ matrices. This procedure is completely classical, and after it is carried out we can quantize the system just like any other classical system with a finite number of degrees of freedom. Now we review the case with S^2 topology, but the argument can be generalized to arbitrary topologies.

The world-sheet of the membrane surface at fixed time can be described by a unit sphere with an $SO(3)$ invariant canonical symplectic form. Functions on the membrane depend on the Cartesian coordinates ξ_1, ξ_2, ξ_3 on the unit sphere, i.e $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$. The Poisson brackets of these functions are given by:

$$\{\xi_A, \xi_B\} = \epsilon_{ABC}\xi_C \quad A, B, C = 1, 2, 3. \tag{A.18}$$

Observe that (A.18) is the same as the algebra of $SU(2)$ generators. Thus, we can associate these coordinates on S^2 with matrices generating $SU(2)$, i.e.

$$\xi_A \to \frac{2}{N} J_A, \tag{A.19}$$

where $J_A, A = 1, 2, 3$ are generators of the N-dimensional representation of $SU(2)$, satisfying the commutation relations,

$$-i\,[J_A, J_B] = \epsilon_{ABC} J_C. \tag{A.20}$$

An arbitrary function on the membrane can be expressed in terms of spherical harmonics, as follows:

$$f(\xi_1, \xi_2, \xi_3) = \sum_{l,m} c_{lm} Y_{lm}(\xi_1, \xi_2, \xi_3). \tag{A.21}$$

The spherical harmonics can be written as:

$$Y_{lm}(\xi_1, \xi_2, \xi_3) = \sum t^{(lm)}_{A_1,\ldots,Al} \xi_{A_1} \ldots \xi_{A_l}, \tag{A.22}$$

where the coefficient $t^{(lm)}_{A_1,\ldots,Al}$ are symmetric and traceless (because $\xi_A \xi_A = 1$). Using (A.19) in (A.22) we obtain the correspondence,

$$Y_{lm}(\xi_1, \xi_2, \xi_3) \to \mathbf{Y}_{lm} = \left(\frac{2}{N}\right)^l \sum t^{(lm)}_{A_1,\ldots,Al} J_{A_1} \ldots J_{A_l}. \tag{A.23}$$

For finite N, only the spherical harmonics with $l < N$ can be constructed because the $J_{A_i}, i = 1, .., l$ are $N \times N$ matrices and the product of N or more J_A's is linearly dependent. Using (A.23) a function on the membrane can be replaced by the matrix,

$$f(\xi_1, \xi_2, \xi_3) \rightarrow F = \sum_{l<N,m} c_{lm} \mathbf{Y}_{lm}. \tag{A.24}$$

Based on these correspondence, Goldstone and Hoppe showed that the Poisson brackets in the membrane theory can be replaced by matrix commutators as follows:

$$\{f, g\} \rightarrow \frac{-iN}{2}[F, G], \tag{A.25}$$

and the integral over the membrane at fixed time can be replaced with matrix trace,

$$\frac{1}{4\pi} \int d^2\sigma f \rightarrow \frac{1}{N} TrF. \tag{A.26}$$

As mentioned in chapter two, the fluid Lagrangian (2.1) is invariant under area preserving diffeomorphisms. The Goldstone-Hoppe prescription can be interpreted as a regularization of the area preserving diffeomorphisms in terms of the $U(N)$ $(N = \infty)$ rotations of matrices [44], in which spatial coordinates are mapped into $N \times N$ hermitian matrices.

A.4 The low-energy effective action of D0-branes in String theory

In this appendix, we consider ten-dimensional Minkowski space, with a time coordinate x^0 and space coordinates $x^1, ..., x^9$. In ref [84], Witten investigated the existence of bound states in an stack of n Dp-branes and proposed the effective low-energy action for such objects, which is the ten-dimensional $U(n)$ supersymmetric gauge theory dimensionally reduced to $p+1$ dimensions.

A p-brane is an object that modifies the boundary conditions of open strings. In addition to Neumann boundary conditions, a p-brane introduces Dirichlet boundary conditions in $(d-p-1)$ directions, as follows:

$$\begin{aligned} X^{p+1}(\sigma,t) &= .... = X^9(\sigma,t) = 0 \quad \text{(Dirichlet boundary condition)}, \\ \partial_\sigma X^1(\sigma,t) &= .... = \partial_\sigma X^p(\sigma,t) = 0 \quad \text{(Neumann boundary conditions)}. \end{aligned} \tag{A.27}$$

Due to (A.27), the zero modes X^j with $j > p$ are frozen, and the massless particles are functions only of $X_1, .., X_p$. The massless bosons $A_i(X^s)(i, s = 0, ..., p)$ propagate as U(1) gauge bosons on the p-brane, while the other components $\phi_j(X^s)(j > p, s = 0, ..., p)$ become scalars fields on the p-brane.

The vertex operators are given by:

$$\begin{aligned} V_A &= \sum_{i=0}^{p} A_i(X^s)\partial_\tau X^i, \\ V_\phi &= \sum_{j>p} \phi_j(X^s)\partial_\sigma X^j. \end{aligned} \tag{A.28}$$

For ϕ_j =constant, the boundary integral of V_ϕ imply the change $X^j \to X^j + \phi^j$ for $j > p$. Thus the scalars $\phi_j, j > p$ can be interpreted as the position coordinates of the p-brane.

The theory on the $(p+1)$ dimensional brane world-volume is naturally the ten-dimensional $U(1)$ supersymmetric gauge theory dimensionally reduced to $(p+1)$ dimensions.

A.4.1 Bound States of Dp-branes

Bound states of n parallel Dirichlet p-branes can be described by the low-energy limit when the branes are nearby. We consider the case of two parallel Dirichlet p-branes, one at $X^j = 0$, and one at $X^j = a^j$ (j>p). The branes are connecting by strings. They can start and end on the same brane and give a $U(1) \times U(1)$ gauge theory (with one $U(1)$ living one each p-brane) or they can start in the first brane and end in the second (or viceversa). In this last case the strings have $U(1) \times U(1)$ charges. The ground state of this configuration has an energy $T \mid a \mid$, with T and $\mid a \mid$ the tension and length of the string respectively. When $\mid a \mid \rightarrow 0$ the charged vector bosons become massless and the $U(1) \times U(1)$ gauge symmetry is enlarged to a $U(2)$ symmetry. In the same way, as n parallel branes become coincident, one has a $U(n)$ gauge symmetry on the p-brane.

The field content in the effective action is given by the $U(n)$ gauge field $A_j(X^s, t), s, j = 1, ..., p$, and the scalar fields $\phi_j(X^s, t), j > p, s = 1, ..., p$, in the adjoint representation of $U(n)$.

The effective potential for ϕ^j is obtained by dimensional reduction of the Yang-Mills theory:

$$V = \frac{T^2}{2} \sum_{i,j=p+1}^{9} Tr \left[\phi^i, \phi^j\right]^2 . \tag{A.29}$$

The supersymmetric classical ground states have $[\phi^i, \phi^j] = 0$, for all i, j, and so the ϕ's can be diagonalized simultaneously, giving $\phi^i = diag(a^i_{(1)}, ..., a^i_{(n)})$. Witten interpreted the $a^i_{(k)}$ for $k = 1, ..., n$ as the position of the k-th brane in the stack of n parallel p-branes. This interpretation is confirmed by expanding V around the given classical solution that gives $T \mid a_\lambda - a_\mu \mid, 1 \leq \lambda < \mu \leq n$, corresponding with masses of strings with one end at a_λ an one at a_μ as showed in the previous example of two p-branes.

In the $p = 0$ case we have D0-branes (space points). The effective action is given by the ten-dimensional supersymmetric Yang-Mills theory dimensionally reduced to one-dimension. The fields in the theory are a $U(n)$ gauge field $A_0(t)$ and the scalar fields $\phi_j(t)$ $(j = 1, ..., 9)$ that transform in the adjoint of $U(n)$ and can be represented as $n \times n$ matrices. The reduction to one dimension of the bosonic sector of the theory is obtained as follows: from the Lagrangian,

$$L_{YM} = -\frac{1}{4g^2} Tr\left(F^{\mu\nu} F_{\mu\nu}\right) = -\frac{1}{4g^2} Tr\left(\left[D_\mu, D_\nu\right]\left[D^\mu, D^\nu\right]\right), \tag{A.30}$$

with $D_\mu = \partial_\mu + igA_\mu$, the only non zero elements of $[D_\mu, D_\nu]$, after the dimensional reduction, are:

$$\begin{aligned} [D_0, D_j] &= ig\left[\partial_0, \phi_j\right] - g^2\left[A_0, \phi_j\right] = igD_0\phi_j, \quad j = 1, ..., 9 \\ [D_i, D_j] &= -g^2\left[\phi_i, \phi_j\right], \quad i \neq j = 1, ..., 9. \end{aligned} \tag{A.31}$$

Using (A.31) in (A.30) we obtain the dimensional reduction of L_{YM},

$$L'_{YM} = \frac{1}{4} Tr \left(\sum_{j=1}^{9} D_0 \phi_j D_0 \phi_j - \sum_{i \neq j=1}^{9} [\phi_i, \phi_j]^2 \right) . \tag{A.32}$$

This is the action of D0-branes used in our work (cf. (3.1)). The inclusion of the Chern-Simons kinetic term can be obtained by adding further D7-branes, that create a magnetic field for the D0-branes as explained in Ref.[67].

A.5 Projections of matrix Landau states

In this appendix we prove some properties of the projections of states described in section 3.2.3. Let us first show that the general solution of the second Landau level projection (3.30),

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^2 \Phi(\overline{B}, \overline{A}, \psi) \;=\; 0 \;, \qquad \forall\; a, b \;. \tag{A.33}$$

and of the $k = 1$ Gauss law (3.20) is given by the expression (3.32). We start from the gauge invariant expressions involving N fields ψ,

$$\Phi = \varepsilon^{a_1 \dots a_N} \;\left(M_1\psi\right)_{a_1} \cdots \left(M_N\psi\right)_{a_N} \;, \tag{A.34}$$

where the M_i are polynomials of $\overline{B}$ and $\overline{A}$. In this appendix, we repeatedly use the graphical description of these expressions in terms of bushes as shown in Fig.3.1. Upon expanding (A.34) into monomials, we get a sum of bushes:

$$\Phi = \varepsilon^{a_1 \dots a_N} \left(P_1\psi\right)_{a_1} \cdots \left(P_N\psi\right)_{a_N} + b\, \varepsilon^{a_1 \dots a_N} \left(Q_1\psi\right)_{a_1} \cdots \left(Q_N\psi\right)_{a_N} + c\,\varepsilon \dots R_1 \cdots R_N + \cdots \;, \tag{A.35}$$

where the monomials in a bush, e.g. the $\{P_i\}$, are all different among themselves, and two sets of monomials, e.g. $\{P_i\}$ and $\{Q_j\}$, differ in one monomial (stem) at least.

The two derivatives in (A.33) act in all possible ways on the stems of the bushes, and can be represented by primed expressions, i.e. $P_i', P_i'', \dots$. Let us momentarily take bushes made of two stems, i.e. $N = 2$:

$$\begin{aligned} \Phi'' &= \varepsilon_{ab} \left(P_1''\psi_a \; P_2\psi_b + 2P_1'\psi_a \; P_2'\psi_b + P_1\psi_a \; P_2''\psi_b\right) \\ &+ b\; \varepsilon_{ab} \left(Q_1''\psi_a \; Q_2\psi_b + 2Q_1'\psi_a \; Q_2'\psi_b + Q_1\psi_a \; Q_2''\psi_b\right) + \cdots \;. \end{aligned} \tag{A.36}$$

We check the possibility of cancellations between terms belonging to two different bushes: these cannot occur between terms with the same pattern of derivatives, i.e. $P_1 \; P_2''$ and $Q_1 \; Q_2''$, because at least one monomial is different between the two bushes: $P_1 \neq Q_1$ or $P_2 \neq Q_2$. There can be cancellations between terms that have different derivatives, i.e. $P_1'' \; P_2 + b \; Q_1' \; Q_2' = 0$, but then the symmetric term would not cancel, $P_1 \; P_2'' + b \; Q_1' \; Q_2' \neq 0$. We conclude that there cannot be complete cancellations between two bushes and that each bush should vanish independently.

Consider now the action of derivatives on the stems of a single bush; the terms with two derivatives, i.e $P_i''\psi$, should vanish independently, because the stems in bush are all different. Thus, there cannot be more than one $\overline{A}$ per stem. Next, we distribute one derivative per stem: each of them cuts the $\overline{A}_{ab}$ from the stem leaving fixed indices at the end points, leading to the expression,

$$\left(\frac{\partial}{\partial \overline{A}_{ab}}\right)^2 \Phi = \varepsilon_{cd}\, \overline{B}^{n_1}_{ca}\, \overline{B}^{n_2}_{da} \quad \left(\overline{B}^{m_1}\psi\right)_b \left(\overline{B}^{m_2}\psi\right)_b \ . \tag{A.37}$$

This vanishes by antisymmetry of the epsilon tensor, provided that $n_1 = n_2$, i.e. that the $\overline{A}$ matrices appear at the same level on the two stems. Furthermore, for $N > 2$ one can repeat the argument, having $N-2$ spectator stems over which the derivatives do not act; the height condition should then applies for any pair of stems that have one $\overline{A}$. In conclusion, all $\overline{A}$ should appear in the stems at the same height, leading to the general solution (3.32).

A.5.1 States obeying the $A^3 = 0$ projection

We now discuss the solution of the $A^3 = 0$ condition. Bushes can have one, two, three $\overline{A}$'s per stem and more: we consider each case in turn. For three $\overline{A}$'s and more, $A^3 = 0$ can act on a single stem and not vanish: this limits the number of $\overline{A}$'s per stem to two.

A) For bushes that have single-$\overline{A}$ stems only, we should examine the action of A^3 an all triples of stems (1-1-1 action). This vanishes by antisymmetry (cf. (A.37)) if for any triple considered, two $\overline{A}$'s are at the same height. It follows that on single-$\overline{A}$ bushes, the $\overline{A}$'s can stay at two heights, i.e. form two bands.

B) For bushes with double-$\overline{A}$ stems only, A^3 can act on pairs (2-1 action (B_1)) or on triples (1-1-1 action (B_2)) of stems.

B_1) Consider the action 2-1 on the pair:

$$\begin{aligned}\left(\tfrac{\partial}{\partial \overline{A}_{ab}}\right)^3 \quad & \varepsilon_{ij}\left(C\overline{A}D\overline{A}E\right)_i \left(F\overline{A}G\overline{A}H\right)_j \\ &= \varepsilon_{ij}\left[\left(C_{ia}D_{ba}E_b\right)\left(F_{ja}G\overline{A}H_b\right) + \left(C_{ia}D_{ba}E_b\right)\left(F\overline{A}G_{ja}H_b\right)\right. \\ &\quad + \left.\left(C_{ia}D\overline{A}E_b\right)\left(F_{ja}G_{ba}H_b\right) + \left(C\overline{A}D_{ia}E_b\right)\left(F_{ja}G_{ba}H_b\right)\right]\end{aligned} \tag{A.38}$$

The first and third term in this equation vanish independently when $C = F$ due to the earlier identity $\varepsilon_{ij}u_iu_j = 0$; the sum of the second and fourth term vanishes for $D = G$ due to the possibility of factorizing an expression of the type, $\varepsilon_{ij}\left(u_iv_j + u_jv_i\right) = 0$. We thus found that the double-$\overline{A}$ stems should have $\overline{A}$ located on two heights (two bands).

B_2) There are $2^3 = 8$ possible actions 1-1-1 on triples of stems involving two $\overline{A}$'s each. Having already enforced condition (B_1), their $\overline{A}$'s are located on two bands. The 8 terms generated by the action of A^3 are found vanish by the same two mechanism found in (A.38). Therefore, there are no new conditions.

C) For bushes involving both double- and single-$\overline{A}$ stems, we should again consider the actions 2-1 on pairs (C_1) and 1-1-1 on triples (C_2) of stems.

C_1) We consider the pair made by one double-$\overline{A}$ stem and one single-$\overline{A}$ stem; the double derivative acts necessarily on the former stem, thus producing a unique term. This vanishes as $\varepsilon_{ij} u_i v_j = 0$ if $u = v$, namely if the $\overline{A}$ on the single stem-$\overline{A}$ is located at the same height of the lower $\overline{A}$ in the double-$\overline{A}$ stem. It implies that the $\overline{A}$ form again two bands, but those on single-$\overline{A}$ stems should stay in the lower band.

C_2) The three derivatives act 1-1-1 on triples of stems with number of $\overline{A}$'s equal to (2,1,1) or (2,2,1), yielding 2 and 4 terms respectively. All these terms vanish independently, because single-$\overline{A}$ stems already have their $\overline{A}$ on the lowest band by condition C_1.

In summary, the $A^3 = 0$ projection allows two $\overline{A}$ per stem at most, that should form two bands. If both single- and double-$\overline{A}$ stems are present in the same bush, the $\overline{A}$ on single stems should stay on the lower band. All these features have been checked on the computer for small-N examples.

A.5.2 States with $A^4 = 0$ projection

Again the action of the four derivatives on a single stem is not vanishing and requires three $\overline{A}$ per stem at most. Hereafter we list the possible actions of the four derivatives.

A) If there are single-$\overline{A}$ stems only, the derivative action is 1-1-1-1: for every four-plet of stems, two $\overline{A}$ should be at the same height; thus, three bands of $\overline{A}$ can be formed on bushes.

B) If there are double-$\overline{A}$ stems only, there can be: (B_1) 2-2 action on pairs of stems; (B_2) 2-1-1 action on triples of stems, (B_3) 1-1-1-1 action on four-plets of stems.

B_1) There is a single term that vanishes if the lower $\overline{A}$ are at the same height.

B_2) There are 12 terms that vanish by the same two mechanisms of B_1 in the previous $A^3 = 0$ case, provided the upper $\overline{A}$ form another band, i.e. stay at the same height.

B_3) All terms vanish once the previous conditions are enforced.

In summary, double-$\overline{A}$ stems should have their $\overline{A}$'s on two bands.

C) If there are triple-$\overline{A}$ stems only, there can be: (C_1) 3-1 and 2-2 actions on pairs; (C_2) 2-1-1 action on triples; (C_3) 1-1-1-1 on four-plets.

C_1) There are 6 terms for the 3-1 action and 9 for the 2-2 action: these cancel individually or in pairs by the two mechanisms of B_1 in the previous $A^3 = 0$ case, provided that all $\overline{A}$'s form three bands.

C_2) The action 2-1-1 on 3-$\overline{A}$ stems generates 81 terms, that are satisfied once C_1 has been enforced.

C_3) The terms generated by the 1-1-1-1 action vanish because there are at least two derivatives of $\overline{A}$ at the same height.

In summary, triple-$\overline{A}$ stems should have their $\overline{A}$'s on three bands.

D) Consider now the case of stems having two or one $\overline{A}$ each, as for states filling the second and third Landau level. From the previous analysis we know that the double-$\overline{A}$ form 2 bands (case (B)) and the triple-$\overline{A}$ stems can have 3 bands (case (C)). We should only consider the new cases when the four derivatives act on stems of mixed type. There can be: (D_{11}) 2-1-1 action on 2-2-1 stems; (D_{12}) 2-1-1 action on 2-1-1 stems; (D_{21}) 1-1-1-1 action on 2-2-2-1 stems; (D_{22}) 1-1-1-1 action on 2-2-1-1 stems; (D_{23}) 1-1-1-1 action on 2-1-1-1 stems;

D_{11}) Given that one derivative acts on the single-$\overline{A}$ stem, the remaining three derivatives cancel as in case (B_1) of $A^3 = 0$, on stems already having two $\overline{A}$ bands. No new conditions.

D_{12}) The condition is that on any pair of single-$\overline{A}$ stems, one of them has the $\overline{A}$ on the lowest band of the double-$\overline{A}$ stems. This allows single-$\overline{A}$ to stay on any of the two bands, with some exceptions.

D_{21}) It is satisfied. D_{22}) It yields the same condition as (D_{12}). D_{23}) For every triple of single-$\overline{A}$ stems, two should be on the same band. The solution is that each of the 2 bands of double-$\overline{A}$ stems are allowed (weaker than (D_{12})).

In summary, mixed double- and single-$\overline{A}$ stems should have their $\overline{A}$'s forming two bands with some exceptions.

E) The most relevant case for Jain's ground state at $\nu^* = 4$ is for mixed stems with one, two and three $\overline{A}$'s. Owing to the previous conditions, each individual type is already organized in 3, 2 and 3 bands respectively. The possible new actions of the four derivatives are the following ones: (E_{11}) 3-1 and 2-2 actions on pairs of type 3-2; (E_{12}) 3-1 action on pairs of type 3-1; (E_{21}) 2-1-1 action on triples of type 3-3-2, (E_{22}) on triples 3-2-2, (E_{23}) on triples 3-2-1 and (E_{24}) on triples 3-1-1 ; (E_4) 1-1-1-1 actions on all stem types.

E_{11}) There is cancelation by the usual two mechanisms ((B_1) of $A^3 = 0$) provided that the $\overline{A}$'s of double-$\overline{A}$ stems stay in the lowest of the three bands of triple-$\overline{A}$ stems.

E_{12}) As before, the $\overline{A}$ of single-$\overline{A}$ stems should all align on the lowest of the three bands of the triple-$\overline{A}$ stems.

Once these two conditions are enforced, the other E-type actions are checked. In summary, mixed triple-, double- and single-$\overline{A}$ stems should have their $\overline{A}$'s forming three bands, with the condition that stems with less that three $\overline{A}$'s should align their $\overline{A}$'s on the lowest available bands. This is the condition enforcing the uniqueness of the state with maximal filling $\nu^* = 4$ as explained in section 3.2.3. The same mechanism works for the $A^m = 0$ ground states with $\nu^* = m$ (3.39) that contain stems of any number of $\overline{A}$'s.

A.6 Gauge invariance of the projection

Here is an explicit proof that the projection $(A_{ab})^m\Psi$ (cf 3.30) is a gauge invariant condition on quantum states. Consider the more general relation for $m = 2$:

$$A_{ab}\ A_{a',b'}\ \Psi\left(\overline{A},\overline{B}\right)\ =\ M_{bb'}\left(\overline{A},\overline{B}\right)\ V_a W_{a'}\ . \tag{A.39}$$

The wave function is assumed to be gauge invariant: $\Psi(\overline{A},\overline{B}) = \Psi(U\overline{A}U^\dagger, U\overline{B}U^\dagger)$. The form in the r.h.s. is specific of the bush states of section 3.2.2, but this is not relevant for the argument. The matrix $M_{bb'}$ vanishes for $a = a', b = b'$ because Ψ is assumed to be one solution of the constraint. In general, there are several terms in the r.h.s. with that structure, but the matrices $M_{bb'}$ should all vanish independently because they are multiplied by monomials $V_a W_{a'}$ that are all independent [13].

Let us now multiply by unitary matrices and sum over indices on both sides to realize a gauge transformation of the two A's:

$$\begin{aligned}\left(UAU^\dagger\right)_{ab}\ \left(UAU^\dagger\right)_{a',b'}\ \Psi\left(\overline{A},\overline{B}\right)\ &=\ M_{\tilde{b}\tilde{b}'}\left(\overline{A},\overline{B}\right)\ U^\dagger_{\tilde{b}b}\ U^\dagger_{\tilde{b}'b'}\ (UV)_a(UW)_{a'}\ ,\\ &=\ M_{bb'}\left(U\overline{A}U^\dagger, U\overline{B}U^\dagger\right)\ (UV)_a(UW)_{a'}\ . \end{aligned} \tag{A.40}$$

The resulting expression of $M(U\overline{A}U^\dagger, U\overline{B}U^\dagger)_{bb'}$ vanishes whenever $M(\overline{A},\overline{B})_{bb'}$ does, i.e. for $b = b'$, because both vanish by polynomial identities that do not depend on the specific vales of the variables. Therefore, a solution of the constraint remains a solution after gauge transformation.

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